Proceedings of the Edinburgh Mathematical Society (1983) **26**, 25–28 ©

ON QUASINORMAL SUBGROUPS OF CERTAIN FINITELY GENERATED GROUPS

by JOHN C. LENNOX

(Received 20th March 1981)

A subgroup Q of a group G is called *quasinormal* in G if Q permutes with every subgroup of G. Of course a quasinormal subgroup Q of a group G may be very far from normal. In fact, examples of Iwasawa show (for a convenient reference see [**8**]) that we may have Q core-free and the normal closure Q^G of Q in G equal to G so that Q is not even subnormal in G. We note also that the core of Q in G, Q_G, is of infinite index in Q^G in this example. If G is finitely generated then any quasinormal subgroup Q is subnormal in G [**8**] and although Q is not necessarily normal in G we have that $|Q^G:Q|$ is finite and Q^G/Q_G is a nilpotent group of finite exponent [**5**].

It is our object here to show that if, in addition, G is soluble by finite then Q^G/Q_G is in fact finite. Thus we have the following theorem.

Theorem A. *Suppose that G is a finitely generated soluble by finite group and that Q is a core-free quasinormal subgroup of G. Then the normal closure of Q in G is finite.*

This means that a quasinormal subgroup of a finitely generated soluble by finite group G is sandwiched between two normal subgroups of G each of finite index in the other. We leave the obvious questions about finitely generated groups open.

We note that in [**1**] it is shown that if G is a soluble minimax group (not necessarily finitely generated) and Q is a core-free quasinormal subgroup of G closed in the profinite topology on G then Q^G is finite. The closure hypothesis on Q given here cannot be dropped as a consideration of the examples of Iwasawa mentioned above reveals.

In [**6**] Maier and Schmid prove that a core-free quasinormal subgroup of a finite group is contained in the hypercentre of that group. As a consequence of Theorem A we extend their result to the following theorem.

Theorem B. *Suppose that G is a finitely generated soluble by finite group which is residually finite and that Q is a core-free quasinormal subgroup of G. Then $Q \leqq \zeta_n(G)$ for some positive integer n.*

Here $\zeta_n(G)$ denotes the nth term of the upper central series of the group G.

Thus, in particular, core-free quasinormal subgroups of polycyclic by finite groups (and, indeed, of finitely generated abelian by polycyclic by finite groups) are hypercentral since such groups are residually finite (see [**4**] and [**7**]).

https://doi.org/10.1017/S0013091500028042 Published online by Cambridge University Press

Again, we leave the obvious question for finitely generated soluble groups open. In another direction Busetto has recently shown in [2] that a torsion locally cyclic core-free quasinormal subgroup is always hypercentral.

Proof of Theorem A. Suppose that G and Q are as in the statement of the theorem.

Suppose first of all that there exists no infinite cyclic subgroup of G which intersects Q trivially.

Thus, if $x \in G$, some power of x lies in Q and hence G/Q^G is periodic. Now finitely generated periodic soluble by finite groups are finite and hence G/Q^G is finite.

Application of a result of Stonehewer [8, Theorem B] yields $|Q^G:Q|$ is finite since G is finitely generated. Hence $|G:Q|$ is finite. But Q is core-free and hence G is finite.

Hence we may assume that some infinite cyclic subgroup of G intersects Q trivially. Then by a result of Gross [3] we have $Q \lhd Q^G$ and Q is abelian.

Moreover, by Stonehewer [8] Q^G/Q is finite and therefore $(Q^G)^n \leqq Q$ for some positive integer n. Since Q is core-free it follows that Q^G and hence Q has finite exponent.

Let N be the normaliser of Q in G. Suppose that x is an element of infinite order not in N. Then by [8, Lemma 2.1] we have that $\langle x \rangle \cap Q \neq 1$. Hence x has finite order since Q has finite exponent, a contradiction. Hence $x \in G \setminus N$ implies that x has finite order.

We now need a proposition which we state in rather more generality than is necessary since it may be of some independent interest.

Proposition. *Suppose that G is a finitely generated soluble by finite group and that N is a subgroup of G such that each element $x \in G$ has some power in N. Then N is of finite index in G.*

Remark. James Wiegold has pointed out that if A is any perfect group and G is the wreath product $A \operatorname{wr} Z$, where Z is an infinite cyclic group, then the only subgroups of G which contain powers of all elements of G are of finite index. This example demonstrates that the property described in the proposition is not a property which characterises finitely generated soluble by finite groups among finitely generated groups.

