Glasgow Math. J. **53** (2011) 419–426. © Glasgow Mathematical Journal Trust 2011.
doi:10.1017/S0017089511000413.

CHARACTERS OF PRIME DEGREE

EDITH ADAN-BANTE

Department of Mathematical Science, Northern Illinois University, Watson Hall 320, DeKalb, IL 60115-2888, USA
e-mail: EdithAdan@illinoisalumni.org

(Received 4 June 2008; accepted 26 May 2011)

Abstract. Let G be a finite nilpotent group, χ and ψ be irreducible complex characters of G with prime degree. Assume that $\chi(1) = p$. Then, either the product $\chi\psi$ is a multiple of an irreducible character or $\chi\psi$ is the linear combination of at least $\frac{p+1}{2}$ distinct irreducible characters.

2010 *Mathematics Subject Classification.* 20c15

1. Introduction. Let G be a finite group and $\chi, \psi \in \mathrm{Irr}(G)$ be irreducible complex characters of G. We can check that the product $\chi\psi$ of χ and ψ, where $\chi\psi(g) = \chi(g)\psi(g)$ for all g in G, is a character and so it can be expressed as a linear combination of irreducible characters. Let $\eta(\chi\psi)$ be the number of distinct irreducible constituents of the product $\chi\psi$.

Theorem A. *Let G be a finite nilpotent group, χ and ψ be irreducible complex characters of prime degree. Assume that $\chi(1) = p$. Then, one of the following holds:*

(i) $\chi\psi$ is the sum of p^2 distinct linear characters.

(ii) $\chi\psi$ is the sum of p distinct linear characters of G and of $p - 1$ distinct irreducible characters of G with degree p.

(iii) all the irreducible constituents of $\chi\psi$ are of degree p. Also, either $\chi\psi$ is a multiple of an irreducible character, or it has at least $\frac{p+1}{2}$ distinct irreducible constituents and at most p distinct irreducible constituents, i.e.

$$\text{either } \eta(\chi\psi) = 1 \text{ or } \frac{p+1}{2} \leq \eta(\chi\psi) \leq p.$$

(iv) $\chi\psi$ is an irreducible character.

It is proved in Theorem A of [**1**] that given any prime p, any p-group P, any faithful characters $\chi, \psi \in \mathrm{Irr}(P)$, either the product $\chi\psi$ is a multiple of an irreducible, or $\chi\psi$ is the linear combination of at least $\frac{p+1}{2}$ distinct irreducible characters, i.e. either $\eta(\chi\psi) = 1$ or $\eta(\chi\psi) \geq \frac{p+1}{2}$. It is proved in [**4**] that given any prime p and any integer $n > 0$, there exists a p-group P and characters $\varphi, \gamma \in \mathrm{Irr}(P)$ such that $\eta(\varphi\gamma) = n$. Thus, without the hypothesis that the characters in Theorem A of [**1**] are faithful, the result may not hold. In this note, we are proving that if the characters have 'small' degree then the values that $\eta(\chi\psi)$ can take have the same constraint as if they were faithful.

Present address: Department of Mathematics, University of Saint Thomas, 2115 Summit Avenue, Saint Paul, MN 55105-1079, USA

2. Proofs. We are going to use the notation of [**5**]. In addition, we denote by $\mathrm{Lin}(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi(1) = 1\}$ the set of linear characters, and by $\mathrm{Irr}(G \mathrm{mod} N) = \{\chi \in \mathrm{Irr}(G) \mid \mathrm{Ker}(\chi) \geq N\}$ the set of irreducible characters of G that contain in their kernel the subgroup N. Also, denote by $\overline{\chi}$ the complex conjugate of χ, i.e. $\overline{\chi}(g) = \overline{\chi(g)}$ for all g in G.

