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RETRACTS AND INJECTIVES

BY

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ABSTRACT. Embedding theorems are employed to show that many important categories do not possess non-trivial retracts or injectives. E.g., the categories of monoids, groups, rings, rings with unity, polynomial identity rings, nilpotent groups, solvable groups, and several varieties of groups.

Several authors have shown that various categories do not possess non-trivial retracts or injectives; R. Baer [1] for the category of groups, Raphael [14] for the category of rings, and others. In [9] a unified approach was given to demonstrate the non-existence of injectives in various categories. This approach is continued in this paper to show that several categories do not possess non-trivial retracts. It is shown, Theorem 1, that an Ore-like condition implies that retracts and injectives coincide. A functorial inequality is obtained (Lemma 2) which assures that a subgroup A of a group B is not a retract in B. This result is employed to show that several categories of groups do not possess non-trivial retracts. The idea behind the results in [9] is the fact that a "good" embedding theorem often implies non-existence of injectives and retracts. This idea and other techniques are utilized to show that some other categories of groups and rings do not possess non-trivial injectives.

The results obtained here include the results of Baer and Raphael mentioned above, as well as similar facts for categories which seem not to have been previously considered, such as the category of monoids, the category of nilpotent groups, the category of rings with polynomial identities. Another example: it will be shown, Theorem 3(xiv), that the variety of groups satisfying the law $x^n = 1$, $n > 2$, does not possess non-trivial injectives. (The special case $n = 3$ has been recently settled by Meldrum [11].)

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DEFINITION 1. An object A in a category C is said to be a retract in $\mathscr{C}$ if every $\mathscr{C}$-monomorphism $A \rightarrowtail B$ is left invertible in $\mathscr{C}$.

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Evidently an injective object in $\mathscr{C}$ is a retract in $\mathscr{C}$. the converse is not true in general. The following Ore-like regularity condition (see [8], 2.1) implies the converse.

$(A)^*$: For every ξ and η in $\mathscr{C}$ with common domain and ξ monic, there is a common left multiple $\xi = \psi\eta$ with ψ monic.

The condition $(A)^*$ does not assume that there are pushouts in $\mathscr{C}$. However, if $(A)^*$ holds then it holds in particular in pushouts.

THEOREM 1. *In a category with $(A)^*$, retracts coincide with injectives.*

Proof. Let J be a retract and consider any monic $\xi : A \to B$ and $\eta : A \to J$. By $(A)^*$ there are $\varphi : B \to L$ and $\psi : J \to L$ such that $\varphi\xi = \psi\eta$. Since J is a retract there exists $\beta : L \to J$ such that $\beta\psi = 1$. Hence $(\beta\varphi)\xi = \beta\psi\eta = \eta$ and J is injective. The other direction (J injective implies J retract) is true in any category.

It is well-known that both $(A)^*$ and its dual (A) hold in any abelian category. In the category of groups $\mathscr{G}r$ the dual (A) holds and the projectives and coretracts coincide with the free groups. The condition $(A)^*$ does not hold in $\mathscr{G}r$ (2.7), [8]), although injectives and retracts coincide with one-element group, [1]. Another example: the condition (A) holds in the category of compact Hausdorff spaces and the projectives = coretracts are the totally disconnected spaces. Dually, in the category of Banach spaces with mappings of norms ≤ 1, the injectives = retracts are (isometric to spaces) of the form $C(X)$, X compact Hausdorff totally disconnected.

LEMMA 1. *Let $\mathscr{C}$ be a category with a zero object. A subsimple non-zero object in $\mathscr{C}$ cannot be a retract in $\mathscr{C}$. (The terms subsimple and simple were introduced in [9].)*

Proof. Let $\alpha : A \rightarrowtail S$ be non-invertible. If A is a retract, then there exists $\beta : S \to A$ such that $\beta\alpha = 1$. However β cannot be monic (since α is non-invertible), hence S cannot be simple (unless A is zero).