Proof of the proposition. If G is finite there is nothing to prove so we may assume by induction on the derived length of a soluble subgroup of finite index in G that $|G:NA|$ is finite where A is an abelian normal subgroup of G. We may therefore assume that $G = NA$ and since in this case $N \cap A \lhd G$ we may take $N \cap A = 1$. Since G is finitely generated we have a finite set $a_1 n_1, \ldots, a_s n_s$ of generators for G and a second induction allows us to assume that $G = BN$ where $B = \langle a \rangle^N$ for some $a \in A$.

Let $n \in N$ and consider $g = an$. Then $g^r \in N$ for some r so that

$$aa^{n^{-1}} \ldots a^{n^{-r+1}} n^r \in N.$$

Hence

$$aa^{n^{-1}} \ldots a^{n^{-r+1}} \in N \cap A = 1.$$

It follows that

$$[a, n^r] = 1$$

so that $n^r \in C_N(a)$, the centraliser in N of a.

By the induction hypothesis we have that $|N : C_N(a)|$ is finite and hence $[N^m, a] = 1$ for some m. We then obtain

$$[N^m, B] = 1.$$

But B is then a module for N/N^n which is a finite group since N is finitely generated and soluble by finite. Furthermore a has finite order since $a^s \in A \cap N = 1$ for some s. Hence B is finite, since $B/C_B(N)$ is finite and $C_B(N)$ is finitely N-generated and central and so is finite.

It now follows that $|G : N|$ is finite, as required.

Applying the proposition in the situation of Theorem A we have that $|G : N|$ is finite, where $N = N_G(Q)$. Hence Q has at most a finite number of conjugates under G, say $Q, Q^{g_1}, \ldots, Q^{g_r}$. We also have that $|Q^G : Q^{g_i}|$ is finite for $i = 1, 2, \ldots, r$ so that $|QQ^{g_i} : Q^{g_i}|$ is finite. Therefore $|Q : Q \cap Q^{g_i}|$ is finite so that $1 = Q_G = Q \cap Q^{g_1} \cap \ldots \cap Q^{g_r}$ is of finite index in Q. Therefore Q is finite and so Q^G is finite since $|Q^G : Q|$ is finite. This completes the proof of Theorem A.

Deduction of Theorem B from Theorem A. By Theorem A we have Q^G finite and since G is residually finite there exists a normal subgroup N of finite index in G with $N \cap Q^G = 1$.

By Maier–Schmid's result applied to G/N we obtain that the repeated commutator $[Q^G, {}_nG]$ is contained in QN for some n.

Hence

$$[Q^G, {}_nG] \leqq (QN) \cap Q^G = Q(N \cap Q^G) = Q.$$

But $[Q^G, {}_nG] \lhd G$ and so

$$[Q^G, {}_nG] = 1$$

as Q is core-free. Hence $Q \leqq \zeta_n(G)$ as required.

REFERENCES

1. D. C. Brewster and J. C. Lennox, Subnormality and quasinormality in soluble minimax groups, *J. Algebra* **48** (1977), 368–381.

2. G. Busetto, Lattice-theoretical characterizations of hyperabelian groups and of supersoluble groups, To appear.

3. F. Gross, Subnormal, core-free, quasinormal subgroups are solvable, *Bull. London Math. Soc.* **7** (1975), 93–95.

4. A. V. Jategaonkar, Integral group rings of polycyclic groups, *J. Pure and Appl. Algebra* **4** (1974), 337–343.

5. J. C. Lennox, A note on quasinormal subgroups of finitely generated groups, *J. London Math. Soc.*, **24** (1981), 127–128.

6. R. Maier and P. Schmid, The embedding of quasinormal subgroups in finite groups, *Math. Z.* **131** (1973), 269–272.

7. J. E. Roseblade, Applications of the Artin–Rees lemma to group rings, *Symp. Mathematica* **17** (1976), 471–478.

8. S. E. Stonehewer, Permutable subgroups of infinite groups, *Math. Z.* **125** (1972), 1–16.

University College
Cardiff, Wales