ON THE CAUCHY PROBLEM FOR THE DIFFERENTIAL EQUATION $f(t, x, x', \ldots, x^{(k)}) = 0$

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Introduction. In the sequel, given $k, n \in \mathbb{N}$, $p \in [1, \infty]$ and a compact real interval I, we denote by $W^{k,p}(I, \mathbb{R}^n)$ (simply by $W^{k,p}(I)$ if $n = 1$) the space of all functions $u \in C^{k-1}(I, \mathbb{R}^n)$ such that $u^{(k-1)}$ is absolutely continuous in I and $u^{(k)} \in L^p(I, \mathbb{R}^n)$.

Very recently, in **[11]**, J. R. L. Webb and S. C. Welsh obtained the following existence result.

THEOREM A ([**11**, Theorem 1]). *Let $g : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function satisfying the following conditions*:

(1) *there exists a constant $c > 0$ such that*

$$|g(x, y, z)| \le c(1 + |x| + |y| + |z|)$$

for all $x, y, z \in \mathbb{R}$;

(2) *one has*

$$(g(x, y, z_1) - g(x, y, z_2))(z_1 - z_2) \le \alpha(s)\, |z_1 - z_2|^2$$

for all $x, y, z_1, z_2 \in \mathbb{R}$, with $|z_1 - z_2| \ge s$, where $\alpha :]0, \infty[\to [0, 1[$ is some function such that if $\alpha(s) \to 1$ then $s \to 0$.

Then, for every $x_0, y_0 \in \mathbb{R}$, there exist $b > 0$ and $u \in W^{2,2}([0, b])$ such that

$$\begin{cases} u''(t) = g(u(t), u'(t), u''(t)) \text{ a.e. in } [0, b] \\ u(0) = x_0 \\ u'(0) = y_0. \end{cases}$$

Webb and Welsh **[11]** also established the following regularity theorem.

THEOREM B ([**11**, Theorem 2]). *Let the assumptions of Theorem A be satisfied. In addition, assume that there is a function $h : [0, \infty[\to [0, \infty[$, vanishing and continuous at zero, such that*

$$|g(x_1, y_1, z) - g(x_2, y_2, z)| \le h(|x_1 - x_2| + |y_1 - y_2|)$$

for all $x_1, x_2, y_1, y_2, z \in \mathbb{R}$.

Then, any function u, as in the conclusion of Theorem A, belongs to $C^2([0, b])$.

To get Theorem A, Webb and Welsh employed a degree theory argument for A-proper mappings, while a direct *ad hoc* argument was used to prove Theorem B.

The aim of the present paper is to establish Theorem 1 below: a general existence theorem for the problem

$$\begin{cases} f(t, x, x', \ldots, x^{(k)}) = 0 \\ x^{(i)}(0) = x_i, \quad i = 0, 1, \ldots, k-1. \end{cases}$$

Our approach, completely different from that of **[11]**, is based on our preceding works (**[4]**, **[5]**, **[6]**, **[7]**).

In particular, the following two results are immediate consequences of our Theorem 1.

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THEOREM C. *Let* $g:\mathbb{R}^3 \to \mathbb{R}$ *be a continuous function such that, for every* $(x,y) \in \mathbb{R}^2$, *the function* $z \to g(x,y,z) - z$ *changes sign in* $\mathbb{R}$ *and* $\operatorname{int}(\{z \in \mathbb{R} : g(x,y,z) = z\}) = \varnothing$.

Then, the conclusion of Theorem A holds, with $u \in W^{2,\infty}([0,b])$.

THEOREM D. *Let* $g:\mathbb{R}^3 \to \mathbb{R}$ *be as in Theorem C. Moreover, assume that, for every* $(x,y) \in \mathbb{R}^2$, *there is only one* $z \in \mathbb{R}$ *such that* $g(x,y,z) = z$. *Then, the conclusions of Theorems A and B hold.*

Observe now that if a continuous function $g:\mathbb{R}^3 \to \mathbb{R}$ satisfies condition (2) of Theorem A, then, for every $(x,y) \in \mathbb{R}^2$, the function $z \to g(x,y,z) - z$ is strictly decreasing in $\mathbb{R}$ and, by [**12**, p. 241], it vanishes at some point. Consequently, such a g satisfies the assumptions of Theorem D. In other words, we have the following result.

