J. Aust. Math. Soc. **74** (2003), 235–238

ON $\mathfrak{F}$-HYPEREXCENTRIC MODULES FOR LIE ALGEBRAS

DONALD W. BARNES

(Received 23 November 2001; revised 4 February 2002)

Communicated by R. B. Howlett

Abstract

Let $\mathfrak{F}$ be a saturated formation of soluble Lie algebras over the field F, and let $L \in \mathfrak{F}$. Let V and W be $\mathfrak{F}$-hypercentral and $\mathfrak{F}$-hyperexcentric L-modules respectively. Then $V \otimes_F W$ and $\mathrm{Hom}_F(V, W)$ are $\mathfrak{F}$-hyperexcentric and $H^n(L, W) = 0$ for all n.

2000 *Mathematics subject classification*: primary 17B30, 17B56; secondary 17B50, 17B55.
Keywords and phrases: Soluble Lie algebras, cohomology.

1. Introduction

Let $\mathfrak{F}$ be a saturated formation of finite-dimensional soluble Lie algebras over the field F. Let $L \in \mathfrak{F}$ and let W be an $\mathfrak{F}$-excentric irreducible L-module. Results in Barnes and Gastineau-Hills [4] imply that $H^n(L, W) = 0$ for $n \leq 2$, and $H^n(L, W) = 0$ for all n was proved for some special cases, suggesting that this might be true in general. This was proved in Barnes [3] for fields F of characteristic 0. The proof involved a description of the saturated formations over an arbitrary field of characteristic 0. Over a field of non-zero characteristic, the saturated formations are much more complicated and no useful description is available. In this paper, we give a proof independent of the characteristic of the field. All algebras and modules considered are assumed finite-dimensional over F.

An irreducible L-module V is called $\mathfrak{F}$-central if the split extension of V by $L/\mathscr{C}_L(V)$ is in $\mathfrak{F}$ and $\mathfrak{F}$-excentric otherwise. An L-module V is called $\mathfrak{F}$-hypercentral if every composition factor of V is $\mathfrak{F}$-central and is called $\mathfrak{F}$-hyperexcentric if ev-

This work was done while the author was an Honorary Associate of the School of Mathematics and Statistics, University of Sydney.

ery composition factor of V is $\mathfrak{F}$-excentric. We need the following theorems from Barnes [2].

THEOREM 1.1 ([2, Theorem 4.4]). *Suppose $L \in \mathfrak{F}$ and let V be an L-module. Then V is the direct sum of an $\mathfrak{F}$-hypercentral L-module and an $\mathfrak{F}$-hyperexcentric L-module.*

THEOREM 1.2 ([2, Theorem 2.1]). *Let V and W be $\mathfrak{F}$-hypercentral L-modules. Then $V \otimes_F W$ and $\mathrm{Hom}_F(V, W)$ are $\mathfrak{F}$-hypercentral.*

Results showing that $H^n(L, V) = 0$ for $\mathfrak{F}$-excentric irreducible L-modules V are easily extended to $\mathfrak{F}$-hyperexcentric modules by using the cohomology exact sequence and induction over the composition length of the module.

2. $\mathfrak{F}$-hyperexcentric modules

In this section, we obtain a cohomological characterisation of $\mathfrak{F}$-hyperexcentric L-modules. The characterisation needs to use other algebras besides the algebra L from which we start.

DEFINITION 2.1. Suppose $L \in \mathfrak{F}$. The *cone of* L in $\mathfrak{F}$ is the class $(\mathfrak{F}/L)$ of all pairs (M, ϵ) where $M \in \mathfrak{F}$ and $\epsilon : M \to L$ is an epimorphism. We usually omit ϵ from the notation, writing simply $M \in (\mathfrak{F}/L)$.

Any L-module V is an M-module via ϵ for any $M \in (\mathfrak{F}/L)$. Then V is $\mathfrak{F}$-hypercentral as M-module if and only if it is $\mathfrak{F}$-hypercentral as L-module. It follows that if V is an $\mathfrak{F}$-hyperexcentric L-module, then $H^n(M, V) = 0$ for all $M \in (\mathfrak{F}/L)$ and $n \leq 2$. We would like a converse to this.

THEOREM 2.2. *Let $\mathfrak{F}$ be a saturated formation and let $L \in \mathfrak{F}$. Suppose V is an L-module such that for all $M \in (\mathfrak{F}/L)$, $H^1(M, V) = 0$. Then V is $\mathfrak{F}$-hyperexcentric.*

PROOF. V is the direct sum of an $\mathfrak{F}$-hypercentral module and an $\mathfrak{F}$-hyperexcentric module. Thus we may suppose without loss of generality, that V is $\mathfrak{F}$-hypercentral, and we then have to prove $V = 0$. Suppose $V \neq 0$ and let W be a minimal submodule of V. We form the direct sum A of sufficiently many copies of W to ensure that $\dim \mathrm{Hom}_L(A, V) > \dim H^2(L, V)$, and construct the split extension M of A by L. As W is $\mathfrak{F}$-central, $M \in (\mathfrak{F}/L)$. We use the Hochschild-Serre spectral sequence to calculate $H^1(M, V)$. We have

$$E_2^{20} = H^2(M/A, V^A) = H^2(L, V)$$

https://doi.org/10.1017/S1446788700003268 Published online by Cambridge University Press

and

$$E_2^{01} = H^0(M/A, H^1(A, V)) = \mathrm{Hom}_F(A, V)^L = \mathrm{Hom}_L(A, V).$$

