Canad. Math. Bull. Vol. 32 (2), 1989

# A RELATION BETWEEN THE 2-PRIMARY PARTS OF THE MAIN CONJECTURE AND THE BIRCH-TATE-CONJECTURE

BY

MANFRED KOLSTER

ABSTRACT. It is shown that for totally real number fields the Main Conjecture in Iwasawa-Theory for $p = 2$ proposed by Federer implies the 2-primary part of the Birch-Tate-Conjecture in analogy with the case $p$ odd.

Let $E$ be a totally real number field with ring of integers $o$. If $p$ is an odd prime, $F_\infty$ the cyclotomic $\mathbf{Z}_p$-extension of $E(\zeta_p)$ and $A_\infty^-$ the minus-part of the $p$-primary component of the class-group of $F_\infty$, the Main Conjecture relates the characteristic polynomial of the Pontryagin dual of $A_\infty^-$ to $p$-adic $L$-functions. Coates ([1], [2]) has shown that this conjecture implies the $p$-primary part of the Birch-Tate-Conjecture, which relates the order of the tame kernel $K_2(o)$ to the value of the $\zeta$-function of $E$ at $-1$.

In [4] Federer proposed an analogous Main Conjecture for the prime 2. The purpose of this note, which is an addendum to [7], is to show that similarly this conjecture implies the 2-primary part of the Birch-Tate-Conjecture.

Whereas the Main Conjecture for odd primes has been proved at least for abelian number fields by Mazur-Wiles [9], Federer's analog is still open, although some evidence was given in [4]. On the other hand the 2-primary part of the Birch-Tate-Conjecture holds, whenever the 2-Sylow-subgroup of $K_2(o)$ is elementary abelian ([6]), so that the relation between these conjectures may give further evidence to both of them.

Let $F_0 = E(\zeta_4)$ and $e \geqq 2$ be maximal with $F_0 = E(\zeta_{2^e})$. If we define $F_n = E(\zeta_{2^{n+e}})$, $n \geqq 1$, and $F_\infty = \bigcup_n F_n$, the fields $F_n$ are the $n$-th layers in the cyclotomic $\mathbf{Z}_2$-extension $F_\infty$ of $F_0$. Each $F_n$ is a $CM$-field with maximal real subfield $F_n^+$ and $F_\infty^+ = \bigcup_n F_n^+$ is the cyclotomic $\mathbf{Z}_2$-extension of $E$. Let $\Gamma = \mathrm{Gal}(F_\infty/F_0) \cong \mathrm{Gal}(F_\infty^+/E) \cong \mathbf{Z}_2$ and choose a topological generator $\gamma_0$ of $\Gamma$. We define $W = \varinjlim \mu_{2^n}$ and $u \in \mathbf{Z}_2^*$ via the action of $\gamma_0$ on $W : \gamma_0(\zeta) = \zeta^u$. Note that by definition of $e$ we have $u = 1 + 2^e \cdot \epsilon$ with $\epsilon \in \mathbf{Z}_2^*$.

The ring of integers in $F_n$ (resp. $F_n^+$) is denoted by $o_n$ (resp. $o_n^+$) and the group of units by $U_n$ (resp. $U_n^+$). Let $A_n$ (resp. $A_n^+$) denote the 2-primary component of the

Received by the editors March 31, 1988.
AMS Subject Classification 11S40.




https://doi.org/10.4153/CMB-1989-036-8 Published online by Cambridge University Press

class-group of $F_n$ (resp. $F_n^+$), and $A_n^-$ the kernel of the (surjective) norm map from $A_n$ to $A_n^+$. Passing to the limit we let $U_\infty^+ = \varinjlim U_n^+$, $A_\infty^- = \varinjlim A_n^-$. $\mathcal{T} = \varprojlim \mu_{2^n}$ denotes the Tate-module.

The compatibility of the conjectures is based on the following result:

THEOREM 1. *The order of the 2-primary part of the tame kernel is given by*

$$|K_2(o)(2)| = 2^{[E:\mathbf{Q}]} \cdot \left| \left( \mathcal{T} \bigotimes_{\mathbf{Z}_2} A_\infty^- \right)^\Gamma \right|.$$

PROOF. It was shown in [7], Theorem 3.7, that there is an exact sequence of finite groups

$$0 \longrightarrow \left( \mu_2 \bigotimes_{\mathbf{Z}} U_\infty^+ \right)^\Gamma \longrightarrow K_2(o)(2) \longrightarrow \left( \mathcal{T} \bigotimes_{\mathbf{Z}_2} A_\infty^- \right)^\Gamma \longrightarrow H^1\left( \Gamma, \mu_2 \bigotimes_{\mathbf{Z}} U_\infty^+ \right) \longrightarrow 0,$$

hence our claim is equivalent to

$$\left| \left( \mu_2 \bigotimes_{\mathbf{Z}} U_\infty^+ \right)^\Gamma \right| = 2^{[E:\mathbf{Q}]} \cdot \left| H^1\left( \Gamma, \mu_2 \bigotimes_{\mathbf{Z}} U_\infty^+ \right) \right|.$$