Lemma 2.1. *Let G be a finite group and $\chi, \psi \in \mathrm{Irr}(G)$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$, for some $n > 0$, be the distinct irreducible constituents of the product $\chi\psi$ and $a_1, a_2, \ldots, a_n$ be the unique positive integers such that*

$$\chi\psi = \sum_{i=1}^{n} a_i \alpha_i.$$

If $\alpha_1(1) = 1$, then $\psi\overline{\alpha_1} = \overline{\chi}$. Hence, the distinct irreducible constituents of the character $\chi\overline{\chi}$ are 1_G, $\overline{\alpha_1}\alpha_2$, $\overline{\alpha_1}\alpha_2, \ldots, \overline{\alpha_1}\alpha_n$, and

$$\chi\overline{\chi} = a_1 1_G + \sum_{i=2}^{n} a_i(\overline{\alpha_1}\alpha_i).$$

Proof. See Lemma 4.1 of [**3**]. □

Lemma 2.2. *Let G be a finite p-group for some prime p and $\chi \in \mathrm{Irr}(G)$ be a character of degree p. Then, one of the following holds:*

(i) $\chi\overline{\chi}$ is the sum of p^2 distinct linear characters.

(ii) $\chi\overline{\chi}$ is the sum of p distinct linear characters of G and of $p-1$ distinct irreducible characters of G with degree p.

Proof. See Lemma 5.1 of [**2**]. □

Lemma 2.3. *Let G be a finite p-group, for some prime p, and $\chi, \psi \in \mathrm{Irr}(G)$ be characters of degree p. Then, either $\eta(\chi\psi) = 1$ or $\eta(\chi\psi) \geq \frac{p+1}{2}$.*

Proof. Assume that the lemma is false. Let G and $\chi, \psi \in \mathrm{Irr}(G)$ be a counterexample of the statement, i.e. $\chi(1) = \psi(1) = p$ and $1 < \eta(\chi\psi) < \frac{p+1}{2}$.

Working with the group $G/(\mathrm{Ker}(\chi) \cap \mathrm{Ker}(\psi))$, by induction on the order of G, we may assume that $\mathrm{Ker}(\chi) \cap \mathrm{Ker}(\psi) = \{1\}$. Set $n = \eta(\chi\psi)$. Let $\theta_i \in \mathrm{Irr}(G)$, for $i = 1, \ldots, n$, be the distinct irreducible constituents of $\chi\psi$. Set

$$\chi\psi = \sum_{i=1}^{n} m_i \theta_i \tag{2.4}$$

where $m_i > 0$ is the multiplicity of θ_i in $\chi\psi$.

If $\chi\psi$ has a linear constituent, then by Lemmas 2.1 and 2.2 we have that $\eta(\chi\psi) \geq p$. If $\chi\psi$ has an irreducible constituent of degree p^2, then $\chi\psi \in \mathrm{Irr}(G)$ and so $\eta(\chi\psi) = 1$. Thus, we may assume that $\theta_i(1) = p$ for $i = 1, \ldots, n$.

Since G is a p-group, there must exist a subgroup H and a linear character ξ of H such that $\xi^G = \chi$. Then, $|G : H| = \chi(1) = p$ and thus H is a normal subgroup. By Clifford theory, we have then

$$\chi_H = \sum_{i=1}^{p} \xi_i \tag{2.5}$$

for some $\xi_1 = \xi, \ldots, \xi_p$ distinct linear characters of H.

Claim 2.6. *H is an abelian group.*

Proof. Suppose that $\psi_H \in \mathrm{Irr}(H)$. Since $(\xi\psi_H)^G = \chi\psi$ by Exercise 5.3 of [**5**], and $\xi\psi_H \in \mathrm{Irr}(H)$, it follows that either $\xi\psi_H$ induces irreducibly, and thus $\eta(\chi\psi) = 1$, or $\xi\psi_H$ extends to G and thus $(\xi\psi_H)^G$ is the sum of the p distinct extensions of $\xi\psi_H$, i.e. $\eta(\chi\psi) = p$. Therefore, $\psi_H \notin \mathrm{Irr}(G)$ and since H is normal in G of index p and $\psi(1) = p$, ψ is induced from some $\tau \in \mathrm{Lin}(H)$.

