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Abstract

Katz's middle convolution algorithm provides a description of rigid connections on $\mathbb{P}^1$ with regular singularities. We extend the algorithm by adding the Fourier transform to it. The extended algorithm provides a description of rigid connections with arbitrary singularities.

1. Introduction

In [Kat96], Katz suggested a new method of studying a local system $\mathcal{L}$ on an open subset $U = \mathbb{P}^1 - \{x_1, \ldots, x_n\}$: the middle convolution algorithm. He defined the middle convolution of local systems on $\mathbb{P}^1$, and showed that for a Kummer local system $\mathcal{K}$, the operation of middle convolution with $\mathcal{K}$ is invertible:

$$\mathcal{L} = (\mathcal{L} \star_{\mathrm{mid}} \mathcal{K}) \star_{\mathrm{mid}} \mathcal{K}^{-1}.$$

Here $\star_{\mathrm{mid}}$ is the middle convolution. Usually, $\mathrm{rk}(\mathcal{L} \star_{\mathrm{mid}} \mathcal{K}) \neq \mathrm{rk}\, \mathcal{L}$.

To apply Katz's middle convolution algorithm to $\mathcal{L}$, one looks for a rank-one local system ℓ_1 and a Kummer local system $\mathcal{K}_1$ such that the middle convolution

$$\mathcal{L}_1 = (\mathcal{L} \otimes \ell_1) \star_{\mathrm{mid}} \mathcal{K}_1$$

has strictly smaller rank. The process is repeated until one arrives at the local system $\mathcal{L}_k$ whose rank can no longer be decreased by this operation. Note that $\mathcal{L}$ can be reconstructed from a smaller rank local system $\mathcal{L}_k$ and the sequence of rank-one local systems $\{(\ell_1, \mathcal{K}_1), \ldots, (\ell_k, \mathcal{K}_k)\}$ used in the algorithm. The isomorphism class of $\mathcal{L}$ is encoded by the isomorphism class of $\mathcal{L}_k$ and the monodromies of ℓ_i and $\mathcal{K}_i$.

Katz applied the algorithm to rigid local systems (a local system is *rigid* if it is determined up to isomorphism by the conjugacy classes of its local monodromies). He showed that any rigid irreducible local system $\mathcal{L}$ is reduced by the algorithm to a rank-one system $\mathcal{L}_k$. This describes rigid irreducible local systems using collections of numbers (the monodromies of $\mathcal{L}_k$, ℓ_i, and $\mathcal{K}_i$). Since then, the algorithm has found numerous applications to both rigid and non-rigid local systems, see [Sim09] for a summary.

Katz's middle convolution algorithm applies to the following 'flavors' of local systems:

– representations of the fundamental group of $\pi_1(\mathbb{P}^1_{\mathbb{C}} - \{x_1, \ldots, x_n\})$ ('Betti flavor');
– tamely ramified l-adic local systems on $\mathbb{P}^1_{\Bbbk} - \{x_1, \ldots, x_n\}$ for any field $\Bbbk$, where l is a prime distinct from $\mathrm{char}(\Bbbk)$;

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– vector bundles with connections on $\mathbb{P}^1_{\Bbbk} - \{x_1, \dots, x_n\}$ with regular singularities at the punctures $x_1, \dots, x_n$ for any field $\Bbbk$ of characteristic zero ('de Rham flavor'); in classical language, one works with linear ordinary differential equations with Fuchsian singularities.

In this paper, we take the de Rham point of view. We extend the middle convolution algorithm to connections with irregular singularities by using two operations: the middle convolution and the Fourier transform. We call this extension the *irregular Katz's algorithm*. It is described in § 4, but a short summary is given here.

For a bundle with connection $\mathcal{L}$ on an open set $U \subset \mathbb{A}^1$, denote its Fourier transform by $\mathcal{L}^\wedge$. The Fourier transform is a bundle with connection $\mathcal{L}^\wedge$ on an open subset $U^\wedge \subset \mathbb{A}^1$; usually $U^\wedge \neq U$ and $\operatorname{rk} \mathcal{L}^\wedge \neq \operatorname{rk} \mathcal{L}$.

On the each step of the irregular Katz's algorithm, we try to lower the rank of $\mathcal{L}$ by one of the following two operations:

(i) replacing $\mathcal{L}$ with the middle convolution

$$\mathcal{L}_1 = (\mathcal{L} \otimes \ell) \star_{\mathrm{mid}} \mathcal{K}^\lambda$$

for appropriate choices of a line bundle with connection ℓ and a Kummer local system $\mathcal{K}^\lambda$;

(ii) replacing $\mathcal{L}$ with the Fourier transform

$$\mathcal{L}_1 = (\mathcal{L} \otimes \ell)^\wedge$$

for appropriate choice of a line bundle with connection ℓ and the choice of the infinity $\infty \in \mathbb{P}^1$ (the point $\infty \in \mathbb{P}^1$ plays a special role in the Fourier transform, and we use it as a parameter in the operation).

Both operations (i) and (ii) are invertible, so $\mathcal{L}$ is determined up to isomorphism by $\mathcal{L}_1$ and the numerical parameters used in the operation. We repeat this procedure to decrease the rank of $\mathcal{L}$ as much as possible.

In this paper, we work with rigid irreducible bundles with connections $\mathcal{L}$. By definition, irreducible $\mathcal{L}$ is *rigid* if it is determined up to isomorphism by the formal types of its singularities. The main result is that the irregular Katz's algorithm always reduces such $\mathcal{L}$ to a rank-one bundle with connection; this yields a recursive description of irreducible rigid connections.

Our result answers the question posed by Katz in [Kat96, p. 10]. Also, in the introduction to [BE04], Bloch and Esnault express hope that their result (see Theorem 2.4) can be used to classify rigid connections with irregular singularities; our paper provides such a classification.

We hope that the irregular Katz's algorithm has other applications. Two examples are discussed in §§ 5.1 and 5.2.

2. Main results

Fix the ground field $\Bbbk$, which is algebraically closed of characteristic zero.

2.1 Connections and rigidity

By definition, a bundle with connection on a non-empty open set $U \subset \mathbb{P}^1$ is a pair $\mathcal{L} = (E_{\mathcal{L}}, \nabla_{\mathcal{L}})$, where $E_{\mathcal{L}}$ is a vector bundle on U and the connection $\nabla_{\mathcal{L}} : E_{\mathcal{L}} \to E_{\mathcal{L}} \otimes \Omega_U$ is a $\Bbbk$-linear map that satisfies the Leibniz identity

$$\nabla_{\mathcal{L}}(fs) = f \nabla_{\mathcal{L}}(s) + s \otimes df \quad (f \in \mathcal{O}_U, s \in E_{\mathcal{L}}).$$

We simply say that $\mathcal{L}$ is a connection on U. In this paper we use the 'de Rham' point of view, so the terms 'local system on U' and 'connection on U' are interchangeable.

For a closed point $x \in \mathbb{P}^1$, let K_x denote the ring of formal Laurent series at x. A choice of local coordinate z identifies K_x with $\Bbbk((z))$. Let $\mathcal{H}ol(\mathcal{D}_{K_x})$ be the category of holonomic $\mathcal{D}$-modules on the punctured formal neighborhood of x. Explicitly, objects of $\mathcal{H}ol(\mathcal{D}_{K_x})$ are pairs $\mathcal{V} = (E_\mathcal{V}, \nabla_\mathcal{V})$, where $E_\mathcal{V}$ is a finite-dimensional vector space over K_x and

$$\nabla_\mathcal{V} : E_\mathcal{V} \to E_\mathcal{V} \otimes \Omega_{K_x} = \mathcal{V}\, dz$$

is a $\Bbbk$-linear map satisfying the Leibniz identity. We sometimes call $\mathcal{V}$ a connection on the punctured formal disk.

For two connections $\mathcal{L}, \mathcal{L}'$ on an open set $U \subset \mathbb{P}^1$, we denote by $\mathcal{H}om(\mathcal{L}, \mathcal{L}')$ the local system of morphisms from $\mathcal{L}$ to $\mathcal{L}'$; equivalently, $\mathcal{H}om(\mathcal{L}, \mathcal{L}') = \mathcal{L}' \otimes \mathcal{L}^\vee$. By definition, $\mathcal{E}nd(\mathcal{L}) = \mathcal{H}om(\mathcal{L}, \mathcal{L})$. We use similar notation $\mathcal{H}om(\mathcal{V}, \mathcal{W})$, $\mathcal{E}nd(\mathcal{V})$ for $\mathcal{V}, \mathcal{W} \in \mathcal{H}ol(\mathcal{D}_{K_x})$, $x \in \mathbb{P}^1$.

A connection $\mathcal{L}$ on U yields an object $\Psi_x(\mathcal{L}) \in \mathcal{H}ol(\mathcal{D}_{K_x})$ for any $x \in \mathbb{P}^1$. Essentially, $\Psi_x(\mathcal{L})$ is the restriction of $\mathcal{L}$ to the punctured formal neighborhood of x: $\Psi_x(\mathcal{L}) = \mathcal{L} \otimes K_x$. One can view $\Psi_x(\mathcal{L})$ as the nearby cycles of $\mathcal{L}$.

