Appendix J
RPA solution of the pairing Hamiltonian

In this appendix we shall derive in detail the properties of the collective modes associated with the pairing Hamiltonian

$$H = H_{\mathrm{sp}} + H_{\mathrm{p}},$$

where

$$H_{\mathrm{sp}} = \sum_{\nu} (\epsilon_{\nu} - \lambda) a_{\nu}^{\dagger} a_{\nu}$$

and

$$H_{\mathrm{p}} = -G \sum_{\nu,\nu'} a_{\nu}^{\dagger} a_{\bar{\nu}}^{\dagger}\, a_{\bar{\nu}'} a_{\nu'}.$$

This Hamiltonian becomes, in the quasiparticle basis (Högaasen–Feldman (1961), Bes and Broglia (1966)),

$$N = \sum_i E_i N_i - \frac{1}{4} G \left(\sum_i \sqrt{\Omega_i}\, f_i \left(\Gamma_i^{\dagger} + \Gamma_i \right) \right)^2 + \frac{1}{4} G \left(\sum_i \sqrt{\Omega_i} \left(\Gamma_i^{\dagger} - \Gamma_i \right) \right)^2 \tag{J.1}$$

neglecting terms of the order of 1 and of order $\sqrt{\Omega_i}$, where

$$\Omega_i = \frac{2 j_i + 1}{2},$$

as well as terms proportional to the quasiparticle number operator

$$N_i = \sum_m \alpha_{im}^{\dagger} \alpha_{im}. \tag{J.2}$$

Consistent with this approximation we shall also neglect the Pauli principle among quasiparticles as expressed in the conmutation relation (see equations (A.72) and (I.50)),

$$\left[\Gamma_i, \Gamma_j^{\dagger} \right] = \delta(i, j) \left(1 - \frac{N_i}{\Omega_i} \right), \tag{J.3}$$

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i.e. assume that

$$\left[\Gamma_i, \Gamma_j^\dagger\right] = \delta(i, j). \tag{J.4}$$

This is a good approximation to the extent that the number of quasiparticle excitations is much smaller than Ω_i, the pair degeneracy of the system.

In the above equations the definitions and relations

$$E_i = \sqrt{(\epsilon_i - \lambda)^2 + \Delta^2}, \tag{J.5}$$

$$\Gamma_j^\dagger = \frac{1}{\sqrt{\Omega_j}} \sum_{m>0} (-1)^{j-m} \alpha_{j,m}^\dagger \alpha_{j,-m}^\dagger, \tag{J.6}$$

$$f_i = U_i^2 - V_i^2 \tag{J.7}$$

and

$$\left[N_i, \Gamma_j^\dagger\right] = \delta(i, j) 2\Gamma_j^\dagger \tag{J.8}$$

have been used.

In what follows we shall also use the phonon-creation operator

$$\Gamma_n^\dagger = \sum_i a_{ni} \Gamma_i^\dagger + \sum_i b_{ni} \Gamma_i. \tag{J.9}$$

We can separate the residual interaction into two parts, one with matrix elements which are odd with respect to the Fermi energy,

$$H_{\text{p}}' = -\frac{1}{4} G \left(\sum_i \sqrt{\Omega_i} f_i \left(\Gamma_i^\dagger + \Gamma_i \right) \right)^2,$$

which give rise to pairing vibrations, and one which is even,

$$H_{\text{p}}'' = \frac{1}{4} G \left(\sum_i \sqrt{\Omega_i} \left(\Gamma_i^\dagger - \Gamma_i \right) \right)^2,$$

and which is connected with the Anderson–Goldstone–Nambu mode of the system (see Chapther 4). We are first going to treat both parts separately and then later linearize the whole Hamiltonian.