Let $\mathcal{E}_\infty^+ = U_\infty^+/\mu_2$ be the free part of $U_\infty^+$. The cohomology sequence attached to the exact sequence

$$0 \longrightarrow \mu_2 \bigotimes_{\mathbf{Z}} \mu_2 \longrightarrow \mu_2 \bigotimes_{\mathbf{Z}} U_\infty^+ \longrightarrow \mu_2 \bigotimes_{\mathbf{Z}} \mathcal{E}_\infty^+ \longrightarrow 0$$

yields

$$\frac{\left| \left( \mu_2 \otimes_{\mathbf{Z}} U_\infty^+ \right)^\Gamma \right|}{\left| H^1\left( \Gamma, \mu_2 \otimes_{\mathbf{Z}} U_\infty^+ \right) \right|} = \frac{\left| \left( \mu_2 \otimes_{\mathbf{Z}} \mathcal{E}_\infty^+ \right)^\Gamma \right|}{\left| H^1\left( \Gamma, \mu_2 \otimes_{\mathbf{Z}} \mathcal{E}_\infty^+ \right) \right|}.$$

Since $\mathcal{E}_\infty^+$ is free abelian, squaring yields an exact sequence

$$0 \longrightarrow \mathcal{E}_\infty^+ \longrightarrow \mathcal{E}_\infty^+ \longrightarrow \mu_2 \bigotimes_{\mathbf{Z}} \mathcal{E}_\infty^+ \longrightarrow 0,$$

hence an exact sequence in cohomology

$$\begin{aligned} 0 \longrightarrow \mathcal{E}_\infty^{+\Gamma} \longrightarrow \mathcal{E}_\infty^{+\Gamma} \longrightarrow \left( \mu_2 \bigotimes_{\mathbf{Z}} \mathcal{E}_\infty^+ \right)^\Gamma \longrightarrow H^1(\Gamma, \mathcal{E}_\infty^+) \\ \xrightarrow{2} H^1(\Gamma, \mathcal{E}_\infty^+) \longrightarrow H^1\left( \Gamma, \mu_2 \bigotimes_{\mathbf{Z}} \mathcal{E}_\infty^+ \right) \longrightarrow H^2(\Gamma, \mathcal{E}_\infty^+) \xrightarrow{2} H^2(\Gamma, \mathcal{E}_\infty^+) \longrightarrow 0. \end{aligned}$$

The structure of the cohomology groups $H^i(\Gamma, \mathcal{E}_\infty^+)$, $i = 1, 2$, has been revealed by Iwasawa ([5], Proposition 2)

$$\begin{aligned} H^1(\Gamma, \mathcal{E}_\infty^+) &\cong B \oplus (\mathbf{Q}_2/\mathbf{Z}_2)^r, B \text{ fin. group} \\ H^2(\Gamma, \mathcal{E}_\infty^+) &\cong (\mathbf{Q}_2/\mathbf{Z}_2)^{r-1} \end{aligned}$$

for some $r$, $1 \leqq r \leqq d$, where $d$ is the number of dyadic primes in $F_\infty^+$.

Let $2^s = |{}_2B| = |B/2B|$. Since multiplication by 2 is onto on the divisible part of $H^1(\Gamma, \mathcal{E}_\infty^+)$, we obtain

$$\frac{\left|\left(\mu_2 \otimes_{\mathbf{Z}} \mathcal{E}_\infty^+\right)^\Gamma\right|}{\left|\mu_2 \otimes_{\mathbf{Z}} \mathcal{E}_\infty^{+\Gamma}\right|} = 2^{s+r}$$

and $|H^1(\Gamma, \mu_2 \otimes_{\mathbf{Z}} \mathcal{E}_\infty^+)| = 2^{s+r-1}$.

Now $\mathcal{E}_\infty^{+\Gamma}$ is the free part of the group of units in $E$, hence by the unit theorem $\mu_2 \otimes_{\mathbf{Z}} \mathcal{E}_\infty^{+\Gamma}$ has order $2^{[E:\mathbf{Q}]-1}$, which yields the claim.