Definition 2.1. The *formal type* $[\Psi_x(\mathcal{L})]$ of $\mathcal{L}$ *at* $x \in \mathbb{P}^1$ is the isomorphism class of $\Psi_x(\mathcal{L})$. The *formal type* of $\mathcal{L}$ is the collection

$$\{[\Psi_x(\mathcal{L})]\}_{x \in \mathbb{P}^1}.$$

Remark. For $x \in U$, the restriction $\Psi_x(\mathcal{L})$ is trivial, so $[\Psi_x(\mathcal{L})]$ is given by $\mathrm{rk}(\mathcal{L})$. Therefore, the formal type of $\mathcal{L}$ is determined by the collection

$$\{[\Psi_x(\mathcal{L})]\}_{x \in \mathbb{P}^1 - U}$$

(excluding the case when $U = \mathbb{P}^1$ and $\mathcal{L}$ is trivial).

Definition 2.2. A connection $\mathcal{L}$ on U is *rigid* if $\mathcal{L}$ is determined by its formal type up to isomorphism: any bundle with connection $\mathcal{L}'$ on U such that $\Psi_x(\mathcal{L}) \simeq \Psi_x(\mathcal{L}')$ for all $x \in \mathbb{P}^1$ is isomorphic to $\mathcal{L}$.

Example 2.3. Suppose $\Bbbk = \mathbb{C}$ and that $\mathcal{L}$ has regular singularities; that is, $\mathcal{L} = (E_\mathcal{L}, \nabla_\mathcal{L})$ can be extended to a vector bundle $\overline{E}_\mathcal{L}$ on $\mathbb{P}^1$ equipped with a connection $\overline{\nabla}_\mathcal{L}$ that has first-order poles:

$$\overline{\nabla}_\mathcal{L} : \overline{E}_\mathcal{L} \to \overline{E}_\mathcal{L} \otimes \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_n),$$

where $\{x_1, \ldots, x_n\} = \mathbb{P}^1 - U$. Then $[\Psi_x(\mathcal{L})]$ is determined by the monodromy of $\nabla_\mathcal{L}$ around x. The monodromy is defined up to conjugation, for instance, it can be given in the Jordan form. Therefore, for regular connections on $\mathbb{P}^1_\mathbb{C}$, Definition 2.2 reduces to the notion of rigidity given in the introduction.

2.2 Fourier transform

Recall the Fourier transform for $\mathcal{D}_{\mathbb{A}^1}$-modules. We can identify $\mathcal{D}_{\mathbb{A}^1}$-modules with modules over the algebra of polynomial differential operators $\Bbbk\langle z, d/dz\rangle$ (*the Weyl algebra*). Here z is the coordinate on $\mathbb{A}^1$. Consider the Fourier automorphism

$$F : \Bbbk\left\langle z, \frac{d}{dz}\right\rangle \to \Bbbk\left\langle z, \frac{d}{dz}\right\rangle : \quad F(z) = -\frac{d}{dz}, \quad F\left(\frac{d}{dz}\right) = z.$$

It yields an autoequivalence of the category of $\Bbbk\langle z, d/dz\rangle$-modules (*the Fourier transform*)

$$M \mapsto \mathfrak{F}(M),$$

where $\mathfrak{F}(M)$ is isomorphic to M as a vector space, but $\Bbbk\langle z, d/dz\rangle$ acts on it through F.

Now let $\mathcal{L}$ be a connection on an open set $U \subset \mathbb{A}^1$. Assume that $\mathcal{L}$ is irreducible. Viewing $\mathcal{L}$ as a $\mathcal{D}_U$-module, we obtain the Goresky–MacPherson extension $j_{!*}\mathcal{L}$, where $j : U \hookrightarrow \mathbb{A}^1$ is the open embedding. Here $j_{!*}\mathcal{L}$ is an irreducible $\mathcal{D}_{\mathbb{A}^1}$-module, therefore its Fourier transform $\mathfrak{F}(j_{!*}\mathcal{L})$ is also irreducible.

The $\mathcal{D}_{\mathbb{A}^1}$-module $\mathfrak{F}(j_{!*}\mathcal{L})$ is smooth on a non-empty open subset $U^\wedge \subset \mathbb{P}^1$; that is, it gives a connection $\mathcal{L}^\wedge$ on $U^\wedge$. Let us exclude the (essentially trivial) case when $\mathcal{L}$ has rank one and its only singularity is a second-order pole at infinity: in this case, $\mathfrak{F}(j_{!*}\mathcal{L})$ is supported at a single point, and $\mathcal{L}^\wedge = 0$. Then

$$\mathfrak{F}(j_{!*}\mathcal{L}) = j^\wedge_{!*}(\mathcal{L}^\wedge), \quad j^\wedge : U^\wedge \hookrightarrow \mathbb{A}^1$$

and $\mathcal{L}^\wedge$ is an irreducible connection on $U^\wedge$. When it does not cause confusion, we call $\mathcal{L}^\wedge$ the Fourier transform of $\mathcal{L}$.

Fourier transform preserves rigidity. In l-adic settings, this was proved by Katz using the local Fourier transform constructed by Laumon in [Lau87]. In the settings of bundles with connections, the local Fourier transform was constructed by Bloch and Esnault in [BE04].

Theorem 2.4 (Bloch and Esnault [BE04]). *Suppose that $\mathcal{L}$ is irreducible and rigid. Then so is its Fourier transform $\mathcal{L}^\wedge$.* □

2.3 Middle convolution

Fix $\lambda \in \Bbbk - \mathbb{Z}$. The corresponding *Kummer local system* is

$$\mathcal{K}^\lambda = \left(\mathcal{O}_{\mathbb{A}^1-\{0\}}, d + \lambda\frac{dz}{z}\right).$$

Up to isomorphism, $\mathcal{K}^\lambda$ depends only on the image of λ in $\Bbbk/\mathbb{Z}$.

Let $\mathcal{L}$ be an irreducible connection on an open subset $U \subset \mathbb{P}^1$. Shrinking U if necessary, we may assume that $U \subset \mathbb{A}^1$. We then define the middle convolution $\mathcal{L} \star_{\mathrm{mid}} \mathcal{K}^\lambda$ to be the inverse Fourier transform of $\mathcal{L}^\wedge \otimes \mathcal{K}^{-\lambda}$:

$$(\mathcal{L} \star_{\mathrm{mid}} \mathcal{K}^\lambda)^\wedge = \mathcal{L}^\wedge \otimes \mathcal{K}^{-\lambda}. \tag{2.1}$$

This definition uses the isomorphism $(\mathcal{K}^\lambda)^\wedge = \mathcal{K}^{-\lambda}$ to rewrite the convolution as a tensor product.

Remark. [Kat96] contains a direct definition of middle convolution that does not use the Fourier transform. The equivalence between this definition and (2.1) is [Kat96, Proposition 2.10.5]. Another approach to middle convolution (2.1) is sketched in § 6.1.

Let us make (2.1) explicit. Consider again the Fourier transform $\mathfrak{F}(j_{!*}\mathcal{L})$. It is a $\mathcal{D}_{\mathbb{A}^1}$-module. The tensor product $\mathfrak{F}(j_{!*}\mathcal{L}) \otimes \mathcal{K}^{-\lambda}$ is a $\mathcal{D}$-module on $\mathbb{A}^1 - \{0\}$. Consider $j_0 : \mathbb{A}^1 - \{0\} \hookrightarrow \mathbb{A}^1$, and take

$$\mathfrak{F}^{-1}(j_{0,!*}(\mathfrak{F}(j_{!*}\mathcal{L}) \otimes \mathcal{K}^{-\lambda})).$$

This is a $\mathcal{D}_{\mathbb{A}^1}$-module, and $\mathcal{L} \star_{\mathrm{mid}} \mathcal{K}^\lambda$ is the corresponding connection.

Again, exclude the essentially trivial case when $\mathcal{L}$ is a rank-one connection which is either trivial or has two simple poles at ∞ and some point $x \in \mathbb{A}^1$ with residues equal to λ and $-\lambda$,

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respectively. Then $\mathcal{L} \star_{\mathrm{mid}} \mathcal{K}^\lambda$ is again an irreducible connection. Theorem 2.4 immediately implies that $\mathcal{L}$ is rigid if and only if so is $\mathcal{L} \star_{\mathrm{mid}} \mathcal{K}^\lambda$. Clearly,

$$\mathcal{L} \simeq (\mathcal{L} \star_{\mathrm{mid}} \mathcal{K}^\lambda) \star_{\mathrm{mid}} \mathcal{K}^{-\lambda}.$$

2.4 Main theorem

Here is the main result of this paper, which is proved in § 4.