J.1 Diagonalization of the $H_0 + H_{\text{p}}'$ Hamiltonian (odd solution)

We shall diagonalize the Hamiltonian

$$H = \sum_i E_i N_i - \frac{1}{4} G \left(\sum_i \sqrt{\Omega_i} f_i \left(\Gamma_i^\dagger + \Gamma_i \right) \right)^2 = H_0 + H_{\text{p}}', \tag{J.10}$$

in the harmonic approximation, thus requiring that (see equation (A.68))

$$\left[H, \Gamma_n^\dagger\right] = W_n \Gamma_n^\dagger.$$

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Let us first calculate the commutation with H_0,

$$\begin{aligned}\left[H_0, \Gamma_n^\dagger\right] &= \left[\sum_i E_i N_i\ ,\ \left(\sum_j a_{nj}\Gamma_j^\dagger + \sum_j b_{nj}\Gamma_j\right)\right] \\ &= \sum_{i,j} E_i\ a_{nj}\ \delta(i,j)\ 2\Gamma_j^\dagger - \sum_{i,j} E_i\ b_{nj}\ \delta(i,j)\ 2\Gamma_j,\end{aligned}$$

leading to

$$\left[H_0, \Gamma_n^\dagger\right] = 2\sum_i E_i a_{ni}\Gamma_i^\dagger - 2\sum_i E_i b_{ni}\Gamma_i. \qquad \text{(J.11)}$$

Making use of the Hamiltonian

$$\begin{aligned}H_\text{p}' &= -\frac{1}{4}G\left(\sum_i \sqrt{\Omega}_i f_i(\Gamma_i^\dagger + \Gamma_i)\right)^2 \\ &= -\frac{1}{4}G\left(\sum_i \sqrt{\Omega}_i f_i(\Gamma_i^\dagger + \Gamma_i)\right)\left(\sum_j \sqrt{\Omega}_j f_j(\Gamma_j^\dagger + \Gamma_j)\right),\end{aligned}$$

we calculate the commutation relation

$$\begin{aligned}\left[H_\text{p}', \Gamma_n^\dagger\right] &= -\frac{1}{4}G\left[\left(\sum_i \sqrt{\Omega}_i f_i(\Gamma_i^\dagger + \Gamma_i)\right)\right. \\ &\qquad \times \left.\left(\sum_j \sqrt{\Omega}_j f_j(\Gamma_j^\dagger + \Gamma_j)\right), \left(\sum_k a_{nk}\Gamma_k^\dagger + \sum_k b_{nk}\Gamma_k\right)\right] \\ &= -\frac{1}{4}G\left(\sum_i \sqrt{\Omega}_i f_i(\Gamma_i^\dagger + \Gamma_i)\right) \\ &\quad \times \left[\left(\sum_j \sqrt{\Omega}_j f_j(\Gamma_j^\dagger + \Gamma_j)\right), \left(\sum_k a_{nk}\Gamma_k^\dagger + \sum_k b_{nk}\Gamma_k\right)\right] \\ &\quad -\frac{1}{4}G\left[\left(\sum_i \sqrt{\Omega}_i f_i(\Gamma_i^\dagger + \Gamma_i)\right), \left(\sum_k a_{nk}\Gamma_k^\dagger + \sum_k b_{nk}\Gamma_k\right)\right] \\ &\quad \times \left(\sum_j \sqrt{\Omega}_j f_j(\Gamma_j^\dagger + \Gamma_j)\right) \\ &= -\frac{1}{4}G\left(\sum_i \sqrt{\Omega}_i f_i(\Gamma_i^\dagger + \Gamma_i)\right) \\ &\quad \times \sum_j \sqrt{\Omega}_j f_j \left\{\sum_k a_{nk}\left[\Gamma_j, \Gamma_k^\dagger\right] + \sum_k b_{nk}\left[\Gamma_j^\dagger, \Gamma_k\right]\right\} \\ &\quad -\frac{1}{4}G\sum_i \sqrt{\Omega}_i f_i \left\{\sum_k a_{nk}\left[\Gamma_i, \Gamma_k^\dagger\right] + \sum_k b_{nk}\left[\Gamma_i^\dagger, \Gamma_k\right]\right\}\left(\sum_j \sqrt{\Omega}_j f_j(\Gamma_j^\dagger + \Gamma_j)\right)\end{aligned}$$

