J. Austral. Math. Soc. (*Series A*) **53** (1992), 116–119

PERIODIC GROUPS WITH MANY PERMUTABLE SUBGROUPS

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(Received 27 June 1990)

Communicated by H. Lausch

Abstract

Groups in which every infinite set of subgroups contains a pair that permute were studied by M. Curzio, J. Lennox, A. Rhemtulla and J. Wiegold. The question whether periodic groups in this class were locally finite was left open. Here we show that if the generators of such a group G are periodic then G is locally finite. This enables us to get the following characterisation. A finitely generated group G is centre-by-finite if and only if every infinite set of subgroups of G contains a pair that permute.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 20 E 15.

Introduction

Let G be a group such that every infinite set of subgroups of G contains a pair that permute. Such groups were called pseudo-Hamiltonian and referred to as PH-groups [1]. Among the results shown in that paper are the following.

Every finitely generated soluble PH-group is centre-by-finite (Theorem 1), and all torsion-free PH-groups are abelian (Theorem 2).

The question as to whether periodic PH-groups are locally finite was left open. We are able to answer this and show that a finitely generated group G is a PH-group if and only if it is centre-by-finite. The two results in this note are as follows.

THEOREM 1. *Let* $G = \langle x_1, \dots, x_n \rangle$ *be a PH-group such that each* x_i *is of finite order. Then* G *is finite.*

THEOREM 2. *Let* G *be a finitely generated group. Then* G *is a PH-group if and only if* G *is centre-by-finite.*

Of course all centre-by-finite groups are PH-groups but the converse is not true for non-finitely generated groups, as was pointed out in [1].

Proofs

We begin by showing that the "only if" part of Theorem 2 is largely a consequence of Theorem 1. Clearly it is enough to show that, in a finitely generated PH-group G, the central factor group $G/Z(G)$ is periodic. This fact will also be of use in proving Theorem 1 and is established as follows.

Let $G = \langle x_1, \dots, x_n \rangle$ and suppose that G is a PH-group. If $g \in G$ is an element of infinite order, then by considering the sequence $\langle gx_i \rangle, \langle g^2 x_i \rangle, \dots$, we see that, for some $r \neq s$, we must have $\langle g^r x_i \rangle \langle g^s x_i \rangle = \langle g^s x_i \rangle \langle g^r x_i \rangle$. Thus as a permutable product of abelian groups, the group $K = \langle g^r x_i, g^s x_i \rangle$ is metabelian. By [1, Theorem 1] it is centre-by-finite. Now g^{r-s} lies in K and hence g^{t_i} lies in the centre of K, for some $t_i > 0$.

Thus $[g^{t_i}, x_i] = 1$. Now let $t = \text{l.c.m.}\{t_1, \dots, t_n\}$. Then $g^t \in Z(G)$, and $G/Z(G)$ is periodic, as claimed.

PROOF OF THEOREM 1. Let $G = \langle x_1, \dots, x_n \rangle$ be a PH-group, where each x_i is of finite order. Firstly, we note that, if H is any maximal subgroup of G (such subgroups exist since G is finitely generated) then H is of finite index in G. For if not, then H is its own normalizer and has infinitely many conjugates, a pair of which must therefore permute. It follows that $HH^g = H^g H$ for some $g \notin H$. This gives $G = HH^g$, a contradiction since no group is the product of two conjugates of a proper subgroup.

Now suppose, for a contradiction, that G is not finite. By a result of Baer (see [2, p. 171]) we may assume that every proper quotient of G is finite. Let J be the intersection of all non-trivial normal subgroups of G. If $J \neq 1$ then it is of finite index in G and therefore finitely generated. By the observation made above, J has finite non-trivial quotients and hence G has normal subgroups of finite index strictly contained in J, a contradiction. Thus $J = 1$ and G is residually finite.

Let Z be the centre of G. If $Z \neq 1$ then G is centre-by-finite and hence G' is finite. But G/G' is in any case finite. This gives a contradiction and so $Z = 1$. It now follows from our earlier remarks that G is periodic.