Since both ξ and τ are linear characters, we have that $\mathrm{Ker}(\xi) \cap \mathrm{Ker}(\tau) \geq [H, H]$. Observe that $\mathrm{core}_G(\mathrm{Ker}(\xi) \cap \mathrm{Ker}(\tau)) = \mathrm{core}_G(\mathrm{Ker}(\xi)) \cap \mathrm{core}_G(\mathrm{Ker}(\tau)) = \mathrm{Ker}(\chi) \cap \mathrm{Ker}(\psi)$. Since H is a normal subgroup of G, so is $[H, H]$ and thus $\{1\} = \mathrm{Ker}(\chi) \cap \mathrm{Ker}(\psi) \geq [H, H]$. Therefore, H is abelian. □

By the previous claim, observe that ψ is also induced by some linear character τ of H and thus

$$\psi_H = \sum_{i=1}^{p} \tau_i \tag{2.7}$$

for some $\tau_1 = \tau, \ldots, \tau_p$ distinct linear characters of H. Observe also that the centre of both χ and ψ is contained in H.

Claim 2.8. $\mathbf{Z}(G) = \mathbf{Z}(\chi) = \mathbf{Z}(\psi)$.

Proof. Suppose that $\mathbf{Z}(\chi) \neq \mathbf{Z}(\psi)$. Set $U = \mathbf{Z}(\chi) \cap \mathbf{Z}(\psi)$. Either U is properly contained in $\mathbf{Z}(\chi)$, or it is properly contained in $\mathbf{Z}(\psi)$. We may assume that $U < \mathbf{Z}(\psi)$ and thus we may find a subgroup $T \leq \mathbf{Z}(\psi)$ such that T/U is chief factor of G. Since H is abelian, $\mathbf{Z}(\psi) < H$ and $\tau^G = \psi$, then $\psi_T = p\tau_T$ and so $(\tau_i)_T = \tau_T$ for $i = 1, \ldots, p$. Because $\xi^G = \chi$, $\xi \in \mathrm{Lin}(H)$ and $T \nleq \mathbf{Z}(\chi)$, the stabilizer of ξ_T is H. Thus, the stabilizer of $\xi_T\tau_T$ in G is H. By Clifford theory, we have that $\xi\tau_i \in \mathrm{Lin}(H)$ induces irreducibly and $\xi\tau_i$ are distinct characters for $i = 1, \ldots, p$. By (2.7), we have that $\chi\psi = (\xi\psi_H)^G = (\xi(\tau_1 + \cdots + \tau_p))^G = (\xi\tau_1)^G + \cdots + (\xi\tau_p)^G$, and thus $\eta(\chi\psi) = p$. We conclude that such T cannot exist and so $\mathbf{Z}(\chi) = \mathbf{Z}(\psi)$.

Given any $z \in \mathrm{Z}(\chi)$ and $g \in G$, we have $z^g \cong z \pmod{\mathrm{Ker}(\chi)}$ since $\mathrm{Z}(G/\mathrm{Ker}(\chi)) = \mathrm{Z}(\chi)/\mathrm{Ker}(\chi)$. Hence, $[z, g] = z^{-1}z^g$ lies in $\mathrm{Ker}(\chi)$. This same z lies in $\mathrm{Z}(\psi) = \mathrm{Z}(\chi)$. Hence, $[z, g]$ also lies in $\mathrm{Ker}(\psi)$. Therefore, $[z, g] \in \mathrm{Ker}(\chi) \cap \mathrm{Ker}(\psi) = 1$ for every $z \in \mathrm{Z}(\chi) = \mathrm{Z}(\psi)$ and every $g \in G$. This implies that $\mathrm{Z}(\chi) = \mathrm{Z}(\psi) = \mathrm{Z}(G)$. □

Set $Z = \mathbf{Z}(G)$. Since Z is the centre of G, $\xi^G = \chi$ and $\tau^G = \psi$, we have

$$\chi_Z = p\xi_Z \text{ and } \psi_Z = p\tau_Z. \tag{2.9}$$

Because $\chi_Z\psi_Z = p^2\xi_Z\tau_Z$, (2.4) implies that

$$(\theta_i)_Z = p\xi_Z\tau_Z \tag{2.10}$$

for all $i = 1, \ldots, n$.