DEFINITION 2. For groups $A < B$ we say that A is *a retract in B* if there exists a homomorphism $\beta : B \to A$ satisfying $\beta|_A = 1_A$.

LEMMA 2. *Let $\mathscr{K}$ be a full subcategory of $\mathscr{G}r$ and let $F : \mathscr{K} \to \mathscr{K}$ be a subfunctor of the identity on $\mathscr{K}$ (i.e., F assigns to every A in $\mathscr{K}$ in a subgroup $F(A) < A$ such that $\alpha F(A) < F(B)$ for all $\alpha : A \to B$ in $\mathscr{K}$). Let A, B be objects in $\mathscr{K}$. If A is a retract in B then $F(A) = A \cap F(B)$. Consequently, if $F(A) \neq A$ and $A < F(B)$ then A is not a retract in B.*

Proof. Clearly $F(A) < F(B)$, and so $F(A) < A \cap F(B)$. Since A is a retract in B, there exists $\beta : B \to A$ with $\beta|_A = 1_A$. This implies $\beta(A \cap F(B)) = A \cap F(B)$, and $\beta F(B) < F(A)$. Hence $A \cap F(B) < F(A)$.

Observe that although Definition 2 and Lemma 2 are formulated for groups, it is easy to generalize in order to apply them to other categories of algebraic structures.

LEMMA 3. *Let H be an abelian subgroup of a wreath product $AWrB$ and let K be the subgroup of $AWrB$ generated by H and A. Then the n-th derived subgroup $K^{(n)}$ is a subgroup of $(A^{(n)})^B$ for every positive integer n. (Notation as in* [13].)

Proof. For $n=1$ it suffices to show that all the commutators $[h, a]$, $h \in H$, $a \in A$ are in $(A')^B$. Now $h=bf$, $b \in B$, $f \in A^B$, so $h^{-1}a^{-1}ha = b^{-1}(f^{-1})^{b^{-1}}a^{-1}bfa = f^{-1}a^{-1}fa \in (A')^B$. Assume that $K^{(n)} < (A^{(n)})^B$ and let $u, v \in K^{(n)}$. Then $[u, v] \in (A^{(n+1)})^B$ since $[u, v](t) = [u(t), v(t)]$ for all $t \in B$.

LEMMA 4. *Let $\mathcal{K}$ be a full subcategory of $\mathcal{G}r$, containing a free group of rank n. If $n=1$ then every injective in $\mathcal{K}$ is divisible. If $n>1$ there are no non-trivial injectives in $\mathcal{K}$.*

Proof. Let J be injective in $\mathcal{K}$, and suppose that $\mathcal{K}$ possesses an infinite cyclic group $\langle x \rangle$. For any element $a \in J$ and an arbitrary positive integer n, let $h:\langle x \rangle \to \langle x \rangle$ be the homorphism induced by $x \mapsto x^n$, and $f:\langle x \rangle \to J$ the homomorphism induced by $x \mapsto a$. There exists a homomorphism $g:\langle x \rangle \to J$ satisfying $gh=f$. Hence $a=f(x)=gh(x)=g(x^n)=g(x)^n$, and J is divisible.

Suppose that $\mathcal{K}$ contains the free group $F=\langle u_0, \ldots, u_{n-1} \rangle$, with $n>1$. Let T be any free, non-cyclic group in $\mathcal{K}$ and let x, y, $x \neq y$, be elements in a set of free generators for T. The homomorphism $h:F \to T$ defined by $h(u_j)=y^{-j}xy^j$, $j=0, \ldots, n-1$, is monic. For an arbitrary $a \in J$, let $f:F \to J$ be a homomorphism satisfying $f(u_0)=a$, $f(u_1)=1$. There exists a homomorphism $g:T \to J$ satisfying $gh=f$. Hence (with $z=g(y)$), $1=f(u_1)=gh(u_1)=g(y^{-1}xy)=z^{-1}g(x)z=z^{-1}gh(u_0)z=z^{-1}f(u_0)z=z^{-1}az$, and so $a=1$.