THEOREM E. *Let* $g:\mathbb{R}^3 \to \mathbb{R}$ *be a continuous function satisfying condition* (2) *of Theorem A. Then, the conclusions of Theorems A and B hold.*

It is worth noticing that if g satisfies the hypotheses of Theorem D (in particular, that of Theorem E) then, the function, say φ, which to each $(x,y) \in \mathbb{R}^2$ associates the unique fixed point of $g(x,y,\cdot)$, is continuous. This follows directly from Theorem 1.1 of [**5**], a key tool in proving Theorem 1. Consequently, in this case, the differential equation $x'' = g(x,x',x'')$ is equivalent to $x'' = \varphi(x,x')$ to which the classical Peano existence theorem applies.

Finally, we remark again that our Theorem 1 is a genuine existence result: it does not give any information on uniqueness. On the contrary, [**11**] was mainly concerned with uniqueness. It is just in the uniqueness setting that condition (2) of Theorem A, together with some other assumption, plays an effective and crucial role ([**11**, Theorem 3]).

The present paper is arranged into two sections. Section 1 is devoted to the proof of Theorem 1, while Section 2 contains some remarks on it.

1. The main result. In the sequel, a is a positive real number; k, n are two positive integers; Y is a non-empty closed connected and locally connected subset of $\mathbb{R}^n$; f is a real function defined in $S \times Y$, where $S = [0,a] \times (\mathbb{R}^n)^k$.

Given $x_0, x_1, \ldots, x_{k-1} \in \mathbb{R}^n$, we consider the following Cauchy problem

$$\text{(CP)} \qquad \begin{cases} f(t,x,x',\ldots,x^{(k)}) = 0, \\ x^{(i)}(0) = x_i \quad \text{for} \quad i = 0,1,\ldots,k-1. \end{cases}$$

If $b \in]0,a]$, a function $u:[0,b] \to \mathbb{R}^n$ is said to be a *generalized* (resp. *classical*) solution of Problem (CP) in $[0,b]$ provided that $u \in W^{k,1}([0,b],\mathbb{R}^n)$ (resp. $u \in C^k([0,b],\mathbb{R}^n)$), $u^{(i)}(0) = x_i$ for $i = 0,1,\ldots,k-1$, and $u^{(k)}(t) \in Y$, $f(t,u(t),u'(t),\ldots,u^{(k)}(t)) = 0$ for almost every (resp. for every) $t \in [0,b]$.

Our main result follows.

THEOREM 1. *Let the following conditions be satisfied*:

(i) *there exists a set* $D \subseteq Y \times Y$, *with* $\bar{D} = Y \times Y$, *such that, for every* $(y,z) \in D$, *the set*

$$\{(t,\xi) \in S : f(t,\xi,y) < 0 < f(t,\xi,z)\}$$

is open in S;

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(ii) *for every* $(t, \xi) \in S$, *the function* $f(t, \xi, \cdot)$ *is continuous*, $0 \in \operatorname{int}(f(t, \xi, Y))$ *and the set*

$$V(t, \xi) = \{y \in Y : f(t, \xi, y) = 0\}$$

has empty interior in Y.

Under such hypotheses, for every $x_0, x_1, \ldots, x_{k-1} \in \mathbb{R}^n$, *the following assertions hold*:

(β) *There exists* $b \in]0, a]$ *such that Problem* (*CP*) *has at least one generalized solution in* $[0, b]$ *belonging to* $W^{k,\infty}([0, b], \mathbb{R}^n)$.

(γ) *If* Y *is convex and there exists a set* $\Gamma \subseteq S$ *of topological dimension zero, such that* $V(t, \xi)$ *is convex, for all* $(t, \xi) \in (S \setminus \Gamma) \cup \{(0, x_0, x_1, \ldots, x_{k-1})\}$, *then for each non-empty bounded set* $E \subseteq V(0, x_0, x_1, \ldots, x_{k-1})$ *there is* $b_E \in]0, a]$ *such that for every* $y_0 \in E$ *there exists a classical solution* u *of Problem* (*CP*) *in* $[0, b_E]$ *satisfying* $u^{(k)}(0) = y_0$.