THEOREM A. *Let $\mathcal{L}$ be a connection on an open subset $U \subset \mathbb{P}^1$. Suppose that $\mathcal{L}$ is irreducible and rigid, and that $\mathrm{rk}(\mathcal{L}) > 1$. Then at least one of the following conditions hold:*

(i) *for appropriate $\lambda \notin \mathbb{Z}$ and a rank-one connection ℓ on $U - \{\infty\}$, the middle convolution $\mathcal{H}om(\ell, \mathcal{L}) \star_{\mathrm{mid}} \mathcal{K}^\lambda$ has rank smaller than $\mathcal{L}$;*

(ii) *for appropriate choice of $\infty \in \mathbb{P}^1 - U$ and a rank-one connection ℓ on U, the Fourier transform of $\mathcal{H}om(\ell, \mathcal{L})$ has rank smaller than $\mathcal{L}$.*

Remark. The Fourier transform can be thought of as a middle convolution on the multiplicative group. In this way, both cases (i) and (ii) involve middle convolution.

In case (ii), we use Fourier transform corresponding to some choice of $\infty \in \mathbb{P}^1$; the choice depends on $\mathcal{L}$. Equivalently, we might fix $\infty \in \mathbb{P}^1$, and use Möbius transformations to shift the connection $\mathcal{L}$. We then reformulate case (ii) as follows:

(ii′) *there is a rank one connection ℓ on U and a Möbius transformation $\phi : \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1$ such that*

$$\mathrm{rk}(\phi^* \mathcal{H}om(\ell, \mathcal{L}))^\wedge < \mathrm{rk}(\mathcal{L}).$$

Theorem A yields a connection $\mathcal{L}_1$ given by one of the two rules

$$\mathcal{L}_1 = \begin{cases} \mathcal{H}om(\ell, \mathcal{L}) \star_{\mathrm{mid}} \mathcal{K}^\lambda, & \text{case (i) of Theorem A} \\ \mathcal{H}om(\ell, \mathcal{L})^\wedge, & \text{case (ii) of Theorem A} \end{cases}$$

such that $\mathrm{rk}(\mathcal{L}_1) < \mathrm{rk}\,\mathcal{L}$. Note that $\mathcal{L}_1$ is again irreducible and rigid (by Theorem 2.4), so either $\mathrm{rk}(\mathcal{L}_1) = 1$, or $\mathrm{rk}(\mathcal{L}_1)$ can be decreased further by Theorem A. Iterating, we eventually get to a rank-one connection. This proves the following claim.

COROLLARY 2.5. *Any rigid connection $\mathcal{L}$ on an open subset $U \subset \mathbb{P}^1$ can be reduced to the trivial connection $(\mathcal{O}, d)$ by iterating the following three operations:*

- *tensor multiplication by a rank-one connection ℓ: $\mathcal{L} \mapsto \mathcal{L} \otimes \ell$;*
- *change of variable by a Möbius transformation ϕ: $\mathcal{L} \mapsto \phi^* \mathcal{L}$;*
- *Fourier transform: $\mathcal{L} \mapsto \mathcal{L}^\wedge$.*

Of course, $\mathcal{L}$ can also be obtained from $(\mathcal{O}, d)$ by these operations. □

3. Connections and Fourier transform

In this section, we recall the necessary statements about bundles with connections.

3.1 Euler–Poincaré formula

Fix a point $x \in \mathbb{P}^1$. For $\mathcal{V} \in \mathcal{H}ol(\mathcal{D}_{K_x})$, we denote by $\mathrm{irreg}(\mathcal{V})$ the irregularity of $\mathcal{V}$ and by

$$\mathrm{slope}(\mathcal{V}) = \frac{\mathrm{irreg}(\mathcal{V})}{\mathrm{rk}(\mathcal{V})}$$

the slope of $\mathcal{V}$. It is also convenient to introduce the following quantity:

$$\delta(\mathcal{V}) = \mathrm{irreg}(\mathcal{V}) + \mathrm{rk}(\mathcal{V}) - \mathrm{rk}(\mathcal{V}^{\mathrm{hor}}), \tag{3.1}$$

where $\mathcal{V}^{\mathrm{hor}} \subset \mathcal{V}$ is the maximal subbundle on which the connection is trivial. In other words,

$$\mathrm{rk}(\mathcal{V}^{\mathrm{hor}}) = \dim_{\Bbbk} H^0_{\mathrm{dR}}(K_x, \mathcal{V}), \quad H^0_{\mathrm{dR}}(K_x, \mathcal{V}) = \ker(\nabla_{\mathcal{V}} : E_{\mathcal{V}} \to E_{\mathcal{V}} \otimes \Omega).$$

Let $\mathcal{L}$ be a connection on an open subset $U \subset \mathbb{P}^1$. Consider the $\mathcal{D}_{\mathbb{P}^1}$-module $j_{!*}(\mathcal{L})$ for $j : U \hookrightarrow \mathbb{P}^1$. Denote by $H^i_{\mathrm{dR}}(\mathbb{P}^1, j_{!*}(\mathcal{L}))$ its de Rham cohomology groups and by

$$\chi(j_{!*}(\mathcal{L})) = \sum_{i=0}^{2} (-1)^i \dim H^i_{\mathrm{dR}}(\mathbb{P}^1, j_{!*}(\mathcal{L}))$$

its Euler characteristic.

We need the Euler–Poincaré formula for the Euler characteristic.

Proposition 3.1. *Let $\mathcal{L}$ and $j : U \hookrightarrow \mathbb{P}^1$ be as above. Then*

$$\chi(j_{!*}(\mathcal{L})) = 2\mathrm{rk}(\mathcal{L}) - \sum_{x \in \mathbb{P}^1 - U} \delta(\Psi_x(\mathcal{L})). \qquad \square$$

3.2 Rigidity index

Definition 3.2. For $\mathcal{L}$ as above, the *rigidity index* of $\mathcal{L}$ is given by

$$\mathrm{rig}(\mathcal{L}) = \chi(j_{!*}\, \mathcal{E}nd(\mathcal{L})).$$

Remark 3.3. It is well known that $\mathrm{rig}(\mathcal{L})$ is even. Indeed, by the Verdier duality the vector spaces $H^0_{\mathrm{dR}}(\mathbb{P}^1, j_{!*}(\mathcal{E}nd(\mathcal{L})))$ and $H^2_{\mathrm{dR}}(\mathbb{P}^1, j_{!*}(\mathcal{E}nd(\mathcal{L}))$ are dual (and therefore have equal dimension), while $H^1_{\mathrm{dR}}(\mathbb{P}^1, j_{!*}(\mathcal{E}nd(\mathcal{L})))$ carries a symplectic form (and therefore has even dimension).

The following statement is an extension of [Kat96, Theorem 1.1.2] to the case of irregular singularities.

Proposition 3.4 [BE04, Theorems 4.7 and 4.10]. *An irreducible connection $\mathcal{L}$ is rigid if and only if* $\mathrm{rig}(\mathcal{L}) = 2$. $\square$

Remark. For irreducible $\mathcal{L}$, we have

$$\mathrm{rig}(\mathcal{L}) = 2 - \dim H^1_{\mathrm{dR}}(\mathbb{P}^1, j_{!*}(\mathcal{E}nd(\mathcal{L})).$$

Therefore, two is the largest possible value of $\mathrm{rig}(\mathcal{L})$ (and the only positive value). In [Kat96], local systems satisfying $\mathrm{rig}(\mathcal{L}) = 2$ are called *cohomologically rigid*, while those satisfying Definition 2.2 are *physically rigid*.

3.3 Rank of the Fourier transform

Suppose now that $\mathcal{L}$ is a connection on an open subset $U \subset \mathbb{A}^1$. Consider the Fourier transform $\mathfrak{F}(j_{!*}(\mathcal{L}))$ for $j : U \hookrightarrow \mathbb{A}^1$. We want to find the (generic) rank of the Fourier transform, that is, $\mathrm{rk}\, \mathcal{L}^\wedge$.

Proposition 3.5 [Mal91, Proposition V.1.5]. *Denote by $\Psi_\infty(\mathcal{L})^{>1} \subset \Psi_\infty(\mathcal{L})$ the maximal submodule of $\mathcal{L}$ whose irreducible components all have slopes greater than one. Then*

$$\mathrm{rk}(\mathcal{L}^\wedge) = \sum_{x \in \mathbb{A}^1 - U} \delta(\Psi_x(\mathcal{L})) + \mathrm{irreg}(\Psi_\infty(\mathcal{L})^{>1}) - \mathrm{rk}(\Psi_\infty(\mathcal{L})^{>1}). \qquad \square$$

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Similarly, we have a formula for the rank of the middle convolution. In the case of regular singularities, this is [Kat96, Corollary 3.3.7] (in l-adic settings).