COROLLARY. *Let $\mathcal{K}$ be a full subcategory of $\mathcal{G}r$ whose objects are finitely generated. Suppose that $\mathcal{K}$ contains an infinite cyclic group, but no non-trivial perfect groups. Then $\mathcal{K}$ does not possess non-trivial injectives.*

Proof. Let J be an injective in $\mathcal{K}$. By Lemma 4, J is divisible. Hence J/J' is a finitely generated, divisible abelian group, i.e., $J=J'$, and so J is trivial.

LEMMA 5. *Let $\mathcal{K}$ be a full subcategory of $\mathcal{G}r$ satisfying the following*

(i) *$\mathcal{K}$ is closed with respect to cyclic subgroups,*

(ii) *$\mathcal{K}$ does not possess non-trivial perfect groups,*

(iii) *At least one group in $\mathcal{K}$ contains a commutator of infinite order.*

Then $\mathcal{K}$ does not possess non-trivial injectives.

Proof. Let J be an injective in $\mathcal{K}$ and let $x^{-1}y^{-1}xy$ be an infinite order commutator belonging to a group G in $\mathcal{K}$. Let $a \in J$ and let $f:\langle x^{-1}y^{-1}xy \rangle \to J$

be the homomorphism defined by $f(x^{-1}y^{-1}xy)=a$. Extend f to a homomorphism $g:G\to J$. then $a=g(x)^{-1}g(y)^{-1}g(x)g(y)\in J'$ i.e., $J=J'$, and so J is trivial.

We have mentioned above that the category of rings does not possess retracts, Raphael [14]. A very simple proof, involving matrices, is contained in the following lemma.

LEMMA 6. *Let $\mathscr{C}$ be a full subcategory of the category of rings satisfying:*
(i) *For any A in $\mathscr{C}$ there is a matrix ring A_n in $\mathscr{C}$ for some $n>1$, and*
(ii) *the ideals in A_n are of the form K_n, with $K\lhd A$. Then there are no non-trivial retracts in $\mathscr{C}$.*

Proof. Let A be a retract in $\mathscr{C}$. Suppose $A\neq 0$ and define $f:A\to A_n$, $f(a)=\begin{pmatrix} a & & 0\\ & \ddots & \\ 0 & & a\end{pmatrix}$. If $g:A_n\to A$ is a left inverse of f then g cannot be $1-1$, hence $\ker g\neq 0$. But $\ker g$ is of the form K_n, $K\lhd A$ and $a\in K$ implies $\begin{pmatrix} a & & 0\\ & \ddots & \\ 0 & & a\end{pmatrix}\in K_n$. So $a=gf(a)=g\begin{pmatrix} a & & 0\\ & \ddots & \\ 0 & & a\end{pmatrix}$, hence $K=0$, a contradiction.

NOTATION. $\mathscr{A}b$ = the class of abelian groups; $\mathscr{S}$ = the class of solvable groups, $\mathscr{S}_n$ = the class of groups solvable of length $\leq n$; $\mathscr{N}$ = the class of nilpotent groups, $\mathscr{N}_n$ = the class of groups nilpotent of class $\leq n$; $\mathscr{B}_n$ = the variety of groups with the law $x^n=1$; $\mathscr{B}$ = the class of bounded groups. The prefix $\mathscr{F}g$ will stand for "finitely-generated".

THEOREM 2. There are no non-trivial retracts in the following categories: (i) $\mathscr{G}r$; (ii) Finite groups; (iii) Groups of cardinality $\leq\tau$, τ infinite; (iv) Monoids; (v) Monoids of cardinality $\leq\tau$, τ infinite; (vi) Rings without zero divisors; (vii) Algebras over a field; (viii) Near-rings; (ix) Composition rings; (x) $\mathscr{S}$; (xi) $\mathscr{F}g\mathscr{S}_n$; (xii) $\mathscr{B}$; (xiii) Rings; (xiv) Rings with unity; (xv) Simple rings; (xvi) Semisimple rings; (xvii) Prime rings; (xviii) Semiprime rings; (xix) Primitive rings; (xx) Rings with polynomial identity.