(δ) *If for every* $(t, \xi) \in S$ *the set* $V(t, \xi)$ *is a singleton, then for every* $b \in]0, a]$ *any generalized solution of Problem* (*CP*) *in* $[0, b]$ *is a classical one there.*

Proof. We follow the approach of [**6**]. For each $(t, \xi) \in S$, put

$$L(t, \xi) = \{y \in Y : y \text{ is a local extremum for } f(t, \xi, \cdot)\}$$

as well as

$$Q(t, \xi) = V(t, \xi) \setminus L(t, \xi).$$

Our assumptions imply that the multifunction $(t, \xi) \to Q(t, \xi)$ is lower semicontinuous and its values are non-empty and closed (see Theorem 1.1 of [**5**] or Theorem 3.2 of [**7**]). Consequently, if $\|\cdot\|_{\mathbb{R}^n}$ denotes the underlying norm on $\mathbb{R}^n$, the real function $(t, \xi) \to \inf_{y \in Q(t, \xi)} \|y\|_{\mathbb{R}^n}$ is upper semicontinuous (see, for instance, Theorem 1.1 of [**8**]). Fix $x_0, x_1, \ldots, x_{k-1} \in \mathbb{R}^n$ and a compact neighbourhood W of $w = (x_0, x_1, \ldots, x_{k-1})$ in $(\mathbb{R}^n)^k$. Then, we can choose $\mu \in \mathbb{R}$ in such a way that $\inf_{y \in Q(t, \xi)} \|y\| < \mu$ for all $(t, \xi) \in [0, a] \times W$. For $r > 0$, let $B_r = \{y \in \mathbb{R}^n : \|y\|_{\mathbb{R}^n} < r\}$. Now, for each $(t, v_0, v_1, \ldots, v_{k-1}) \in [0, a] \times W$, put

$$F(t, v_0, v_1, \ldots, v_{k-1}) = \{v_1\} \times \{v_2\} \times \ldots \times \{v_{k-1}\} \times \overline{(Q(t, v_0, v_1, \ldots, v_{k-1}) \cap B_\mu)}.$$

Clearly, the multifunction $(t, v_0, v_1, \ldots, v_{k-1}) \to F(t, v_0, v_1, \ldots, v_{k-1})$ is non-empty compact-valued, lower semicontinuous and with bounded range. Consequently, by Theorem 2 of [**1**], there exist $b \in]0, a]$ and a Lipschitzian function $\varphi : [0, b] \to W$ such that

$$\begin{cases} \varphi'(t) \in F(t, \varphi(t)) & \text{a.e. in } [0, b] \\ \varphi(0) = w. \end{cases}$$

Therefore, if $\varphi(t) = (\varphi_0(t), \varphi_1(t), \ldots, \varphi_{k-1}(t))$, we have

$$\begin{cases} \varphi_0'(t) = \varphi_1(t),\ \varphi_1'(t) = \varphi_2(t), \ldots, \varphi_{k-2}'(t) = \varphi_{k-1}(t) & \text{a.e. in } [0, b] \\ \varphi_{k-1}'(t) \in Q(t, \varphi_0(t), \varphi_1(t), \ldots, \varphi_{k-1}(t)) & \text{a.e. in } [0, b] \\ \varphi_i(0) = x_i \quad \text{for } i = 0, 1, \ldots, k-1. \end{cases}$$

From this it follows at once that the function φ_0 is a generalized solution of Problem (CP) in $[0, b]$ belonging to $W^{k,\infty}([0, b], \mathbb{R}^n)$. Thus, ($\beta$) is proved.

Now, let us prove (γ). Under the present additional assumptions, from the proof of Theorem 2.2 of [5], we know that $V(t,\xi)=Q(t,\xi)$ for all $(t,\xi)\in(S\backslash\Gamma)\cup\{(0,x_0,x_1,\ldots,x_{k-1})\}$. Fix any non-empty bounded set $E\subseteq V(0,x_0,x_1,\ldots,x_{k-1})$. Put

$$\tilde{Q}(t,\xi)=\begin{cases}Q(t,\xi) & \text{if } (t,\xi)\in([0,a]\times W)\backslash\{(0,w)\}\\ \overline{\mathrm{conv}}(E) & \text{if } t=0,\ \xi=w,\end{cases}$$