Proposition 3.6. *Denote by $\mathcal{K}^\lambda_\infty \in \mathcal{H}ol(\mathcal{D}_{K_\infty})$ the 'Kummer local system at infinity' given by*

$$\mathcal{K}^\lambda_\infty = \left(K_\infty, d + \lambda\frac{d\zeta}{\zeta}\right),$$

where ζ is a local coordinate at $\infty \in \mathbb{P}^1$. Note that the residue of $\mathcal{K}^\lambda_\infty$ is λ, so $\mathcal{K}^\lambda_\infty \simeq \Psi_\infty(\mathcal{K}^{-\lambda})$. Then

$$\operatorname{rk}(\mathcal{L} \star_{\mathrm{mid}} \mathcal{K}^\lambda) = \sum_{x \in \mathbb{A}^1 - U} \delta(\Psi_x(\mathcal{L})) + \delta(\Psi_\infty(\mathcal{L}) \otimes \mathcal{K}^{-\lambda}_\infty) - \operatorname{rk}\mathcal{L}.$$

Proof. For any point $x \in U$, the fiber of $\mathcal{L} \star_{\mathrm{mid}} \mathcal{K}^\lambda$ over x equals

$$(\mathcal{L} \star_{\mathrm{mid}} \mathcal{K}^\lambda)_x = H^1_{\mathrm{dR}}(\mathbb{P}^1, j_{!*}(\mathcal{L} \otimes s_x^*(\mathcal{K}^\lambda))), \tag{3.2}$$

where $j : U \hookrightarrow \mathbb{P}^1$ is the embedding and $s_x : \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1$ is given by $y \mapsto y + x$ is the shift (so $s_x^*\mathcal{K}^\lambda$ has regular singularities at x and ∞). The formula (3.2) is essentially [Kat96, Corollary 2.8.5]. It is easy to prove if one views the middle convolution as an integral transform, as in § 6.1.

Clearly, $H^i(\mathbb{P}^1, j_{!*}(\mathcal{L} \otimes s_x^*(\mathcal{K}^\lambda))) = 0$ for $i = 0, 2$, so

$$\operatorname{rk}(\mathcal{L} \star_{\mathrm{mid}} \mathcal{K}^\lambda) = -\chi(j_{!*}(\mathcal{L} \otimes s_x^*(\mathcal{K}^\lambda))).$$

It remains to apply Proposition 3.1. □

4. Proof of Theorem A

4.1 Outline of proof

For every singular point $x \in \mathbb{P}^1 - U$, consider the formal type $\Psi_x(\mathcal{L}) \in \mathcal{H}ol(\mathcal{D}_{K_x})$ of $\mathcal{L}$ at x. Choose an irreducible connection $\mathcal{V}_x \in \mathcal{H}ol(\mathcal{D}_{K_x})$ that minimizes

$$\frac{\delta(\mathcal{H}om(\mathcal{V}_x, \Psi_x(\mathcal{L})))}{\operatorname{rk}(\mathcal{V}_x)}$$

(this choice is described in Corollary 4.4).

Example 4.1. Suppose that $\mathcal{L}$ has regular singularities. Then $\operatorname{rk}(\mathcal{V}_x) = 1$, and $\mathcal{V}_x$ has regular singularities. Therefore, $\mathcal{V}_x \simeq (K_x, d + \lambda(dz/z))$, where λ is chosen so as to maximize $\dim \operatorname{Hom}(\mathcal{V}_x, \Psi_x(\mathcal{L}))$. Here z is a local coordinate at x. Explicitly, we can write $\Psi_x(\mathcal{L}) \simeq (K_x^r, d + R(dz/z))$, where R is an $r \times r$ matrix with constant coefficients such that no two eigenvalues of R differ by a non-zero integer. Then λ is the eigenvalue of R with maximal geometric multiplicity (that is, the eigenspace of λ has maximal dimension).

If $\Bbbk = \mathbb{C}$, we can simply say that $\mathcal{V}_x$ is given by the eigenvalue of the monodromy of $\mathcal{L}$ with maximal geometric multiplicity.

Case I: Suppose $\operatorname{rk}(\mathcal{V}_x) = 1$ for all $x \in \mathbb{P}^1 - U$. (By Example 4.1, this is true if $\mathcal{L}$ has regular singularities, so this is the only case appearing in the middle convolution algorithm of [Kat96].) It makes sense to talk about $\operatorname{res}(\mathcal{V}_x) \in \Bbbk/\mathbb{Z}$.

Case Ia: Suppose $\sum_x \operatorname{res}(\mathcal{V}_x) \in \mathbb{Z}$. Then one can find a local system ℓ on U such that $\Psi_x(\ell) \simeq \mathcal{V}_x$. One can easily see from the Euler–Poincaré formula that either $\operatorname{Hom}(\mathcal{L}, \ell)$ or $\operatorname{Hom}(\ell, \mathcal{L})$ is non-zero (Proposition 4.5). Since $\mathcal{L}$ is irreducible, this implies $\mathcal{L} \simeq \ell$, so $\operatorname{rk}(\mathcal{L}) = 1$, which contradicts the assumptions.

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Case Ib: Suppose $\lambda = \sum_x \operatorname{res}(\mathcal{V}_x) \notin \mathbb{Z}$. Shrinking U if necessary, we may assume that $\infty \notin U$. Then there is a rank-one connection ℓ on U such that

$$\Psi_x(\ell) \simeq \begin{cases} \mathcal{V}_x, x \in \mathbb{A}^1 - U \\ \mathcal{V}_\infty \otimes \mathcal{K}_\infty^{-\lambda}, x = \infty, \end{cases} \tag{4.1}$$

for $\mathcal{K}_\infty^\lambda$ as in Proposition 3.6. It follows from Proposition 3.6 that $\mathcal{L}$ satisfies Theorem A(i) for this ℓ (Proposition 4.6).

Case II: Suppose that $\operatorname{rk}(\mathcal{V}_x) > 1$ for some $x \in \mathbb{P}^1 - U$. We show (Proposition 4.10) that there is unique x with this property. Choose it as ∞. Then choose ℓ to be a rank-one connection on U that satisfies $\Psi_x(\ell) \simeq \mathcal{V}_x$ for $x \in \mathbb{A}^1 - U$, and such that

$$\mathcal{H}om(\Psi_\infty(\ell), \mathcal{V}_\infty) \in \mathcal{H}ol(\mathcal{D}_{K_\infty})$$

has non-integer slope. It follows from Proposition 3.5 that $\mathcal{L}$ satisfies Theorem A(ii) for this ℓ (Proposition 4.11). □

Remark 4.2. Let us discuss the condition on ℓ in Case II. Choose $\ell_\infty \in \mathcal{H}ol(\mathcal{D}_{K_\infty})$ to be a rank-one connection which is the 'best approximation' of $\mathcal{V}_\infty$ in the sense that it minimizes

$$\operatorname{slope} \mathcal{H}om(\ell_\infty, \mathcal{V}_\infty).$$

It is clear that the minimal slope is not an integer. Note that ℓ_∞ is not unique; in particular, it can be tensored by a rank-one bundle with regular connection. This means that $\operatorname{res}(\ell_\infty)$ is unrestricted, and so ℓ_∞ can be chosen so that

$$\operatorname{res} \ell_\infty + \sum_{x \in \mathbb{A}^1 - U} \operatorname{res} \mathcal{V}_x \in \mathbb{Z}.$$

We can then find ℓ with $\Psi_x(\ell) \simeq \mathcal{V}_x$ for $x \in \mathbb{A}^1 - U$ and $\Psi_\infty(\ell) \simeq \ell_\infty$.

More explicitly, let ζ be a local coordinate at ∞, and $r = \operatorname{rk}(\mathcal{V}_\infty)$. We apply the well-known description of connections on a punctured formal disk (see, for instance, [Mal91, Theorem III.1.2]). Since $\mathcal{V}_\infty$ is irreducible, there exists a ramified extension

$$\Bbbk((\zeta^{1/r})) \supset \Bbbk((\zeta)) = K_\infty$$

and a differential form $\mu \in \Bbbk((\zeta^{1/r}))\, d\zeta$ such that

$$\mathcal{V}_\infty \simeq (\Bbbk((\zeta^{1/r})), d + \mu).$$

Choose a differential form $\mu_\ell \in \Bbbk((\zeta))\, d\zeta$ as a 'best approximation' of μ in the sense that the leading term of $\mu - \mu_\ell$ is a fractional power of ζ. Then take

$$\ell_\infty = (K_\infty, d + \mu_\ell).$$

4.2 Details of the proof: Case I

Let us fill in the gaps in the above outline. We start with some local calculations. Fix $x \in \mathbb{P}^1$.

Recall that for $\mathcal{V} \in \mathcal{H}ol(\mathcal{D}_{K_x})$, $\delta(\mathcal{V})$ is defined by (3.1). It is obvious that δ is semiadditive.