Let Y/Z be a chief factor of G with $Y \leq H$. Since Z is the centre of G and $Z = \mathbf{Z}(\chi)$, the set $\mathrm{Lin}(\, Y \mid \xi_Z\,)$ of all extensions of ξ_Z to linear characters is $\{(\xi_1)_Y = \xi_Y, (\xi_2)_Y, \ldots, (\xi_p)_Y\}$ and it is a single G-conjugacy class. By Clifford theory, we have

https://doi.org/10.1017/S0017089511000413 Published online by Cambridge University Press

that

$$\chi_Y = \sum_{i=1}^{p} (\xi_i)_Y. \tag{2.11}$$

Since H is the stabilizer of τ_Y in G and $\psi(1) = p$, as before we have that the set $\text{Lin}(Y \mid \tau_z) = \{(\tau_1)_Y = \tau_Y, (\tau_2)_Y, \ldots, (\tau_p)_Y\}$ and

$$\psi_Y = \sum_{i=1}^{p} (\tau_i)_Y. \tag{2.12}$$

Claim 2.13. *The stabilizer* $G_{\xi_Y \tau_Y} = \{g \in G \mid (\xi_Y \tau_Y)^g = \xi_Y \tau_Y\}$ *of* $\xi_Y \tau_Y \in \text{Lin}(Y)$ *in* G *is* H.

Proof. Assume notation (2.4). Since H is an abelian subgroup of index p in G, we have that $G_{\xi_Y \tau_Y} \geq H$ and thus either $G_{\xi_Y \tau_Y} = H$ or $G_{\xi_Y \tau_Y} = G$. Suppose $\xi_Y \tau_Y$ is a G-invariant character, i.e. $G_{\xi_Y \tau_Y} = G$. Since $|Y : Z| = p$ and $\xi_Y \tau_Y$ is an extension of $\xi_Z \tau_Z$, it follows then that all the extensions of $\xi_Z \tau_Z$ to Y are G-invariant. Thus, by (2.4) and (2.10), given any i, there exists some extension $\upsilon_i \in \text{Lin}(Y)$ of $\xi_Z \tau_Z$ such that $(\theta_i)_Y = p\upsilon_i$. Thus, $(\chi\psi)_Y = (\sum_{i=1}^n m_i \theta_i)_Y = \sum_{i=1}^n m_i (\upsilon_i)_Y = \sum_{i=1}^n m_i p \upsilon_i$ has at most $n < \frac{p+1}{2}$ distinct irreducible constituents. On the other hand, by (2.11) and (2.12) we have

$$(\chi\psi)_Y = \chi_Y \psi_Y = \left(\sum_{i=1}^{p} (\xi_i)_Y \right) \left(\sum_{j=1}^{p} (\tau_j)_Y \right) = p \sum_{j=1}^{p} \xi_Y (\tau_j)_Y,$$

and so $(\chi\psi)_Y$ has p distinct irreducible constituents. That is a contradiction and thus $G_{\xi_Y \tau_Y} = H$. □

By Clifford theory and the previous claim, we have that for each $i = 1, \ldots, n$, there exists a unique character $\sigma_i \in \text{Lin}(H \mid \xi_Y \tau_Y)$ such that

$$\theta_i = (\sigma_i)^G. \tag{2.14}$$

If $Y = H$, then $|G : Z| = |G : H||H : Z| = p^2$. Since $\chi(1) = \psi(1) = p$, by Corollary 2.30 of [**5**] we have that χ and ψ vanish outside Z. Since $\theta_i(1) = p$ for all i and $|G : Z| = |G : \mathbf{Z}(\theta_i)| = p^2$, it follows that there exists a unique irreducible character lying above $\xi_Z \tau_Z$ and thus $\eta(\chi\psi) = 1$.