Let $\Lambda = \mathbf{Z}_2[[T]]$ and let $f(T)$ be the characteristic polynomial of the Pontryagin dual $\check{A}_\infty^- = \mathrm{Hom}_{\mathbf{Z}_2}(A_\infty^-, \mathbf{Q}_2/\mathbf{Z}_2)$ of $A_\infty^-$. Then $f(u^{-1}(1+T)-1)$ is the characteristic polynomial of the dual $\check{A}_\infty^-(-1)$ of $\mathcal{T} \otimes_{\mathbf{Z}_2} A_\infty^-$ (cf. [8], Lemma 4.1). Since $A_\infty^-$ has no non-trivial finite $\Lambda$-submodules (cf. [4]), the order of $(\mathcal{T} \otimes_{\mathbf{Z}_2} A_\infty^-)^\Gamma$, which is the same as the order of $(\check{A}_\infty^-(-1))_\Gamma$, is as usual determined by evaluating the characteristic polynomial at $T = 0$. Hence, if we use the notation $a \sim b$ for two 2-adic integers having the same 2-adic valuation, we obtain

LEMMA 2. $|(\mathcal{T} \otimes_{\mathbf{Z}_2} A_\infty^-)^\Gamma| \sim f(u^{-1}-1)$.

Let $\chi_0$ denote the trivial character of $\mathrm{Gal}(F_0/E)$. There is a unique power series $G(T)$, such that the 2-adic $L$-function $L_2(\chi_0, s)$ is given by

$$L_2(\chi_0, s) = G(u^s - 1)/u^s - u \quad \text{(cf. [4], [2]).}$$

Furthermore from [3] it is known that $G(T)$ is contained in $2^{[E:\mathbf{Q}]} \cdot \Lambda$. This motivated the following Main Conjecture for $p = 2$ (cf. [4]):

*Conjecture* 3 (Federer): $G(T)$ and $2^{[E:\mathbf{Q}]} \cdot f(T)$ generate the same ideal in $\Lambda$.

Let $\zeta_E$ denote the $\zeta$-function of $E$ and let $w_2(E)$ be the maximal natural number $n$, such that $\mathrm{Gal}(E(\zeta_n)/E)$ has exponent 2. The 2-primary part of the Birch-Tate-Conjecture states:

*Conjecture* 4 (Birch-Tate):

$$|K_2(o)| \sim w_2(E) \cdot \zeta_E(-1).$$

The relation between these conjectures is given by

THEOREM 5. *Conjecture* 3 *implies Conjecture* 4.

PROOF. By definition of the number $e$ we obtain $w_2(E) \sim 2^{e+1} \sim u^{-1} - u$. Since $L_2(\chi_0, -1) \sim \zeta_e(-1)$, we get $w_2(E) \cdot \zeta_E(-1) \sim G(u^{-1}-1) \overset{(3)}{\sim} 2^{[E:\mathbf{Q}]} \cdot f(u^{-1}-1) \overset{(2)}{\sim} 2^{[E:\mathbf{Q}]} \cdot |(\mathcal{T} \otimes_{\mathbf{Z}_2} A_\infty^-)^\Gamma| \overset{(1)}{\sim} |K_2(o)|$. □

https://doi.org/10.4153/CMB-1989-036-8 Published online by Cambridge University Press

## References

1. J. Coates, *On $K_2$ and some classical conjectures in algebraic number theory*, Ann. of Math. **95** (1972), 99–116.

2. ——— *p-adic L-functions and Iwasawa's theory*, in A. Fröhlich (ed.): Algebraic Number Fields, Academic Press, London (1977), 227–286.

3. P. Deligne and K. Ribet, *Values of abelian L-functions at negative integers over totally real fields*, Inv. math. **59** (1980), 227–286.

4. L. J. Federer, *Regulators, Iwasawa Modules, and the Main Conjecture for $p = 2$*, in N. Koblitz (ed.): Number Theory related to Fermat's Last Theorem, Birkhäuser, Basel (1982), 289–296.

5. K. Iwasawa, *On cohomology groups of units for $\mathbf{Z}_p$-extensions*, Am. J. of Math. **105** (1983), 189–200.

6. M. Kolster, *The structure of the 2-Sylow-subgroup of $K_2(o)$*, I, Comm. Math. Helv. **61** (1986), 376–388.

7. ——— *The structure of the 2-Sylow-subgroup of $K_2(o)$*, II, $K$-theory, **1** (1987), 467–479.

8. S. Lichtenbaum, *On the values of zeta and L-functions*, I, Ann. of Math. **96** (1972), 388–360.

9. B. Mazur and A. Wiles, *Class fields of abelian extensions of* $\mathbf{Q}$, Inv. Math. **76** (1984), 179–330.

*Westf. Wilhelms-Universität*
*Math. Institut*
*Einsteinstr. 62 D-4400 Münster*
*West-Germany*

https://doi.org/10.4153/CMB-1989-036-8 Published online by Cambridge University Press