Lemma 4.3. *For a short exact sequence*

$$0 \to \mathcal{V}_1 \to \mathcal{V} \to \mathcal{V}_2 \to 0$$

in $\mathcal{H}ol(\mathcal{D}_{K_x})$, we have

$$\delta(\mathcal{V}) \geqslant \delta(\mathcal{V}_1) + \delta(\mathcal{V}_2).$$

Proof. Indeed,

$$\operatorname{rk}(\mathcal{V}) = \operatorname{rk}(\mathcal{V}_1) + \operatorname{rk}(\mathcal{V}_2),$$
$$\operatorname{irreg}(\mathcal{V}) = \operatorname{irreg}(\mathcal{V}_1) + \operatorname{irreg}(\mathcal{V}_2),$$
$$\dim H^0_{\mathrm{dR}}(K_x, \mathcal{V}) \leqslant \dim H^0_{\mathrm{dR}}(K_x, \mathcal{V}_1) + \dim H^0_{\mathrm{dR}}(K_x, \mathcal{V}_2).$$ □

COROLLARY 4.4. *For any $\mathcal{V} \in \mathcal{H}ol(\mathcal{D}_{K_x})$, there is irreducible $\mathcal{V}' \in \mathcal{H}ol(\mathcal{D}_{K_x})$ such that*

$$\delta(\mathcal{E}nd(\mathcal{V})) \geqslant \frac{\operatorname{rk}(\mathcal{V})}{\operatorname{rk}(\mathcal{V}')}\delta(\mathcal{H}om(\mathcal{V}', \mathcal{V})).$$

Proof. Let $\mathcal{V}_1, \dots, \mathcal{V}_k$ be the irreducible components of $\mathcal{V}$ (with multiplicity). Take $\mathcal{V}'$ to be the $\mathcal{V}_i$ that minimizes

$$\frac{\delta(\mathcal{H}om(\mathcal{V}', \mathcal{V}))}{\operatorname{rk}\mathcal{V}'}. \tag{4.2}$$

Semiadditivity of δ implies that

$$\delta(\mathcal{E}nd(\mathcal{V})) \geqslant \sum_i \delta(\mathcal{H}om(\mathcal{V}_i, \mathcal{V})),$$

and Corollary 4.4 follows. □

Remark. It is easy to see that this choice of $\mathcal{V}'$ actually minimizes (4.2) over all $\mathcal{V}' \in \mathcal{H}ol(\mathcal{D}_{K_x})$. Also, if irreducible $\mathcal{V}' \in \mathcal{H}ol(\mathcal{D}_{K_x})$ minimizes (4.2), then $\mathcal{V}'$ is a component of $\mathcal{V}$.

Now let $\mathcal{L}$ be as in Theorem A; for every $x \in \mathbb{P}^1 - U$, Corollary 4.4 yields an irreducible object $\mathcal{V}_x \in \mathcal{H}ol(\mathcal{D}_{K_x})$ such that

$$\delta(\mathcal{E}nd(\Psi_x(\mathcal{L}))) \geqslant \frac{\operatorname{rk}(\mathcal{L})}{\operatorname{rk}\mathcal{V}_x}\delta(\mathcal{H}om(\mathcal{V}_x, \Psi_x(\mathcal{L}))).$$

It remains to prove the following statements.

PROPOSITION 4.5 (Theorem A, Case Ia). *Suppose that there is a rank-one local system ℓ on U such that $\mathcal{V}_x = \Psi_x(\ell)$ for every $x \in \mathbb{P}^1 - U$. Then either* $\operatorname{Hom}(\ell, \mathcal{L})$ *or* $\operatorname{Hom}(\mathcal{L}, \ell)$ *is non-zero.*

Proof. It suffices to show that $\chi_{\mathrm{dR}}(j_{!*}(\mathcal{H}om(\ell, \mathcal{L}))) > 0$. Indeed, by Proposition 3.1, we have

$$\begin{aligned}\chi_{\mathrm{dR}}(j_{!*}(\mathcal{H}om(\ell, \mathcal{L}))) &= 2\operatorname{rk}(\mathcal{L}) - \sum \delta(\mathcal{H}om(\mathcal{V}_x, \Psi_x(\mathcal{L})))\\ &\geqslant 2\operatorname{rk}(\mathcal{L}) - \frac{1}{\operatorname{rk}(\mathcal{L})}\sum \delta(\mathcal{E}nd(\Psi_x(\mathcal{L})))\\ &= \frac{\operatorname{rig}(\mathcal{L})}{\operatorname{rk}\mathcal{L}} = \frac{2}{\operatorname{rk}(\mathcal{L})} > 0.\end{aligned}$$ □

PROPOSITION 4.6 (Theorem A, Case Ib). *Suppose that there is $\lambda \in \Bbbk - \mathbb{Z}$ and a rank-one connection ℓ on U satisfying (4.1). Then* $\operatorname{rk}(\mathcal{H}om(\ell, \mathcal{L}) \star_{\mathrm{mid}} \mathcal{K}^\lambda) < \operatorname{rk}(\mathcal{L})$.

Proof. By Proposition 3.6, we have

$$\begin{aligned}\operatorname{rk}(\mathcal{H}om(\ell, \mathcal{L}) \star_{\mathrm{mid}} \mathcal{K}^\lambda) &= \sum_{x \in \mathbb{P}^1 - U} \delta(\mathcal{H}om(\mathcal{V}_x, \Psi_x(\mathcal{L}))) - \operatorname{rk}(\mathcal{L})\\ &\leqslant \frac{1}{\operatorname{rk}(\mathcal{L})}\sum \delta(\mathcal{E}nd(\Psi_x(\mathcal{L}))) - \operatorname{rk}(\mathcal{L})\\ &= \operatorname{rk}(\mathcal{L}) - \frac{\operatorname{rig}(\mathcal{L})}{\operatorname{rk}(\mathcal{L})} = \operatorname{rk}(\mathcal{L}) - \frac{2}{\operatorname{rk}(\mathcal{L})} < \operatorname{rk}(\mathcal{L}).\end{aligned}$$ □

4.3 Details of the proof: Case II

Again, we start with some local results at fixed $x \in \mathbb{P}^1$.

Lemma 4.7. *Suppose that $\mathcal{V}, \mathcal{W} \in \mathcal{H}ol(\mathcal{D}_{K_x})$ are irreducible.*

(i) *If* $\operatorname{slope}(\mathcal{V}) \neq \operatorname{slope}(\mathcal{W})$, *then*

$$\operatorname{slope}(\mathcal{H}om(\mathcal{V},\mathcal{W})) = \max(\operatorname{slope}(\mathcal{V}), \operatorname{slope}(\mathcal{W})),$$
$$\frac{\delta(\mathcal{H}om(\mathcal{V},\mathcal{W}))}{\operatorname{rk}(\mathcal{V})\operatorname{rk}(\mathcal{W})} = 1 + \max(\operatorname{slope}(\mathcal{V}), \operatorname{slope}(\mathcal{W})).$$

(ii) *If* $\operatorname{slope}(\mathcal{V}) = \operatorname{slope}(\mathcal{W})$ *has denominator d, then*

$$\operatorname{slope}(\mathcal{H}om(\mathcal{V},\mathcal{W})) \geqslant \left(1 - \frac{1}{d}\right)\operatorname{slope}(\mathcal{W}),$$
$$\frac{\delta(\mathcal{H}om(\mathcal{V},\mathcal{W}))}{\operatorname{rk}(\mathcal{V})\operatorname{rk}(\mathcal{W})} \geqslant 1 - \frac{1}{d^2} + \left(1 - \frac{1}{d}\right)\operatorname{slope}(\mathcal{W}).$$

Proof. We use classification of connections on a formal disk (see, for instance, [Mal91, Theorem III.1.2]). There exists a ramified extension

$$\widetilde{K}_x = \Bbbk((z^{1/r})) \supset K_x = \Bbbk((z))$$

and isomorphisms

$$\mathcal{V} \otimes \widetilde{K}_x \simeq (\widetilde{K}_x^{\operatorname{rk}\mathcal{V}}, d + \operatorname{diag}(\mu_{\mathcal{V}}^{(1)}, \dots, \mu_{\mathcal{V}}^{(\operatorname{rk}\mathcal{V})})), \quad \mu_{\mathcal{V}}^{(i)} \in \Omega_{\widetilde{K}_x} = \Bbbk((z^{1/r}))\,dz;$$
$$\mathcal{W} \otimes \widetilde{K}_x \simeq (\widetilde{K}_x^{\operatorname{rk}\mathcal{W}}, d + \operatorname{diag}(\mu_{\mathcal{W}}^{(1)}, \dots, \mu_{\mathcal{W}}^{(\operatorname{rk}\mathcal{W})})), \quad \mu_{\mathcal{W}}^{(j)} \in \Omega_{\widetilde{K}_x}.$$

Let us denote by $\operatorname{ord}(\mu)$ the order of $\mu \in \Omega_{\widetilde{K}}$ in z (which might be fractional). Then

$$\operatorname{ord}(\mu_{\mathcal{V}}^{(i)}) = -1 - \operatorname{slope}(\mathcal{V}) \quad (i = 1, \dots, \operatorname{rk}(\mathcal{V})),$$
$$\operatorname{ord}(\mu_{\mathcal{W}}^{(j)}) = -1 - \operatorname{slope}(\mathcal{W}) \quad (j = 1, \dots, \operatorname{rk}(\mathcal{W})).$$

Then

$$\operatorname{irreg}(\mathcal{H}om(\mathcal{V},\mathcal{W})) = \sum_{i,j} \max(-1 - \operatorname{ord}(\mu_{\mathcal{W}}^{(j)} - \mu_{\mathcal{V}}^{(i)}), 0). \tag{4.3}$$