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$$= -\frac{1}{4}G\left(\sum_i \sqrt{\Omega}_i f_i(\Gamma_i^\dagger + \Gamma_i)\right)\sum_j \sqrt{\Omega}_j f_j(a_{nj} - b_{nj})$$
$$-\frac{1}{4}G\sum_i \sqrt{\Omega}_i f_i(a_{ni} - b_{ni})\left(\sum_j \sqrt{\Omega}_j f_j(\Gamma_j^\dagger + \Gamma_j)\right).$$

Consequently,

$$\left[H_\text{p}', \Gamma_n^\dagger\right] = \frac{1}{2}G\sum_j \sqrt{\Omega}_j f_j(b_{nj} - a_{nj})\left(\sum_i \sqrt{\Omega}_i f_i(\Gamma_i^\dagger + \Gamma_i)\right). \tag{J.12}$$

From equations (J.11) and (J.12) we find

$$\begin{aligned}
\left[H, \Gamma_n^\dagger\right] &= 2\sum_i E_i a_{ni}\Gamma_i^\dagger - 2\sum_i E_i b_{ni}\Gamma_i + \frac{1}{2}G\sum_j \sqrt{\Omega}_j f_j(b_{nj} - a_{nj}) \\
&\quad \times\left(\sum_i \sqrt{\Omega}_i f_i(\Gamma_i^\dagger + \Gamma_i)\right) \\
&= \sum_i \left\{2E_i a_{ni} + \frac{1}{2}G\left(\sum_j \sqrt{\Omega}_j f_j(b_{nj} - a_{nj})\right) f_i\sqrt{\Omega}_i\right\}\Gamma_i^\dagger \\
&\quad + \sum_i \left\{-2E_i b_{ni} + \frac{1}{2}G\left(\sum_j \sqrt{\Omega}_j f_j(b_{nj} - a_{nj})\right) f_i\sqrt{\Omega}_i\right\}\Gamma_i^\dagger \\
&= W_n\sum_i a_{ni}\Gamma_i^\dagger + W_n\sum_i b_{ni}\Gamma_i.
\end{aligned}$$

Thus,

$$2E_i a_{ni} + \frac{G}{2}\left(\sum_j \sqrt{\Omega}_j f_j(b_{nj} - a_{nj})\right) f_i\sqrt{\Omega}_i = W_n a_{ni},$$
$$-2E_i b_{ni} + \frac{G}{2}\left(\sum_j \sqrt{\Omega}_j f_j(b_{nj} - a_{nj})\right) f_i\sqrt{\Omega}_i = W_n b_{ni}.$$

Defining

$$\Lambda_n = \frac{G}{2}\left(\sum_j \sqrt{\Omega}_j f_j(a_{nj} - b_{nj})\right), \tag{J.13}$$

one obtains

$$\begin{aligned}
2E_i a_{ni} - \Lambda_n f_i\sqrt{\Omega}_i &= W_n a_{ni}, \\
-2E_i b_{ni} - \Lambda_n f_i\sqrt{\Omega}_i &= W_n b_{ni},
\end{aligned}$$

which lead to

$$a_{ni} = \frac{\Lambda_n f_i\sqrt{\Omega}_i}{2E_i - W_n}, \qquad b_{ni} = -\frac{\Lambda_n f_i\sqrt{\Omega}_i}{2E_i + W_n}. \tag{J.14}$$

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Substituting equations (J.14) in (J.13) one can write

$$\Lambda_n = \frac{G}{2}\sum_j \sqrt{\Omega}_j f_j \left(\frac{\Lambda_n f_j \sqrt{\Omega}_j}{2E_j - W_n} + \frac{\Lambda_n f_j \sqrt{\Omega}_j}{2E_j + W_n}\right)$$
$$= G\sum_j \Omega_j f_j^2 \left(\frac{2E_j}{4E_j^2 - W_n^2}\right)\Lambda_n = G\sum_j \frac{2E_j\Omega_j f_j^2}{4E_j^2 - W_n^2}\Lambda_n,$$