Thus $\dim d_2^{01}(E_2^{01}) \leq \dim H^2(L, V) < \dim E_2^{01}$, so $E_3^{01} = \ker d_2^{01} \neq 0$ and so $H^1(M, V) \neq 0$ contrary to assumption. □

THEOREM 2.3. *Let $\mathfrak{F}$ be a saturated formation and let $L \in \mathfrak{F}$. Suppose V is an $\mathfrak{F}$-hypercentral L-module and let W be an $\mathfrak{F}$-hyperexcentric L-module. Then $V \otimes_F W$ and* $\mathrm{Hom}_F(V, W)$ *are $\mathfrak{F}$-hyperexcentric.*

PROOF. Let $M \in (\mathfrak{F}/L)$. Then V and W are $\mathfrak{F}$-hypercentral and $\mathfrak{F}$-hyperexcentric respectively as M-modules, and every M-module extension X of W by V splits. Thus $H^1(M, \mathrm{Hom}_F(V, W)) = 0$. By Theorem 2.2, $\mathrm{Hom}_F(V, W)$ is $\mathfrak{F}$-hyperexcentric. By Theorem 1.2, the dual module $V^* = \mathrm{Hom}_F(V, F)$ is $\mathfrak{F}$-hypercentral. As

$$V \otimes_F W \simeq V^{**} \otimes_F W \simeq \mathrm{Hom}_F(V^*, W),$$

the result follows. □

This suggests that we could have some sort of $\mathbb{Z}_2$-grading on the class of all L-modules. However, the tensor product of two $\mathfrak{F}$-hyperexcentric modules need not be $\mathfrak{F}$-hypercentral. Anything can happen as is shown by the following examples. Here, $\mathfrak{N}$ denotes the saturated formation of all nilpotent algebras.

EXAMPLE 2.4. Suppose the characteristic of F is not 2. Let $L = \langle e \rangle$ be the 1-dimensional algebra, and let $V = \langle v \rangle$ and $W = \langle w \rangle$ be the modules with action given by $ev = v$ and $ew = w$. Then V and W are $\mathfrak{N}$-excentric and $V \otimes_F W$ is $\mathfrak{N}$-excentric.

EXAMPLE 2.5. Let $L = \langle e \rangle$ be the 1-dimensional algebra, and let $V = \langle v \rangle$ and $W = \langle w \rangle$ be the modules with action given by $ev = v$ and $ew = -w$. Then V and W are $\mathfrak{N}$-excentric and $V \otimes_F W$ is $\mathfrak{N}$-central.

EXAMPLE 2.6. Suppose the characteristic of F is not 2. Let $i \in \bar{F}$ have minimum polynomial $x^2 + 1$. Let $L = \langle e \rangle$ be the 1-dimensional algebra, and let V and W be 2-dimensional modules with the action given by the matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The eigenvalues of the action of e on $V \otimes_F W$ are the sums of the eigenvalues on V and W, thus $2i, 0, 0, -2i$. Thus $V \otimes_F W$ is the direct sum of a 2-dimensional module on which the action is trivial, and a 2-dimensional module on which the action is given by the matrix $2A$. It is thus the sum of an $\mathfrak{N}$-hypercentral and an $\mathfrak{N}$-excentric module.

3. Cohomology of $\mathfrak{F}$-hyperexcentric modules

We can now prove the desired theorem on the cohomology of $\mathfrak{F}$-hyperexcentric modules.

THEOREM 3.1. *Let $\mathfrak{F}$ be a saturated formation and let $L \in \mathfrak{F}$. Let V be an $\mathfrak{F}$-hyperexcentric L-module. Then $H^n(L, V) = 0$ for all n.*

PROOF. By the cohomology exact sequence for a submodule W

$$\cdots \to H^n(L, W) \to H^n(L, V) \to H^n(L, V/W) \to \cdots,$$

we need only consider the case in which V is irreducible. We use induction over $\dim L$. The result holds if $\dim L = 1$, so suppose $\dim L > 1$. Let A be a minimal ideal of L. We use the Hochschild -Serre spectral sequence. We have

$$E_2^{rs} = H^r(L/A, H^s(A, V)).$$

If A acts non-trivially on V, then $V^A = 0$ and $H^s(A, V) = 0$ for all s by Barnes [1, Theorem 1]. If on the other hand, A acts trivially on V, then $H^s(A, V) = \mathrm{Hom}_F(\Lambda^s A, V)$. Now $\Lambda^s A$ is a submodule of the tensor power of A, so is $\mathfrak{F}$-hypercentral by Theorem 1.2. By Theorem 2.3, $\mathrm{Hom}_F(\Lambda^s A, V)$ is $\mathfrak{F}$-hyperexcentric. By induction over $\dim L$, we have $H^r(L/A, H^s(A, V)) = 0$ for all r, s. In either case, we have $H^r(L/A, H^s(A, V)) = 0$ for all r, s. By the Hochschild-Serre spectral sequence, $H^n(L, V) = 0$ for all n. □

References

[1] D. W. Barnes, 'On the cohomology of soluble Lie algebras', *Math. Z.* **101** (1967), 343–349.
[2] ———, 'On $\mathfrak{F}$-hypercentral modules for Lie algebras', *Arch. Math.* **30** (1978), 1–7.
[3] ———, 'Saturated formations of soluble Lie algebras in characteristic 0', *Arch. Math.* **30** (1978), 477–480.
[4] D. W. Barnes and H. M. Gastineau-Hills, 'On the theory of soluble Lie algebras', *Math. Z.* **106** (1968), 343–354.

1 Little Wonga Rd.
Cremorne NSW 2090
Australia
e-mail: donb@netspace.net.au

https://doi.org/10.1017/S1446788700003268 Published online by Cambridge University Press