Proof of (i). The leading terms of $\mu_{\mathcal{W}}^{(j)} - \mu_{\mathcal{V}}^{(i)}$ do not cancel, so

$$\operatorname{ord}(\mu_{\mathcal{W}}^{(j)} - \mu_{\mathcal{V}}^{(i)}) = \min(\operatorname{ord}(\mu_{\mathcal{W}}^{(j)}), \operatorname{ord}(\mu_{\mathcal{V}}^{(i)})).$$

Now (4.3) implies the formula for $\operatorname{slope}(\mathcal{H}om(\mathcal{V},\mathcal{W}))$. To prove the formula for $\delta(\mathcal{H}om(\mathcal{V},\mathcal{W}))$, we note that $H^0(K_x, \mathcal{H}om(\mathcal{V},\mathcal{W})) = \operatorname{Hom}(\mathcal{V},\mathcal{W}) = 0$, so

$$\delta(\mathcal{H}om(\mathcal{V},\mathcal{W})) = \operatorname{irreg}(\mathcal{H}om(\mathcal{V},\mathcal{W})) + \operatorname{rk}(\mathcal{H}om(\mathcal{V},\mathcal{W})).$$

Proof of (ii). Now cancellation in the leading terms of $\mu_{\mathcal{W}}^{(j)} - \mu_{\mathcal{V}}^{(i)}$ is possible. However, $\mathcal{V} \otimes \widetilde{K}_x$ carries an action of the Galois group $\operatorname{Gal}(\widetilde{K}_x/K_x)$. In particular, leading terms of $\mu_{\mathcal{V}}^{(j)}$ come in d-tuples of the form

$$\{\zeta a z^{-1-\operatorname{slope}(\mathcal{V})}\,dz : \zeta \in \Bbbk, \zeta^d = 1\}$$

for some fixed $a \in \Bbbk - \{0\}$. Therefore, among the differences $\mu_{\mathcal{W}}^{(j)} - \mu_{\mathcal{V}}^{(i)}$, not more than one out of every d has cancellation. Now (4.3) implies the formula for $\operatorname{slope}(\mathcal{H}om(\mathcal{V},\mathcal{W}))$. Finally,

$\dim \mathrm{Hom}(\mathcal{V}, \mathcal{W}) \leqslant 1$, and so

$$\begin{aligned}\frac{\delta(\mathcal{H}om(\mathcal{V}, \mathcal{W}))}{\mathrm{rk}(\mathcal{V})\,\mathrm{rk}(\mathcal{W})} &\geqslant \frac{\mathrm{rk}(\mathcal{V})\,\mathrm{rk}(\mathcal{W}) - 1}{\mathrm{rk}(\mathcal{V})\,\mathrm{rk}(\mathcal{W})} + \mathrm{slope}(\mathcal{H}om(\mathcal{V}, \mathcal{W}))\\ &\geqslant 1 - \frac{1}{\mathrm{rk}(\mathcal{V})\,\mathrm{rk}(\mathcal{W})} + \left(1 - \frac{1}{d}\right)\mathrm{slope}(\mathcal{W}).\end{aligned}$$

It remains to note that $\mathrm{rk}(\mathcal{V}), \mathrm{rk}(\mathcal{W}) \geqslant d$. □

LEMMA 4.8. *Suppose $\mathcal{V}, \mathcal{W} \in \mathcal{H}ol(\mathcal{D}_{K_x})$, and that $\mathcal{V}$ is irreducible.*

(i) *If $\mathrm{rk}(\mathcal{V}) > 1$, then $\delta(\mathcal{H}om(\mathcal{V}, \mathcal{W})) \geqslant \mathrm{rk}(\mathcal{V})\,\mathrm{rk}(\mathcal{W})$.*

(ii) *If $\mathrm{slope}(\mathcal{V}) > 2$ is not an integer, then $\delta(\mathcal{H}om(\mathcal{V}, \mathcal{W})) \geqslant 2\mathrm{rk}(\mathcal{V})\,\mathrm{rk}(\mathcal{W})$.*

Proof. By semiadditivity of δ, we may assume that $\mathcal{W}$ is irreducible without losing generality.

(i) Without loss of generality, we may assume that $\mathrm{slope}(\mathcal{V})$ is not an integer. Indeed, we can replace $\mathcal{V}$ and $\mathcal{W}$ with $\mathcal{H}om(\ell, \mathcal{V})$ and $\mathcal{H}om(\ell, \mathcal{W})$ for any $\ell \in \mathcal{H}ol(\mathcal{D}_K)$ of rank one, and we can choose ℓ so that $\mathrm{slope}(\mathcal{H}om(\ell, \mathcal{V}))$ is not an integer (as in Remark 4.2). Now the statement follows from Lemma 4.7(i) if $\mathrm{slope}(\mathcal{V}) \neq \mathrm{slope}(\mathcal{W})$ or from Lemma 4.7(ii) if $\mathrm{slope}(\mathcal{V}) = \mathrm{slope}(\mathcal{W})$.

(ii) If $\mathrm{slope}(\mathcal{V}) \neq \mathrm{slope}(\mathcal{W})$, the statement follows from Lemma 4.7(i). Assume $\mathrm{slope}(\mathcal{V}) = \mathrm{slope}(\mathcal{W})$. Then $\mathrm{slope}(\mathcal{W}) \geqslant 2 + 1/d$, where d is the denominator of $\mathrm{slope}(\mathcal{W})$. Lemma 4.7(ii) implies

$$\frac{\delta(\mathcal{H}om(\mathcal{V}, \mathcal{W}))}{\mathrm{rk}(\mathcal{V})\,\mathrm{rk}(\mathcal{W})} \geqslant 1 - \frac{1}{d^2} + \left(1 - \frac{1}{d}\right) \cdot \left(2 + \frac{1}{d}\right) = 2 + \frac{d^2 - d - 2}{d^2} \geqslant 2.$$ □

LEMMA 4.9. *Suppose that $\mathcal{V} \in \mathcal{H}ol(\mathcal{D}_{K_x})$ is irreducible and $\mathrm{slope}(\mathcal{V}) < 2$ is not an integer. Then for any $\mathcal{W} \in \mathcal{H}ol(\mathcal{D}_{K_x})$,*

$$\delta(\mathcal{H}om(\mathcal{V}, \mathcal{W})) \geqslant (\mathrm{irreg}(\mathcal{W}^{>1}) - \mathrm{rk}(\mathcal{W}^{>1}))\,\mathrm{rk}(\mathcal{V}) + \mathrm{rk}(\mathcal{V})\,\mathrm{rk}(\mathcal{W}).$$

(Recall that $\mathcal{W}^{>1}$ is the maximal submodule of $\mathcal{W}$ whose components all have slopes greater than one.)

Proof. By semiadditivity of δ, we can assume that $\mathcal{W}$ is irreducible (the right-hand side is additive in $\mathcal{W}$). The statement follows from Lemma 4.8(i) if $\mathrm{slope}(\mathcal{W}) \leqslant 1$, so assume $\mathrm{slope}(\mathcal{W}) > 1$. Then $\mathcal{W} = \mathcal{W}^{>1}$, and we have to show that

$$\frac{\delta(\mathcal{H}om(\mathcal{V}, \mathcal{W}))}{\mathrm{rk}(\mathcal{V})\,\mathrm{rk}(\mathcal{W})} \geqslant \mathrm{slope}(\mathcal{W}).$$

If $\mathrm{slope}(\mathcal{V}) \neq \mathrm{slope}(\mathcal{W})$, this follows from Lemma 4.7(i). Suppose therefore that $\mathrm{slope}(\mathcal{V}) = \mathrm{slope}(\mathcal{W})$. Then by Lemma 4.7(ii), we have

$$\frac{\delta(\mathcal{H}om(\mathcal{V}, \mathcal{W}))}{\mathrm{rk}(\mathcal{V})\,\mathrm{rk}(\mathcal{W})} - \mathrm{slope}(\mathcal{W}) \geqslant 1 - \frac{1}{d^2} - \frac{\mathrm{slope}(\mathcal{W})}{d} \geqslant 1 - \frac{1}{d^2} - \frac{1}{d}\left(2 - \frac{1}{d}\right) = 1 - \frac{2}{d} \geqslant 0.$$

Here we have used that $\mathrm{slope}(\mathcal{V}) = \mathrm{slope}(\mathcal{W}) \leqslant 2 - 1/d$. □

Let us return to the second case of Theorem A. We need to verify the following two claims.

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Proposition 4.10. *There is at most one $x \in \mathbb{P}^1 - U$ such that* $\mathrm{rk}(\mathcal{V}_x) > 1$.