leading to

$$\frac{1}{G} = \sum_j \frac{2E_j\Omega_j f_j^2}{4E_j^2 - W_n^2}. \tag{J.15}$$

The normalization condition

$$\left[\Gamma_n, \Gamma_n^\dagger\right] = \left[\left(\sum_i a_{ni}\Gamma_i + \sum_i b_{ni}\Gamma_i^\dagger\right), \left(\sum_j a_{mj}\Gamma_j^\dagger + \sum_j b_{mj}\Gamma_j\right)\right]$$

gives the relation

$$\sum_i a_{ni}a_{mi} - \sum_i b_{ni}b_{mi} = \delta(n, m). \tag{J.16}$$

Consequently

$$\sum_i (a_{ni}^2 - b_{ni}^2) = 1.$$

Inserting in this equation the amplitudes defined in equation (J.14) one obtains

$$\Lambda_n^2 \sum_i \left\{\frac{f_i^2\Omega_i}{(2E_i - W_n)^2} - \frac{f_i^2\Omega_i}{(2E_i + W_n)^2}\right\} = 1,$$

$$\Lambda_n^2 \sum_i \frac{4E_i^2 + 4E_iW_n + W_n^2 - 4E_i^2 + 4E_iW_n - W_n^2}{(2E_i - W_n)^2(2E_i + W_n)^2} f_i^2\Omega_i = 1,$$

leading to

$$\Lambda_n^2 \sum_i \frac{f_i^2\Omega_i 8E_iW_n}{(4E_i^2 - W_n^2)^2} = 1.$$

Thus

$$\Lambda_n = \frac{1}{2}\left[\sum_i \frac{f_i^2\Omega_i 2E_iW_n}{(4E_i^2 - W_n^2)^2}\right]^{-1/2}. \tag{J.17}$$

J.2 Diagonalization of the $H_0 + H_{\rm p}''$ Hamiltonian (even solution)

Let us now consider the Hamiltonian

$$H = H_0 + H_{\rm p}'' = \sum_i E_iN_i + \frac{1}{4}G\left(\sum_i \sqrt{\Omega}_i(\Gamma_i^\dagger - \Gamma_i)\right)^2,$$

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where

$$H''_{\mathrm{p}} = \frac{1}{4}G\left(\sum_i \sqrt{\Omega}_i(\Gamma_i^\dagger - \Gamma_i)\right)^2$$
$$= \frac{1}{4}G\left(\sum_i \sqrt{\Omega}_i(\Gamma_i^\dagger - \Gamma_i)\right)\left(\sum_j \sqrt{\Omega}_j(\Gamma_j^\dagger - \Gamma_j)\right).$$