2.15. Fix a subgroup $X \leq H$ of G such that X/Y is a chief factor of G. Let α, $\beta \in \text{Lin}(X)$ be the linear characters such that

$$\alpha = \xi_X \text{ and } \beta = \tau_X.$$

Since σ_i lies above $\xi_Y \tau_Y \in \text{Lin}(Y)$ for all i and X/Y is a chief factor of a p-group, there is some $\delta_i \in \text{Irr}(X \mod Y)$ such that

$$(\sigma_i)_X = \delta_i \alpha \beta. \tag{2.16}$$

Claim 2.17. *The subgroup* $[X, G]$ *generates* $Y = [X, G]Z$ *modulo* Z.

https://doi.org/10.1017/S0017089511000413 Published online by Cambridge University Press

Proof. Working with the group $\bar{G} = G/\mathrm{Ker}(\chi)$, using the same argument as in the proof of Claim 3.26 of [**1**], we have that $[\bar{X}, \bar{G}]$ generates $\bar{Y} = [\bar{X}, \bar{G}]\bar{Z}$ modulo $\bar{Z}$. Since $Z = \mathbf{Z}(\chi)$, we have that $\mathrm{Ker}(\chi) \leq Z$. Thus, $\bar{Z} = Z/\mathrm{Ker}(\chi)$ and the claim follows. □

2.18. Observe that G/H is cyclic of order p. So, we may choose $g \in G$ such that the distinct cosets of H in G are $H, Hg, Hg^2, \ldots, Hg^{p-1}$.

Since $\chi = \xi^G$ and $\xi_X = \alpha$, it follows from 2.15 that

$$\chi_X = \alpha + \alpha^g + \cdots + \alpha^{g^{p-1}} = \sum_{i=0}^{p-1} \alpha^{g^i}.$$

Similarly, we have that

$$\psi_X = \beta + \beta^g + \cdots + \beta^{g^{p-1}} = \sum_{j=0}^{p-1} \beta^{g^j}.$$

Combining the two previous equations, we have that

$$\chi_X \psi_X = \left(\sum_{j=0}^{p-1} \alpha^{g^j}\right)) \left(\sum_{j=0}^{p-1} \beta^{g^j}\right) = \sum_{i=0}^{p-1}\sum_{j=0}^{p-1} \alpha^{g^i}\beta^{g^j}. \tag{2.19}$$

By (2.4) and (2.16), we have that

$$(\chi\psi)_X = \left(\sum_{i=1}^{n} m_i\theta_i\right)_X = \sum_{i=1}^{n} m_i \left[\sum_{j=0}^{p-1} (\delta_i\alpha\beta)^{g^j}\right]. \tag{2.20}$$

Claim 2.21. *Let $g \in G$ be as in 2.18. For each $i = 0, 1, \ldots, p-1$, there exist $j \in \{0, 1, \ldots, p-1\}$ and $\delta_{g^i} \in \mathrm{Lin}(X \bmod Y)$ such that*

$$\alpha\beta^{g^i} = (\alpha\beta)^{g^j}\delta_{g^i}. \tag{2.22}$$

Also, $|\{\delta_{g^i} \mid i = 0, 1, 2, \ldots, p-1\}| \leq n$.

Proof. See Proof of Claim 3.30 of [**1**]. □

Claim 2.23. *Let $g \in G$ be as in 2.18. Then, there exist three distinct integers $i, j, k \in \{0, 1, 2, \ldots, p-1\}$, and some $\delta \in \mathrm{Irr}(X \bmod Y)$, such that*

$$\alpha\beta^{g^i} = (\alpha\beta)^{g^r}\delta,\ \alpha\beta^{g^j} = (\alpha\beta)^{g^s}\delta \text{ and } \alpha\beta^{g^k} = (\alpha\beta)^{g^t}\delta,$$

for some $r, s, t \in \{0, 1, 2, \ldots, p-1\}$.

