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# A GENERALIZATION OF SPERNER'S THEOREM

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**Abstract**

Some generalizations of Sperner's theorem and of the LYM inequality are given to the case when $A_1, \ldots, A_t$ are $t$ families of subsets of $\{1, \ldots, m\}$ such that a set in one family does not properly contain a set in another.

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In this note we generalize Sperner's theorem [3] that "a Sperner family (or clutter, or antichain) of subsets of the finite set $\{1, \ldots, m\}$ contains at most $\binom{m}{[m/2]}$ sets", to the case where $\mathcal{A}_1, \ldots, \mathcal{A}_t$ are $t$ families of subsets of $\{1, \ldots, m\}$ such that a set in one family does not properly contain a set in another. We also generalize the LYM inequality to the case and give another interesting inequality.

THEOREM. *Let* $t \geqslant 2, m \geqslant 2$. *Let* $\mathcal{A}_1, \ldots, \mathcal{A}_t$ *be* $t$ *sets of subsets of* $\{1, \ldots, m\}$ *such that*

$$A_i \in \mathcal{A}_i, A_j \in \mathcal{A}_j, i \neq j \Rightarrow A_i \text{ does not properly contain } A_j.$$

*Let* $\beta_{ij}$ *be the number of sets of cardinality* $i$ *in* $\mathcal{A}_j$ *and let* $\gamma_i = \beta_{i1} + \cdots + \beta_{it}$. *Then*

(i) $\sum_{i=0}^{m} \gamma_i / \binom{m}{i} \leqslant \max(t, m+1)$,

(ii) $|\mathcal{A}_1| + \cdots + |\mathcal{A}_t| \leqslant \max(2^m, t\binom{m}{[m/2]})$,

(iii) $|\mathcal{A}_1| + \cdots + |\mathcal{A}_t| \leqslant 2^m + st - 2^m\binom{m}{[m/2]}^{-1}s$, *where* $s$ *is the number of subsets of* $\{1, \ldots, m\}$ *which occur in more than one* $\mathcal{A}_i$.





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REMARKS. 1. All the bounds are best possible. If $t \geqslant m + 1$ then we can take each $\mathcal{Q}_i$ to consist of all subsets of some given size $i$. If $m + 1 > t$ we can have $\mathcal{Q}_1$ consisting of all subsets of $\{1, \ldots, m\}$, and $|\mathcal{Q}_2| = \cdots = |\mathcal{Q}_t| = 0$. Then $\gamma_i = \binom{m}{i}$.

2. (ii) follows from (iii) since, by Sperner's theorem

$$s \leqslant \binom{m}{\left[\frac{m}{2}\right]}.$$

However, we give another derivation of (ii) as well.

PROOF OF (i). Let

$$x = \sum_{i=0}^{m} \gamma_i \frac{m!}{\binom{m}{i}}.$$

Then

$$(1) \qquad x = \sum_{j=1}^{t} \sum_{a \in \mathcal{Q}_j} (\text{the number of maximal chains through } a),$$

since the number of maximal chains through a set $a$ of cardinality $i$ is $i(i-1)\cdots 1.\ (m-i)(m-i-1)\cdots 1 = m!/\binom{m}{i}$. The number of maximal chains is $m!$. Therefore

$$x \leqslant m! \text{ (the number of times a maximal chain can be counted in (1))}$$

$$= m! \begin{cases} t \text{ if the maximal chain meets only one } a \in \mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_t, \\ m+1 \text{ if the maximal chain meets more than one} \\ a \in \mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_t, \text{ for then each such } a \text{ must be in the same } \mathcal{Q}_i. \end{cases}$$

$$\leqslant m! \max(t, m+1).$$

(i) now follows by dividing by $m!$.