We start by calculating the commutation relation

$$\begin{aligned}
\left[H''_{\mathrm{p}}, \Gamma_n^\dagger\right] &= \frac{1}{4}G\left[\left(\sum_i \sqrt{\Omega}_i(\Gamma_i^\dagger - \Gamma_i)\right)\left(\sum_j \sqrt{\Omega}_j(\Gamma_j^\dagger - \Gamma_j)\right), \Gamma_n^\dagger\right] \\
&= \frac{1}{4}G\left(\sum_i \sqrt{\Omega}_i(\Gamma_i^\dagger - \Gamma_i)\right)\left[\left(\sum_j \sqrt{\Omega}_j(\Gamma_j^\dagger - \Gamma_j)\right), \Gamma_n^\dagger\right] \\
&\quad + \frac{1}{4}G\left[\left(\sum_i \sqrt{\Omega}_i(\Gamma_i^\dagger - \Gamma_i)\right), \Gamma_n^\dagger\right]\left(\sum_j \sqrt{\Omega}_j(\Gamma_j^\dagger - \Gamma_j)\right) \\
&= \frac{1}{4}G\left(\sum_i \sqrt{\Omega}_i(\Gamma_i^\dagger - \Gamma_i)\right) \\
&\quad \times \left[\left(\sum_j \sqrt{\Omega}_j(\Gamma_j^\dagger - \Gamma_j)\right), \left(\sum_k a_{nk}\Gamma_k^\dagger + \sum_k b_{nk}\Gamma_k\right)\right] \\
&\qquad + \frac{1}{4}G\left[\left(\sum_i \sqrt{\Omega}_i(\Gamma_i^\dagger - \Gamma_i)\right), \left(\sum_k a_{nk}\Gamma_k^\dagger + \sum_k b_{nk}\Gamma_k\right)\right] \\
&\quad \times \left(\sum_j \sqrt{\Omega}_j(\Gamma_j^\dagger - \Gamma_j)\right) \\
&= \frac{1}{4}G\left(\sum_i \sqrt{\Omega}_i(\Gamma_i^\dagger - \Gamma_i)\right) \\
&\quad \times \sum_j \sqrt{\Omega}_j\left\{-\sum_k b_{n,k}\delta(j,k) - \sum_k a_{n,k}\delta(j,k)\right\} \\
&\qquad + \frac{1}{4}G\sum_i \sqrt{\Omega}_i\left\{-\sum_k \delta(i,k)b_{n,k} - \sum_k a_{n,k}\delta(i,k)\right\} \\
&\quad \times \left(\sum_j \sqrt{\Omega}_j(\Gamma_j^\dagger - \Gamma_j)\right) \\
&= -\frac{G}{2}\left(\sum_i \sqrt{\Omega}_i(a_{n,i} + b_{n,i})\right)\left(\sum_j \sqrt{\Omega}_j(\Gamma_j^\dagger - \Gamma_j)\right).
\end{aligned}$$

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That is

$$\left[H_{\rm p}'', \Gamma_n^\dagger\right] = -\frac{G}{2}\left(\sum_i \sqrt{\Omega}_i(a_{n,i} + b_{n,i})\right)\left(\sum_j \sqrt{\Omega}_j(\Gamma_j^\dagger - \Gamma_j)\right). \tag{J.18}$$

Making use of this relation and equation (J.11), one can write

$$\begin{aligned}
\left[H, \Gamma_n^\dagger\right] &= 2\sum_i E_i a_{n,i}\Gamma_i^\dagger - 2\sum_i E_i b_{n,i}\Gamma_i \\
&\quad - \frac{1}{2}G\sum_j \sqrt{\Omega}_j(a_{n,j} + b_{n,j})\sum_i \sqrt{\Omega}_i\Gamma_i^\dagger \\
&\quad + \frac{1}{2}G\sum_j \sqrt{\Omega}_j(a_{n,j} + b_{n,j})\sum_i \sqrt{\Omega}_i\Gamma_i \\
&= \sum_i\left\{2E_i a_{n,i} - \frac{1}{2}G\sum_j \sqrt{\Omega}_j(a_{n,j} + b_{n,j})\right\}\sqrt{\Omega}_i\Gamma_i^\dagger \\
&\quad + \sum_i\left\{-2E_i b_{n,i} + \frac{1}{2}G\sum_j \sqrt{\Omega}_j(a_{n,j} + b_{n,j})\right\}\sqrt{\Omega}_i\Gamma_i \\
&= W_n\sum_i a_{n,i}\Gamma_i^\dagger + W_n\sum_i b_{n,i}\Gamma_i.
\end{aligned}$$

This relation implies that

$$\begin{aligned}
2E_i a_{n,i} - \left(\frac{1}{2}G\sum_j \sqrt{\Omega}_j(a_{n,j} + b_{n,j})\right)\sqrt{\Omega}_i &= W_n a_{n,i}, \\
-2E_i b_{n,i} + \left(\frac{1}{2}G\sum_j \sqrt{\Omega}_j(a_{n,j} + b_{n,j})\right)\sqrt{\Omega}_i &= W_n b_{n,i}.
\end{aligned}$$