Proof. Indeed, by Lemma 4.8(i), whenever $\mathrm{rk}(\mathcal{V}_x) > 1$, we have

$$\delta(\mathcal{E}nd(\Psi_x(\mathcal{L}))) \geqslant \frac{\mathrm{rk}(\mathcal{L})}{\mathrm{rk}(\mathcal{V}_x)} \delta(\mathcal{H}om(\mathcal{V}_x, \Psi_x(\mathcal{L}))) \geqslant \mathrm{rk}(\mathcal{L})^2.$$

However, by rigidity,

$$\sum_x \delta(\mathcal{E}nd(\Psi_x(\mathcal{L}))) = 2\mathrm{rk}(\mathcal{L})^2 - \mathrm{rig}(\mathcal{L}) < 2\mathrm{rk}(\mathcal{L})^2. \qquad \square$$

Proposition 4.11 (Theorem A, Case II). *Suppose that for $\infty \in \mathbb{P}^1 - U$, $\mathrm{rk}(\mathcal{V}_\infty) > 1$. Choose a rank-one connection ℓ on U such that $\mathcal{V}_x \simeq \Psi_x(\ell)$ for $x \in \mathbb{A}^1 - U$, and the slope of $\mathcal{H}om(\Psi_\infty(\ell), \mathcal{V}_\infty)$ is not an integer. Then* $\mathrm{rk}(\mathcal{H}om(\ell, \mathcal{L})^\wedge) < \mathrm{rk}(\mathcal{L})$.

Proof. Indeed, by Proposition 3.5, we have

$$\begin{aligned}\mathrm{rk}(\mathcal{H}om(\ell, \mathcal{L})^\wedge) = &\sum_{x \in \mathbb{A}^1 - U} \delta(\mathcal{H}om(\mathcal{V}_x, \Psi_x(\mathcal{L}))) \\ &+ \mathrm{irreg}(\Psi_\infty(\mathcal{H}om(\ell, \mathcal{L}))^{>1}) - \mathrm{rk}\, \Psi_\infty(\mathcal{H}om(\ell, \mathcal{L}))^{>1}.\end{aligned}$$

It suffices to prove the inequality

$$\begin{aligned}&\mathrm{irreg}(\Psi_\infty(\mathcal{H}om(\ell, \mathcal{L}))^{>1}) - \mathrm{rk}\, \Psi_\infty(\mathcal{H}om(\ell, \mathcal{L}))^{>1} \\ &\quad \leqslant \frac{\delta(\mathcal{H}om(\mathcal{V}_\infty, \Psi_\infty(\mathcal{L})))}{\mathrm{rk}(\mathcal{V}_\infty)} - \mathrm{rk}(\mathcal{L}),\end{aligned} \tag{4.4}$$

and then use the argument of Proposition 4.6.

To prove (4.4), take

$$\mathcal{V} = \mathcal{H}om(\Psi_\infty(\ell), \mathcal{V}_\infty), \quad \mathcal{W} = \mathcal{H}om(\Psi_\infty(\ell), \Psi_\infty(\mathcal{L})).$$

By the argument used to prove Proposition 4.10, $\delta(\mathcal{H}om(\mathcal{V}, \mathcal{W})) < 2\mathrm{rk}(\mathcal{V})\, \mathrm{rk}(\mathcal{W})$. We then see that $\mathrm{slope}(\mathcal{V}) < 2$ by Lemma 4.8(ii). Finally, (4.4) follows from Lemma 4.9. $\square$

5. Applications

5.1 Irregular Deligne–Simpson problem

Irregular Katz's algorithm can be applied to the 'irregular Deligne–Simpson problem' for rigid local systems. In the case of regular singularities, this is explained in [Kat96, § 6.4], and the irregular case is quite similar.

Definition 5.1. A *formal type datum* is a collection of isomorphism classes

$$\{[\mathcal{V}_x]\}_{x \in \mathbb{P}^1}$$

of connections $\mathcal{V}_x \in \mathcal{H}ol(\mathcal{D}_{K_x})$ such that the following conditions hold:

(i) $r = \mathrm{rk}\, \mathcal{V}_x$ does not depend on x;

(ii) for all but finitely many x, $\mathcal{V}_x$ is trivial, $\mathcal{V}_x \simeq (K_x^r, d)$;

(iii) $\sum_x \mathrm{res}(\bigwedge^r \mathcal{V}_x) \in \mathbb{Z}$ (since $\bigwedge^r \mathcal{V}_x \in \mathcal{H}ol(\mathcal{D}_{K_x})$ has rank one, its residue makes sense as an element of $\Bbbk/\mathbb{Z}$).

A solution of the *(irregular) Deligne–Simpson problem* corresponding to $\{[\mathcal{V}_x]\}$ is an irreducible connection $\mathcal{L}$ on an open subset of $\mathbb{P}^1$ with prescribed formal type: $\Psi_x(\mathcal{L}) \simeq \mathcal{V}_x$ for all x.

The *rigidity index* of $\{[\mathcal{V}_x]\}$ is

$$\operatorname{rig}\{[\mathcal{V}_x]\} = 2r^2 - \sum_{x \in \mathbb{P}^1} \delta(\mathcal{E}nd(\mathcal{V}_x)),$$

where δ is defined by (3.1).

Suppose that $\mathcal{L}$ solves the Deligne–Simpson problem for $\{[\mathcal{V}_x]\}$. By Proposition 3.1, $\operatorname{rig}(\mathcal{L}) = \operatorname{rig}\{[\mathcal{V}_x]\}$. In particular, $\operatorname{rig}\{[\mathcal{V}_x]\} \leqslant 2$.

Let $\mathcal{L}^\wedge$ be the Fourier transform of $\mathcal{L}$, and let $\{[\mathcal{V}_x^\wedge]\}$ be the formal type of $\mathcal{L}^\wedge$: $\mathcal{V}_x^\wedge = \Psi_x(\mathcal{L}^\wedge)$. One can check that $\{[\mathcal{V}_x^\wedge]\}$ is determined by $\{[\mathcal{V}_x]\}$; essentially, $\mathcal{V}_x^\wedge$ is given by the local Fourier transform of [BE04] (this is discussed in more detail in [Ari08]). In other words, we obtain a notion of the Fourier transform for formal type data, and $\{[\mathcal{V}_x^\wedge]\}$ is the Fourier transform of $\{[\mathcal{V}_x]\}$.

For arbitrary formal type datum $\{[\mathcal{V}_x]\}$, its Fourier transform $\{[\mathcal{V}_x^\wedge]\}$ might be undefined. Actually, $[\mathcal{V}_x^\wedge]$ (for $x \neq \infty$) is constructed from $\{[\mathcal{V}_x]\}$ in two steps: the local Fourier transform describes the quotient $\mathcal{V}_x^\wedge/(\mathcal{V}_x^\wedge)^{\mathrm{hor}}$ modulo the maximal trivial subconnection, while Proposition 3.5 gives a formula for $\dim(\mathcal{V}_x^\wedge)$. This determines the isomorphism class $[\mathcal{V}_x^\wedge]$, assuming the obvious compatibility condition $\dim(\mathcal{V}_x^\wedge/(\mathcal{V}_x^\wedge)^{\mathrm{hor}}) \leqslant \dim \mathcal{V}_x^\wedge$. If the compatibility condition fails, the Fourier transform of $\{[\mathcal{V}_x]\}$ is undefined.

The Fourier transform $\mathcal{L} \to \mathcal{L}^\wedge$ provides a one-to-one correspondence between solutions to the Deligne–Simpson problems for $\{[\mathcal{V}_x]\}$ and $\{[\mathcal{V}_x^\wedge]\}$. If $\{[\mathcal{V}_x^\wedge]\}$ is undefined, the Deligne–Simpson problem for $\{[\mathcal{V}_x]\}$ has no solutions.

The situation for middle convolution $\mathcal{L} \star_{\mathrm{mid}} \mathcal{K}^\lambda$ is similar to that for the Fourier transform. Again, it makes sense for formal type data, but it is not always defined. If the formal type data are related by the middle convolution, their Deligne–Simpson problems are equivalent. If the middle convolution of a formal type datum is undefined, its Deligne–Simpson problem has no solutions.

Now let us analyze the Deligne–Simpson problem for a formal type datum $\{[\mathcal{V}_x]\}$ in the case $\operatorname{rig}\{[\mathcal{V}_x]\} = 2$. We can run an irregular Katz's algorithm on the level of formal type data. On each step, we decrease the rank of the formal type datum using either the middle convolution or the Fourier transform, assuming that they are defined. After finitely many steps, we arrive at one of the two situations.

- The irregular Katz's algorithm decreases the rank of the formal type datum to one. Then the Deligne–Simpson problem for $\{[\mathcal{V}_x]\}$ is equivalent to the Deligne–Simpson problem for a formal type datum of rank one, which is clearly solvable.
- The output of a step of irregular Katz's algorithm is undefined, and then the Deligne–Simpson problem for $\{[\mathcal{V}_x]\}$ has no solutions.

Remark. In [Sim09], Simpson uses Katz's algorithm to analyze the (regular) Deligne–Simpson problem without restrictions on the rigidity index. We do not know whether the irregular Katz's algorithm can be used for a similar analysis in the irregular case.

5.2 Rigidity index zero and Lax pairs for Painlevé equations

The irregular Katz's algorithm can also be used to classify connections $\mathcal{L}$ of rigidity index 0; the details will be given elsewhere. In the case of regular singularities, such a classification was proved by Kostov [Kos01, Lemma 17].