where $\overline{\mathrm{conv}}(E)$ denotes the closed convex hull of E. Since the set $Q(0,w)$ is closed and convex, one has $\overline{\mathrm{conv}}(E)\subseteq Q(0,w)$. This implies at once that the multifunction $(t,\xi)\to\tilde{Q}(t,\xi)$ is lower semicontinuous. Hence, as seen above, there is $\eta\in\mathbb{R}$ such that $\inf_{y\in\tilde{Q}(t,\xi)}\|y\|_{\mathbb{R}^n}\le\eta$ for all $(t,\xi)\in[0,a]\times W$. Fix $\rho>\max\{\eta,\sup_{y\in E}\|y\|_{\mathbb{R}^n}\}$. By the classical Peano theorem, there exists $b_\rho\in]0,a]$ such that, for any continuous function $\omega:[0,a]\times W\to\mathbb{R}^n$ satisfying

$$\sup_{(t,\xi)\in[0,a]\times W}\|\omega(t,\xi)\|_{\mathbb{R}^n}\le\rho,$$

the problem

$$\begin{cases}x^{(k)}=\omega(t,x,x',\ldots,x^{(k-1)})\\ x^{(i)}(0)=x_i \quad \text{for } i=0,1,\ldots,k-1\end{cases}$$

admits at least one classical solution in $[0,b_\rho]$. Now observe that the set $\overline{\tilde{Q}(t,\xi)\cap B_\rho}$ is non-empty and closed for all $(t,\xi)\in[0,a]\times W$, and convex for all $(t,\xi)\in([0,a]\times W)\backslash\Gamma$. Moreover, $\overline{\tilde{Q}(0,w)\cap B_\rho}=\overline{\mathrm{conv}}(E)$ and, of course, the multifunction $(t,\xi)\to\overline{\tilde{Q}(t,\xi)\cap B_\rho}$ is lower semicontinuous. Consequently, by Theorem 7.1 of [3], for every fixed $y_0\in E$, there exists a continuous function $\tilde{\omega}:[0,a]\times W\to\mathbb{R}^n$ such that $\tilde{\omega}(t,\xi)\in\tilde{Q}(t,\xi)\cap\bar{B}_\rho$ for all $(t,\xi)\in[0,a]\times W$, and $\tilde{\omega}(0,w)=y_0$. Then, the problem

$$\begin{cases}x^{(k)}=\tilde{\omega}(t,x,x',\ldots,x^{(k-1)}),\\ x^{(i)}(0)=x_i, \quad \text{for } i=0,1,\ldots,k-1,\end{cases}$$

has at least one classical solution u in $[0,b_\rho]$. Of course, the function u solves Problem (CP) in $[0,b_\rho]$ and satisfies $u^{(k)}(0)=y_0$. Finally, let us prove (δ). In this case, we have $V(t,\xi)=Q(t,\xi)$ for all $(t,\xi)\in S$. Therefore, the (single-valued) function $(t,\xi)\to V(t,\xi)$ is continuous. Let $b\in]0,a]$. If u is any generalized solution of Problem (CP) in $[0,b]$, then we have

$$u^{(k)}(t)=V(t,u(t),u'(t),\ldots,u^{(k-1)}(t)) \quad \text{a.e. in } [0,b].$$

Thus, since $u^{(k-1)}$ is absolutely continuous, one has

$$u^{(k-1)}(t)=x_{k-1}+\int_0^t V(\tau,u(\tau),u'(\tau),\ldots,u^{(k-1)}(\tau))\,d\tau$$

for all $t\in[0,b]$, and hence $u\in C^k([0,b],\mathbb{R}^n)$. ■

2. Some remarks. As we said in the introduction, this section is devoted to some remarks on Theorem 1. First, it should be observed that, in (β), we have $u\in W^{k,\infty}([0,b],\mathbb{R}^n)$, although the set Y, where $u^{(k)}$ takes its values, is not assumed to be bounded.

Furthermore, observe that assertion (γ) can be used to obtain multiplicity results for (CP). For instance, we have the following result.