Proof. For (i)–(ix) we show that every object is subsimple and employ Lemma 1.

(i) Every group is subsimple.

(ii) Embed any finite group into a finite symmetric group S_n, $n\geq 3$, then embed S_n into the alternative group A_{n+2}, see [10].

(iii) In [5] an embedding of a denumerable group G into a denumerable simple group $G^{(\infty)}$ is given. The procedure described in [5] can be readily generalized to any infinite τ.

(iv)–(v) The embedding of any infinite monoid S into a simple monoid $S^{(\infty)}$ of the same cardinality follows from a direct generalization of constructions in [5].

(vi) A ring without zero-divisors is embeddable into a simple ring without zero-divisors, Cohn [6].

(vii) Any algebra over a field is subsimple, Bokut [4].

(viii) Let A be a near-ring. Embed its additive group $A, +$ properly into a group $G, +$ such that $|G|>2$. Let $T(G)$ be the near-ring of all functions $G \to G$. Embed A into $T(G)$ via $a \mapsto f_a$, $f_a(x)=ax$ for $x \in A$, $f_a(x)=a$ for $x \notin A$. The near-ring $T(G)$ is simple [3].

(ix) For a ring R one can make $T(R)=$ functions $R \to R$, into a composition ring and show that $T(R)$ is simple (at least for R infinite).

(x) Let $F: \mathcal{G}r \to \mathcal{G}r$ be defined by $F(X)=X'$. Let $G \in \mathcal{S}$, $G \neq 1$. Then $G \in \mathcal{S}_n$, for some positive integer n. By a theorem of B. H. Neumann and H. Neumann [13, 4.2] there exists a group $H \in \mathcal{S}_n \circ \mathcal{A}b$ with $G < H'$. Now $\mathcal{S}_n \circ \mathcal{A}b = \mathcal{S}_{n+1}$, so $H \in \mathcal{S}$, and $F(G) \neq G < F(H)$. Hence G is not a retract in H, Lemma 2.

(The above argument shows in effect that a non-trivial group in $\mathcal{S}_n$ is not a retract in $\mathcal{S}_{n+1}$.)

(xi) A theorem of Baumslag, [2, 4.1], states that for every group G and every positive integer m, the elements of G are m-th powers in $GWrC_m$, $C_m = \langle c \rangle$ a cyclic group of order m. Actually, for $g \in G$, $g=(c^{-1}u)^m$, $u: C_m \to G$, $u(1)=g$, $u(t)=1$ for $t \neq 1$. Suppose that G is a retract in $\mathcal{F}g\mathcal{S}_n$. Let $g \in G$, u as above, and let K be the subgroup of $GWrC_m$ generated by $c^{-1}u$ and G. By Lemma 3, K is in $\mathcal{S}_n$. There exists an epimorphism $\varphi: K \to G$ such that $\varphi|_G = 1_G$. Therefore $g=(\varphi(c^{-1}u))^m$, and this shows that G is divisible. Hence G/G' is finitely generated, divisible and abelian, so $G=G'$. Since G is solvable $G=1$.

(xii) Let $F_n: \mathcal{G}r \to \mathcal{G}r$ be defined by $F_n(G)=\langle x^n \mid x \in G \rangle$ i.e., the subgroup of G generated by the n-th powers of elements of G. Let $G \in \mathcal{B}$, $G \neq 1$, so $G \in \mathcal{B}_n$ for some positive integer n. Then $P = GWrC_n \in B_{n^2}$ by [2, 4.1]. Now $1 = F_n(G) \neq G < F_n(P)$ and so G is not a retract in P, Lemma 3.