PROOF OF (ii). We use the following result of Kleitman and Greene [2] (which we have specialized for our purpose). Let $\lambda$ be a real valued function defined on the subsets of $\{1, \ldots, m\}$. Let $\mathcal{C}$ be the set of all maximal chains. If $B$ is any set of subsets of $\{1, \ldots, m\}$, then

$$\sum_{b \in B} \frac{\lambda_b}{\binom{m}{|b|}} \leqslant \max_{C \in \mathcal{C}} \sum_{b \in C \cap B} \lambda_b.$$

To apply this result, let $\lambda_b = \binom{m}{|b|} \times$ (the number of $\mathcal{Q}_i$'s containing $b$). Then

$$|\mathcal{Q}_1| + \cdots + |\mathcal{Q}_t| = \sum_{a \in \cup \mathcal{Q}_i} \frac{\lambda_a}{\binom{m}{|a|}} \leqslant \max_{C \in \mathcal{C}} \sum_{a \in C \cap (\cup \mathcal{Q}_i)} \lambda_a.$$

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If one of the $a$'s in the chain $C$ occurs in more than one $\mathcal{C}_i$ then $\lambda_a \leqslant t$ and there is only one element in $C \cap (\bigcup \mathcal{C}_i)$ so we get

$$|\mathcal{C}_1| + \cdots + |\mathcal{C}_t| \leqslant t \max_{a \in \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_t} \binom{m}{|a|} \leqslant t \binom{m}{\left[\frac{m}{2}\right]}.$$

If each $a$ in $C$ occurs in at most one $\mathcal{C}_i$ then

$$|\mathcal{C}_1| + \cdots + |\mathcal{C}_t| \leqslant \sum_{i=0}^{m} \binom{m}{i} = 2^m.$$

This proves (ii).

PROOF OF (iii). To prove (iii), first we prove the following lemma which is of some interest in its own right.

LEMMA. *Let $T$ be a family of subsets of $\{1, \ldots, m\}$. Then the probability that a given set is in $T$ is not greater than the probability that the given set is in a maximal chain which meets $T$.* ("meets" here means that the maximal chain contains a member of $T$.)

PROOF OF THE LEMMA. For $0 \leqslant i \leqslant m$, let $t_i$ be the number of sets of $T$ of cardinality $i$. Let

$$\mu = \max_{0 < i < m} \frac{t_i}{\binom{m}{i}} = \frac{t_{i_0}}{\binom{m}{i_0}}.$$

Then for $0 \leqslant i \leqslant m$, $t_i \leqslant \mu\binom{m}{i}$ so $\sum_{i=0}^{m} t_i \leqslant \mu \sum_{i=0}^{m} \binom{m}{i} = \mu 2^m$. Therefore

$$\frac{|T|}{2^m} = \frac{\sum_{i=0}^{m} t_i}{2^m} \leqslant \mu = \frac{t_{i_0}}{\binom{m}{i_0}} = \frac{t_{i_0}}{m!}(i_0)!\,(m - i_0)!$$

$$= \frac{1}{m!}\left(\begin{array}{c}\text{the number of maximal chains which meet}\\ \text{a member of } T \text{ of size } i_0.\end{array}\right)$$

$$\leqslant \frac{\text{the number of maximal chains which meet } T}{\text{the total number of maximal chains}}.$$

The lemma now follows.

Now to return to the proof of (iii). Let $S = \{a\colon a$ lies in more than one of $\mathcal{C}_1, \ldots, \mathcal{C}_t\}$ and let $T = \{a\colon a \in (\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_t) \setminus S\}$. Then $S$ is a Sperner family, $|S| = s$ and $S$ and $T$ are incomparable.

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The probability that a set of size $i$ is in a given chain is

$$\frac{1}{\binom{m}{i}} \geqslant \frac{1}{\binom{m}{\left[\frac{m}{2}\right]}}.$$

Therefore, the probability that a given chain meets $S$ is at least

$$\sum_{a \in S} \frac{1}{\binom{m}{\left[\frac{m}{2}\right]}} = \frac{s}{\binom{m}{\left[\frac{m}{2}\right]}}.$$

Since $S$ and $T$ are incomparable, it follows from the lemma that the probability that a given maximal chain meets $S$ or $T$ is at least

$$\frac{|S|}{\binom{m}{\left[\frac{m}{2}\right]}} + \frac{|T|}{2^m},$$

and so it follows that

$$\frac{|S|}{\binom{m}{\left[\frac{m}{2}\right]}} + \frac{|T|}{2^m} \leqslant 1,$$

and therefore

$$|T| \leqslant 2^m - \frac{2^m s}{\binom{m}{\left[\frac{m}{2}\right]}}.$$

Now we have that

$$|\mathcal{C}_1| + \cdots + |\mathcal{C}_t| \leqslant |T| + t|S| \leqslant 2^m + st - \frac{2^m s}{\binom{m}{\left[\frac{m}{2}\right]}}.$$

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