THEOREM 2. *Let Y be a linear subspace of $\mathbb{R}^n$ of dimension greater or equal to 2. Let f be continuous and such that, for every $(t, \xi) \in S$, the function $f(t, \xi, \cdot)$ is affine and non-constant in Y. Then, for every $x_0, x_1, \ldots, x_{k-1} \in \mathbb{R}^n$, there exists $b \in]0, a]$ such that the set of all classical solutions of Problem (CP) in $[0, b]$ has the continuum power.*

Proof. It is immediate to check that f satisfies (i), (ii), and each set $V(t, \xi)$ is, of course, an affine manifold of positive dimension. Hence, (γ) holds. Fix $x_0, x_1, \ldots, x_{k-1} \in \mathbb{R}^n$ as well as any bounded set $E \subseteq V(0, x_0, x_1, \ldots, x_{k-1})$ having the continuum power. Denote by $\mathcal{U}_E$ the set of all classical solutions of Problem (CP) in $[0, b_E]$, where b_E has the property expressed in (γ). For each $y_0 \in E$, choose $u_{y_0} \in \mathcal{U}_E$ such that $u_{y_0}^{(k)}(0) = y_0$. Of course, the mapping $y_0 \to u_{y_0}$ is one-to-one, and so its range has the continuum power. On the other hand, $C^k([0, b_E], \mathbb{R}^n)$ has the continuum power, since, endowed with one of the usual metrics, it is separable (see [**2**, p. 251]). Hence, $\mathcal{U}_E$ has the continuum power and the conclusion follows by taking $b = b_E$. ∎

Let us observe also that the conditions (i) and (ii) are not sufficient to ensure the existence of classical solutions for Problem (CP). For example, let $n = k = 1$, $Y = \mathbb{R}$ and let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \operatorname{arctg} x - \dfrac{y-1}{|y-1|}\left(2 + \sin \dfrac{3}{|y-1|}\right)\exp\left(-\dfrac{1}{|y-1|}\right) & \text{if } x \in \mathbb{R},\ y \in \mathbb{R}\setminus\{1\}), \\ \operatorname{arctg} x & \text{if } x \in \mathbb{R},\ y = 1. \end{cases}$$

It is not hard to check that such an f satisfies (i), (ii) and belongs to $C^\infty(\mathbb{R}^2)$. Nevertheless, for each $b > 0$, the problem

$$\begin{cases} f(x, x') = 0 \\ x(0) = 0 \end{cases}$$

has no classical solution in $[0, b]$. Indeed, assume, on the contrary, that u is a classical solution of this problem in $[0, b]$. Then, one has $u'(0) = 1$, and so, since u' is continuous, there is $b^* \in]0, b]$ such that u is strictly increasing in $[0, b^*]$. In particular, $u(t) > 0$ for all $t \in]0, b^*]$. This implies that there is $\varepsilon > 1$ such that $u'(]0, b^*]) =]1, \varepsilon]$ and that

$$\operatorname{arctg} u(t) = \left(2 + \sin \frac{3}{u'(t) - 1}\right)\exp\left(\frac{1}{1 - u'(t)}\right)$$

for all $t \in]0, b^*]$. Therefore, the function

$$t \to \left(2 + \sin \frac{3}{u'(t) - 1}\right)\exp\left(\frac{1}{1 - u'(t)}\right)$$

is strictly increasing in $]0, b^*]$. From this it follows that the function

$$y \to \left(2 + \sin \frac{3}{y - 1}\right)\exp\left(\frac{1}{1 - y}\right)$$

is one-to-one in $]1, \varepsilon]$, which is clearly false.

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The last remarks concern conditions (i) and (ii). Condition (i) is a very weak continuity assumption which we do not see how to weaken further, unless f is of the form $f_1(t, \xi) - f_2(y)$ ([**9**]).

In (ii), the requirement "$0 \in \operatorname{int}(f(t, \xi, Y))$" is essential. In other words, it is not enough to assume that "$0 \in f(t, \xi, Y)$". To see this, it suffices to take $n = k = 1$, $Y = \mathbb{R}$ and $f(t, y) = |\varphi(t) - y|$, where $\varphi : [0, a] \to \mathbb{R}$ is such that $\varphi|_{[0,b]} \notin L^1([0, b])$ for every $b \in]0, a]$.

Finally, always in (ii), the assumption that the interior of $V(t, \xi)$ in Y is empty, cannot be dropped. In this connection, we refer the reader to a very recent (and rather complicated) example by J. Saint Raymond (see [**10**, Théorème 11]).

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