Proof. See Proof of Claim 3.34 of [**1**]. □

Claim 2.24. *We can choose the element g in 2.18 such that one of the following holds:*

(i) There exists some $j = 2, \ldots, p-1$ such that

$$\alpha\beta^g = (\alpha\beta)^{g^r} \text{ and } \alpha\beta^{g^j} = (\alpha\beta)^{g^s},$$

for some $r, s \in \{0, 1, \ldots, p-1\}$ with $r \neq 1$.

https://doi.org/10.1017/S0017089511000413 Published online by Cambridge University Press

(ii) There exist j and k such that $1 < j < k < p$, *and*

$$\alpha\beta^g = (\alpha\beta)^{g^r}\delta,\ \alpha\beta^{g^j} = (\alpha\beta)^{g^s}\delta \text{ and } \alpha\beta^{g^k} = (\alpha\beta)^{g^t}\delta,$$

for some $\delta \in \mathrm{Irr}(X \bmod Y)$ *and some* $r, s, t \in \{0, 1, \ldots, p-1\}$ *with* $r \neq 1$.

Proof. See Proof of Claim 3.35 of [**1**]. □

Let g be as in Claim 2.24. Since X/Y is cyclic of order p, we may choose $x \in X$ such that $X = Y < x >$. Since H is abelian, we have $[X, H] = 1$. Suppose that $[x, g^{-1}] \in Z$. Then, x centralizes both g^{-1} and H modulo Z. Hence, $xZ \in \mathbf{Z}(G/Z)$ and so $[x, G] \leq Z$. Since Y/Z is a chief section of the p-group G, we have that $[Y, G] \leq Z$ and so $[< x > Y, G] = [X, G] \leq Z$ which is false by Claim 2.17. Hence $[x, g^{-1}] \in Y \setminus Z$ and so

$$Y = Z < y > \text{ is generated over } Z \text{ by } y = [x, g^{-1}]. \tag{2.25}$$

Since $[Y, G] \leq Z$, we have that $z = [y, g^{-1}] \in Z$. If $z = 1$, then $G = H < g >$ centralizes $Y = Z < y >$, since H centralizes $Y < X$ by 2.15, and G centralizes Z. This is impossible because $Z = \mathbf{Z}(G) < Y$. Thus,

$$z = [y, g^{-1}] \text{ is a non-trivial element of } Z. \tag{2.26}$$

By (2.25), we have $y = [x, g^{-1}] = x^{-1}x^{g^{-1}}$. By (2.26), we have $z = [y, g^{-1}] = y^{-1}y^{g^{-1}}$. Finally, $z^{g^{-1}} = z$ since $z \in Z$. Since $X = Z < x, y > \leq H$ is abelian, it follows that

$$z^{g^{-j}} = z,\ y^{g^{-j}} = yz^j \text{ and } x^{g^{-j}} = xy^j z^{\binom{j}{2}}, \tag{2.27}$$

for any integer $j = 0, 1, \ldots, p-1$. Because $g^{-p} \in H$ centralizes X by 2.15, we have

$$z^p = 1 \text{ and } y^p z^{\binom{p}{2}} = 1.$$

Observe that the statement is true for $p \leq 3$ since then $\frac{p+1}{2} \leq 2$. Thus, we may assume that p is odd. Hence, p divides $\binom{p}{2} = \frac{p(p-1)}{2}$ and $z^{\binom{p}{2}} = 1$. Therefore,

$$y^p = z^p = 1. \tag{2.28}$$

It follows that y^i, z^i and $z^{i/2}$ depend only on the residue of i modulo p, for any integer $i \geq 0$.