(xiii)–(xx) Follow immediately from Lemma 6.

THEOREM 3. *There are no non-trivial injectives in the following categories:* (i) $\mathcal{G}r$; (ii) $\mathcal{F}g\mathcal{G}r$; (iii) *n-generator groups, n a positive integer*; (iv) $\mathcal{F}g\mathcal{S}$; (v) $\mathcal{F}g\mathcal{N}$; (vi) *n-generator $\mathcal{N}$-groups, n a positive integer*; (vii) $\mathcal{F}g\mathcal{S}_n$; (viii) $\mathcal{F}g\mathcal{N}_n$; (ix) *$n$-generator $\mathcal{S}_n$-groups, n a positive integer*; (x) *n-generator $\mathcal{N}_n$-groups, n a positive integer*; (xi) $\mathcal{N}$; (xii) $\mathcal{S}_n$; (xiii) $\mathcal{N}_n$, $n>1$; (xiv) $\mathcal{B}_n$, $n>2$.

Proof. (i)–(iii); An immediate consequence of Lemma 4.

(iv)–(x): Follows from Corollary to Lemma 4.

(xi)–(xiii): These classes of groups satisfy the conditions of Lemma 5.

(xiv): Let F be the relatively free group in $\mathcal{B}_n$, "freely generated" by x, y and let H be the subgroup of F "freely generated" by x, $y^{-1}xy$. For J an injective in $\mathcal{B}_n$, and $a \in J$, let $f: H \to J$ be the homomorphism induced by the

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maps $x \to a$, $y^{-1}xy \to 1$. Extend f to a homomorphism $g: F \to J$. Then $1 = g(y)^{-1}ag(y)$, and so $a = 1$.

QUESTION. Are there retracts in $\mathcal{F}g\mathcal{G}r$? Every group is embeddable in a divisible group, Neumann [12, 6.2], and every countable group is embeddable in a 2-generator group, Higman–Neumann–Neumann [7, iv]. Therefore a retract in $\mathcal{F}g\mathcal{G}r$ must be a 2-generator, divisible group. I. Rips has informed us that 2-generator, divisible groups do exist.

REFERENCES

1. R. Baer, *Absolute retracts in group theory*, Bull. AMS **52** (1946), 510–506.
2. G. Baumslag, *Wreath products and p-groups*, Proc. Camb. Phil. Soc. **55** (1959), 224–231.
3. G. Berman and R. J. Silverman, *Simplicity of near-rings of transformations*, Proc. AMS **10** (1959), 456–459.
4. L. A. Bokut, *Some embedding theorems for rings and semigroups*, Sibirsk Mat. Z. **4** (1963), 500–518.
5. W. W. Boone and G. Higman, *An algebraic characterization of groups with soluble word problem*, J. Australian Math. Soc. **18** (1974), 41–53.
6. P. M. Cohn, *Simple rings without zero-divisors and Lie division rings*, Mathematika **6** (1959), 4–18.
7. G. Higman, B. H. Neumann and H. Neumann, *Embedding theorems for groups*, J. London Math. Soc. **24** (1949), 247–254.
8. A. Klein, *Relations, in categories*, Illinois J. Math. **14** (1970), 536–550.
9. A. Klein, *Injectives and simple objects*, J. Pure and App. Alg. **15** (1979), 243–245.
10. J. J. Malone, *A near-ring analogue of a ring embedding theorem*, J. Alg. **16** (1970), 237–238.
11. J. D. P. Meldrum, *Injective near-ring of modules over Z_n*, Proc. AMS **68** (1978), 16–18.
12. B. H. Neumann, *Adjunction of elements to groups*, J. London Math. Soc. **18** (1943), 4–11.
13. B. H. Neumann and H. Neumann, *Embedding theorems for groups*, J. London Math. Soc. **34** (1959), 465–479.
14. R. Raphael, *Injective rings*, Comm. in Alg. **1** (1974), 403–414.

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