We claim that the proof of Theorem A can be modified for such $\mathcal{L}$. It is not true that the rank of $\mathcal{L}$ can always be decreased to one, however, the algorithm's *stopping points* (that is, the connections whose rank cannot be decreased) can be described.

Let us study the moduli spaces of connections. For a fixed formal type datum $\{[\mathcal{V}_x]\}$, consider the moduli space $\mathfrak{M} = \mathfrak{M}_{\{[\mathcal{V}_x]\}}$ of irreducible connections of this formal type. Equivalently, points of $\mathfrak{M}$ are solutions to the Deligne–Simpson problem. Then

$$\dim \mathfrak{M} = -2 - \operatorname{rig}(\{[\mathcal{V}_x]\});$$

in particular, if $\operatorname{rig}(\{\mathcal{V}_x\}) = 0$, then $\mathfrak{M}$ is a surface.

If $\{[\mathcal{V}_x^\wedge]\}$ is the Fourier transform of $\{[\mathcal{V}_x]\}$, we obtain an isomorphism

$$\mathfrak{M}_{\{[\mathcal{V}_x]\}} \xrightarrow{\sim} \mathfrak{M}_{\{[\mathcal{V}_x^\wedge]\}} : \mathcal{L} \mapsto \mathcal{L}^\wedge.$$

Similarly, middle convolution $\mathcal{L} \mapsto \mathcal{L} \star_{\mathrm{mid}} \mathcal{K}^\lambda$ induces an isomorphism between moduli spaces. Therefore, the space $\mathfrak{M}$ does not change as we apply the irregular Katz's algorithm to $\{[\mathcal{V}_x]\}$. In this way, we can always reduce to the case when the formal type $\mathcal{V}_x$ is a stopping point of the algorithm.

Important examples are spaces $\mathfrak{M}$ for $\operatorname{rk} \mathcal{V}_x = 2$. Assume $\operatorname{rig}\{[\mathcal{V}_x]\} = 0$, so $\dim(\mathfrak{M}([\mathcal{V}_x])) = 2$. Note that $\{[\mathcal{V}_x]\}$ is automatically a stopping point of the algorithm, because its rank cannot be decreased to one, as all rank-one systems are rigid. The surface $\mathfrak{M}$ is the space of initial conditions of a Painlevé equation P_*, where index $* = I, II, \ldots, VI$ depends on $\{[\mathcal{V}_x]\}$. Geometrically, P_* controls the isomonodromy deformation of connections. The isomorphisms induced by the Fourier transform and the middle convolution respect the isomonodromy deformations. Therefore, if generalized Katz's algorithm reduces formal type datum $\{[\mathcal{W}_x]\}$ to $\{[\mathcal{V}_x]\}$, the isomonodromy deformation of connections of type $\{[\mathcal{W}_x]\}$ is also given by P_*. In other words, $\{[\mathcal{W}_x]\}$ gives another Lax pair for P_*. In this manner, irregular Katz's algorithm in the case of rigidity index zero can be viewed as a reduction algorithm for Lax pairs for Painlevé equations.

Remark. Only the sixth Painlevé equation P_{VI} appears in the classification of [Kos01, Lemma 17]; the other Painlevé equations correspond to irregular formal types.

6. Remarks

6.1 Middle convolution via twisted differential operators

The middle convolution with Kummer local system is naturally formulated in terms of rings of twisted differential operators (TDOs).

Denote by $\mathcal{D}_1$ the TDO ring acting on $\mathcal{O}_{\mathbb{P}^1}(1)$ (see [BB93] for the definition of a TDO ring). Let us 'scale' $\mathcal{D}_1$ by a fixed number $\lambda \in \Bbbk$, denote the resulting TDO by $\mathcal{D}_\lambda$. Informally, $\mathcal{D}_\lambda$ is the ring of differential operators on $\mathcal{O}_{\mathbb{P}^1}(\lambda)$.

Remark. Consider the natural projection $p : \mathbb{A}^2 - \{0\} \to \mathbb{P}^1$. We can interpret holonomic $\mathcal{D}_\lambda$-modules as $\mathcal{D}$-modules M on $\mathbb{A}^2 - \{0\}$ such that the restriction of M to any fiber $p^{-1}(x)$ is a sum of several copies of $\mathcal{K}^\lambda$. Informally, we require that M is a *monodromic* $\mathcal{D}$-module whose restriction to each fiber has 'monodromy $e^{2\pi i\lambda}$'.

Suppose that $\lambda \in \Bbbk - \mathbb{Z}$. In [DE03], D'Agnolo and Eastwood present an equivalence $\mathcal{R}$ (*the Radon transform*) between the category of $\mathcal{D}_\lambda$-modules and that of $\mathcal{D}_{-\lambda}$-modules. (One should keep in mind that up to equivalence, the category of $\mathcal{D}_\lambda$-modules depends only on the image of λ in $\Bbbk/\mathbb{Z}$.) Here $\mathcal{R}$ can be viewed as a twisted version of the transform defined by Brylinski in [Bry86]. In a sense, it is also a particular case of the Radon transform defined by Braverman and Polishchuk, who consider monodromic sheaves whose monodromy need not be scalar [BP98].

Explicitly, $\mathcal{R}$ can be defined as the integral transform whose kernel is a rank-one $\mathcal{D}_\lambda \boxtimes \mathcal{D}_\lambda$-module on $\mathbb{P}^1 \times \mathbb{P}^1$ with a simple pole along the diagonal (and no other singularities). Alternatively, if one interprets $\mathcal{D}_\lambda$-modules as monodromic $\mathcal{D}$-modules on $\mathbb{A}^2 - \{0\}$, the equivalence is simply the Fourier transform on $\mathbb{A}^2$.

We can view the middle convolution with Kummer local system as a composition of the Goresky–MacPherson extension and the Radon transform as follows. A connection $\mathcal{L}$ on an open set $U \subset \mathbb{A}^1$ can be viewed as a $\mathcal{D}_\lambda|_U$-module using the trivialization $\mathcal{D}_\lambda|_{\mathbb{A}^1} = \mathcal{D}_{\mathbb{A}^1}$. We then extend it to a $\mathcal{D}_\lambda$-module $j_{!*}\mathcal{L}$ for $j : U \hookrightarrow \mathbb{P}^1$. The Radon transform $\mathcal{R}(j_{!*}\mathcal{L})$ is a holonomic $\mathcal{D}_{-\lambda}$-module that is smooth on U. Its restriction to U is a connection which equals $\mathcal{L} \star_{\mathrm{mid}} \mathcal{K}^\lambda$.

The first case of Theorem A can be reformulated.

(i′) *There is a rank-one $\mathcal{D}_{-\lambda}|_U$-module ℓ such that*

$$\operatorname{rk} \mathcal{R}(j_{!*}\, \mathcal{H}om(\ell, \mathcal{L})) < \operatorname{rk}(\mathcal{L}).$$

Note that the point ∞ plays no special role in this formulation. Similarly, if one rewrites Proposition 3.6 using the Radon transform, the special treatment of ∞ is not necessary, essentially because it plays no special role in the definition of $\mathcal{R}$.

Remark. In [Sim09], Simpson studies the middle convolution via the so-called 'convoluters'. One can rewrite Theorem A(i) using the de Rham version of convoluters [Sim09, § 3.3] with irregular singularities. From the viewpoint of TDO rings, the convoluter encodes the $\mathcal{D}_{-\lambda}$-module ℓ from case (i′).

6.2 The l-adic version of the irregular Katz's algorithm

The following observation is due to Deligne.

Most of the proof of Theorem A remains valid in the settings of l-adic sheaves. The only exception is Lemma 4.7. Its first statement still holds (see [Kat88, Lemma 1.3]), but the second statement requires the additional assumption that d (the denominator of the slope) is not divisible by the characteristic of the ground field. Let us make the statement precise.

Let K be the fraction field of a Henselian valuation ring whose residue field is perfect of finite characteristic p. Denote by $I \subset \mathrm{Gal}(K^{\mathrm{sep}}/K)$ the inertia group of K and by $P \subset I$ its Sylow's p-group. For a continuous finite-dimensional representation V of P (over a fixed l-adic field), we denote its break decomposition by

$$V = \bigoplus_{s \in \mathbb{Q}} V(s).$$

One can check the following statement.

LEMMA 6.1. *Let V and W be continuous finite-dimensional representations of* $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. *Fix $s \in \mathbb{Q}$ with denominator d, and suppose that p does not divide d. Then*

$$\dim(\mathrm{Hom}(V, W)(s)) \geqslant \dim V(s) \dim W(s) \left(1 - \frac{1}{d}\right). \tag{6.1}$$

Remark. Since Lemma 6.1 holds for all V and W, one can replace them by $V(s)$ and $W(s)$. In other words, in (6.1) one can replace $\mathrm{Hom}(V, W)(s)$ with the image of $\mathrm{Hom}(V(s), W(s))$ in this space.

In particular, (6.1) holds if either $\dim(V)$ or $\dim(W)$ is less than p. This implies that the extension of Katz's algorithm works for wild l-adic local systems whose rank does not exceed the characteristic p of the ground field.

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