2.29. Observe that $\mathrm{Ker}(\xi_Z) \cap \mathrm{Ker}(\tau_Z) \leq \mathrm{Ker}(\chi) \cap \mathrm{Ker}(\psi) = 1$ implies that z is not in both $\mathrm{Ker}(\xi_Z)$ and $\mathrm{Ker}(\tau_Z)$. Without loss of generality, we may assume that $\tau_Z(z) \neq 1$. Since β is an extension of τ_Z, we may assume that $\beta(z) \neq 1$.

Claim 2.30. *$\xi_Z\tau_Z(z)$ is primitive pth root of unit.*

Proof. Suppose that $(\xi_Z\tau_Z)(z) = 1$. Then, $(\xi_Z\tau_Z)([y, g^{-1}]) = 1$ and so $(\xi_Z\tau_Z)^g(y) = (\xi_Z\tau_Z)(y)$. Since H is abelian, $|G : H| = p$, θ_i lies above $\xi_Z\tau_Z$ for all i and $g \in G \setminus H$, it follows that $Y =< y, \mathbf{Z}(G) >$ is contained in $\mathbf{Z}(\theta_i)$. This is contradiction with Claim 2.13. Thus, $(\xi_Z\tau_Z)(z) \neq 1$. Since z is of order p and $\xi_Z\tau_Z$ is a linear character, the claim follows. □

Claim 2.31. *Suppose that*

$$\alpha\beta^g = (\alpha\beta)^{g^r}\delta, \tag{2.32}$$

https://doi.org/10.1017/S0017089511000413 Published online by Cambridge University Press

and

$$\alpha\beta^{g^j} = (\alpha\beta)^{g^s}\delta, \tag{2.33}$$

for some $j \in \{0, 1, \ldots, p-1\}$*,* $j \neq 1$*, some* $\delta \in \mathrm{Irr}(X \bmod Y)$ *and some* $r, s \in \{0, 1, \ldots, p-1\}$*. Then,*

$$\delta(x) = \beta(z)^{hj(r-1)}, \tag{2.34}$$

where $2h \equiv 1 \bmod p$*.*

Proof. By Claim 2.30 and the same argument as in the proof of Claim 3.40 of [**1**], the statement follows. □

Suppose that Claim 2.24 (ii) holds. Then, by Claim 2.31, we have that $\delta(x) = \beta(z)^{hj(r-1)}$ and $\delta(x) = \beta(z)^{hk(r-1)}$. Since $\beta(z) = \tau_Z(z)$ is a primitive pth root of unit by 2.29, we have that $hj(r-1) \equiv hk(r-1) \bmod p$. Since $r \not\equiv 1 \bmod p$ and $2h \equiv 1 \bmod p$, we have that $k \equiv j \bmod p$, which is a contradiction. Thus, Claim 2.24 (i) must hold.

We now apply Claim 2.31 with $\delta = 1$. Thus, $1 = \delta(x) = \beta(z)^{hj(r-1)}$. Therefore, $hj(r-1) \equiv 0 \bmod p$. Since $2h \equiv 1 \bmod p$, either $j \equiv 0 \bmod p$ or $r - 1 \equiv 0 \bmod p$. Neither is possible. That is our final contradiction and Lemma 2.3 is proved. □

Proof of Theorem A. Since G is a nilpotent group, G is the direct product $G_1 \times G_2$ of its Sylow p-subgroup G_1 and its Hall p'-subgroup G_2. We can then write $\chi = \chi_1 \times \chi_2$ and $\psi = \psi_1 \times \psi_2$ for some characters $\chi_1, \psi_1 \in \mathrm{Irr}(G_1)$ and some characters $\chi_2, \psi_2 \in \mathrm{Irr}(G_2)$. Since $\chi(1) = p$, we have that $\chi_2(1) = 1$ and thus $\chi_2\psi_2 \in \mathrm{Irr}(G_2)$. If $\psi(1) \neq p$, since $\psi(1)$ is a prime number, we have that $\psi_1(1) = 1$ and thus $\chi_1\psi_1$ is an irreducible. Therefore, $\chi\psi \in \mathrm{Irr}(G)$ and (iv) holds. We may assume then that $\psi(1) = p$ and thus $\psi_2(1) = 1$. Then, $\chi_2\psi_2$ is a linear character and so we may assume that G is a p-group.