Let a_1 be any non-identity element of G and pick $a_2 \in G$ such that the pair $\langle a_1 \rangle$, $\langle a_2 \rangle$ is not permutable. Then, if possible, pick $a_3 \in G$ such that no two of the subgroups $\langle a_1 \rangle$, $\langle a_2 \rangle$, $\langle a_3 \rangle$ permute. Continue this process. It ends after a finite number, say, r, of steps since G is a PH-group. We thus have a set $\langle a_1 \rangle, \dots, \langle a_r \rangle$ of subgroups no two of which permute but such that if $g \in G$ then $\langle g \rangle \langle a_i \rangle = \langle a_i \rangle \langle g \rangle$ for some $i \in \{1, \dots, r\}$.

Let $C = C_G(a_1) \cup \cdots \cup C_G(a_r)$ and let $Q = G \backslash C$, the set of those elements of G not in C. As above, let $\{\langle b_1 \rangle, \dots, \langle b_s \rangle\}$ be a maximal set of pairwise non-permutable subgroups with each $b_i \in Q$. Now, for every $h \in Q$, there exists $j \in \{1, \dots, s\}$ such that $\langle h \rangle \langle b_j \rangle = \langle b_j \rangle \langle h \rangle$. If the set Q is empty then take the set $\{\langle b_1 \rangle, \dots, \langle b_s \rangle\}$ to be the empty set.

Since G is residually finite, there exists $1 \neq N \lhd G$ such that $N \cap (\langle a_1 \rangle \cup \cdots \cup \langle a_r \rangle \cup \langle b_1 \rangle \cup \cdots \cup \langle b_s \rangle) = 1$. Suppose $g \in Q \cap N$. Then there exists $i \in \{1, \dots, r\}$ such that $\langle g \rangle \langle a_i \rangle = \langle a_i \rangle \langle g \rangle$ and $j \in \{1, \dots, s\}$ such that $\langle g \rangle \langle b_j \rangle = \langle b_j \rangle \langle g \rangle$. For some α, β, therefore, $gb_j = b_j^{\alpha} g^{\beta}$ and so $b_j^{\alpha - 1} = b_j^{-1} g b_j g^{-\beta} \in N \cap \langle b_j \rangle = 1$. Thus $b_j^{-1} g b_j = g^{\beta} \in \langle g \rangle$ and $b_j \in N_G \langle g \rangle$. Similarly $a_i \in N_G \langle g \rangle$. Since the automorphism group of a cyclic group is abelian, it follows that $[a_i, b_j] \in C_G(g)$.

Let $D = \bigcup C_G([a_i, b_j])$ and notice that $[a_i, b_j] \neq 1$ since $b_j \notin C_G(a_i)$. On the other hand, if $Q \cap N$ is empty, let D be the empty set also. In either case we have $N \leq C \cup D$, a union of finitely many subgroups of G of the form $C_G(a_i)$ or $C_G([a_i, b_j])$. Thus, by a result of B. H. Neumann (see [3, Theorem 7.6]), one of these subgroups has finite index in N and hence in G. This means that the FC-centre F of G is non-trivial and therefore of finite index in G. Thus F is finitely generated periodic and hence finite, giving once more the contradiction that G is finite.

Proof of Theorem 2. We have already seen that the implications in one direction follows from Theorem 1. Conversely, if G is a centre-by-finite group and $H_1, H_2, \dots$ is an infinite sequence of subgroups of G then $H_i Z = H_j Z$ for some $i \neq j$, with $Z = Z(G)$, so that H_i is normalized by H_j and certainly $H_i H_j = H_j H_i$. This completes the proof.

Note that if $G/Z(G)$ is finite then there are only finitely many distinct subgroups of G containing $Z(G)$. If this number is say n, then for any sequence $H_1, \dots, H_{n+1}$ of subgroups of G, there is a pair that normalize each other. Thus if G is a finitely generated PH-group, then there exists an integer n such that any set of more than n subgroups contains a pair that normalize each other.

https://doi.org/10.1017/S1446788700035448 Published online by Cambridge University Press

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https://doi.org/10.1017/S1446788700035448 Published online by Cambridge University Press