If $\chi\psi$ has a linear constituent, by Lemmas 2.1 and 2.2, we have that (i) or (ii) holds. So, we may assume that all the irreducible constituents of $\chi\psi$ are of degree at least p. If $\chi\psi$ has an irreducible constituent of degree p^2, then $\chi\psi \in \mathrm{Irr}(G)$ and (iv) holds. We may assume then that all the irreducible constituents of $\chi\psi$ have degree p. Since $\chi\psi(1) = p^2$, it follows that $\eta(\chi\psi) \leq p$. By Lemma 2.3, we have that either $\eta(\chi\psi) = 1$ or $\eta(\chi\psi) \geq \frac{p+1}{2}$, and so (iii) holds. □

Examples. Fix a prime $p > 2$

(i) Let E be an extraspecial group of order p^3 and $\phi \in \mathrm{Irr}(E)$ of degree p. We can check that the product $\phi\overline{\phi}$ is the sum of all the linear characters of E.

(ii) In the proof of Proposition 6.1 of [**2**], an example is constructed of a p-group G and a character $\chi \in \mathrm{Irr}(G)$ such that $\chi\overline{\chi}$ is the sum of p distinct linear characters and of $p-1$ distinct irreducible characters of degree p.

(iii) Given an extraspecial group E of order p^3, where $p > 2$, and $\phi \in \mathrm{Irr}(E)$ a character of degree p, we can check that $\phi\phi$ is a multiple of an irreducible. In Proposition 6.1 of [**1**], an example is provided of a p-group G and a character $\chi \in \mathrm{Irr}(G)$ such that $\eta(\chi\chi) = \frac{p+1}{2}$. In [**6**], an example is provided of a p-group P and two faithful characters $\delta, \epsilon \in \mathrm{Irr}(P)$ of degree p such that $\eta(\delta\epsilon) = p - 1$.

Let G be the wreath product of a cyclic group of order p^2 with a cyclic group of order p. Thus, G has a normal abelian subgroup N of order $(p^2)^p$ and index p. Let $\lambda \in \mathrm{Lin}(N)$ be a nontrivial character. We can check that $\chi = \lambda^G$ and $\psi = (\lambda^2)^G$ are irreducible characters of degree p and $\chi\psi$ is the sum of p distinct irreducible characters of degree p.

We wonder if there exists a p-group P with characters $\chi, \psi \in \mathrm{Irr}(P)$ of degree p such that $\frac{p+1}{2} < \eta(\chi\psi) < p-1$.

(iv) Let Q be a p-group and $\kappa \in \mathrm{Irr}(Q)$ be a character of degree p. Set $P = Q \times Q$, $\chi = \kappa \times 1_G$ and $\psi = 1_G \times \kappa$. Observe that χ, ψ and $\chi\psi$ are irreducible characters of P.

ACKNOWLEDGEMENTS. I would like to thank Professor Everett C. Dade for his suggestions. Also, I thank Irene S. Suarez for her encouragement.

REFERENCES

1. E. Adan-Bante, Products of characters and finite p-groups, *J. Algebra* **277** (2004) 236–255.

2. E. Adan-Bante, Products of characters and finite p-groups II, *Arch. Math.* **82**(4) (2004), 289–297.

3. E. Adan-Bante, Products of characters and derived length II, *J. Group Theory* **8** (2005), 453–459.

4. E. Adan-Bante, Restriction of characters and products of characters, *Israel J. Math* **174**(1) (2009), 221–225.

5. I. M. Isaacs, *Character theory of finite groups* (Academic Press, New York-San Francisco–London, 1976)

6. M. Loukaki and A. Moreto, On the number of constituents of products of characters, *Algebra Colloq.* **14**(2) (2007), 207–208.