Defining the quantity

$$\Lambda_n = \frac{1}{2}G\sum_j \sqrt{\Omega}_j(a_{n,j} + b_{n,j}), \tag{J.19}$$

the above equations can be written as

$$\begin{aligned}
2E_i a_{n,i} - \Lambda_n\sqrt{\Omega_i} &= W_n a_{n,i}, \\
-2E_i b_{n,i} + \Lambda_n\sqrt{\Omega_i} &= W_n b_{n,i},
\end{aligned}$$

leading to

$$\begin{aligned}
(2E_i - W_N)a_{n,i} &= \Lambda_n\sqrt{\Omega_i}, \\
(2E_i + W_N)b_{n,i} &= \Lambda_n\sqrt{\Omega_i}.
\end{aligned}$$

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The colletive phonon forwards-going and backwards-going amplitudes (see Fig. 8.11) are thus

$$a_{ni} = \frac{\Lambda_n \sqrt{\Omega_i}}{2E_i - W_n}, \qquad b_{ni} = \frac{\Lambda_n \sqrt{\Omega_i}}{2E_i + W_n}. \tag{J.20}$$

Replacing these amplitudes in equation (J.19) leads to the relation

$$\Lambda_n = \frac{1}{2} G \sum_i \sqrt{\Omega_i} \left\{ \frac{\Lambda_n \sqrt{\Omega_i}}{2E_i - W_n} + \frac{\Lambda_n \sqrt{\Omega_i}}{2E_i + W_n} \right\},$$
$$\Lambda_n = \frac{1}{2} G \Lambda_n \sum_i \frac{2E_i + W_n + 2E_i - W_n}{4E_i^2 - W_n^2} \, \Omega_i,$$

and thus to the dispersion relation

$$\frac{1}{G} = \sum_i \frac{2E_i \Omega_i}{4E_i^2 - W_n^2}. \tag{J.21}$$

It can be seen that this equation has, as the lowest root, $W_1 = 0$. In fact, in this case the above expression leads to

$$\frac{2}{G} = \sum_i \frac{\Omega_i}{E_i},$$

which is the BCS gap equation.

From the normalization condition,

$$\begin{aligned} 1 = \sum_i (a_{n,i}^2 - b_{n,i}^2) &= \Lambda_n^2 \sum_i \left\{ \frac{\Omega_i}{(2E_i - W_n)^2} - \frac{\Omega_i}{(2E_i + W_n)^2} \right\} \\ &= \Lambda_n^2 \sum_i \frac{4E_i^2 + 4E_i W_n + W_n^2 - 4E_i^2 + 4E_i W_n - W_n^2}{(4E_i^2 - W_n^2)^2} \Omega_i \\ &= \Lambda_n^2 \sum_i \frac{8E_i W_n \Omega_i}{(4E_i^2 - W_n^2)^2}, \end{aligned}$$

one obtains

$$\Lambda_n = \frac{1}{2} \left[\sum_i \frac{2E_i W_n \Omega_i}{(4E_i^2 - W_n^2)^2} \right]^{-1/2}. \tag{J.22}$$

J.3 Diagonalization of the full Hamiltonian $H = H_0 + H'_p + H''_p$

We consider now the complete Hamiltonian

$$H = H_0 + H'_p + H''_p = \sum_i E_i N_i - \frac{1}{4} G \left(\sum_i \sqrt{\Omega}_i f_i (\Gamma_i^\dagger + \Gamma_i) \right)^2 + \frac{1}{4} G \left(\sum_i \sqrt{\Omega}_i (\Gamma_i^\dagger - \Gamma_i) \right)^2,$$

and linearize it, i.e. impose

$$[H, \Gamma_n^\dagger] = [H_0, \Gamma_n^\dagger] + [H'_p, \Gamma_n^\dagger] + [H''_p, \Gamma_n^\dagger] = W_n \sum_i a_{ni} \Gamma_i^\dagger + W_n \sum_i b_{ni} \Gamma_i.$$

From equations (J.11), (J.12) and (J.18) we get,

$$\begin{aligned}
&\sum_i 2E_i a_{ni} \Gamma_i^\dagger - \sum_i 2E_i b_{ni} \Gamma_i \\
&\quad + \frac{G}{2} \sum_j \sqrt{\Omega}_j f_j (b_{nj} - a_{nj}) \left(\sum_i \sqrt{\Omega}_i f_i (\Gamma_i^\dagger + \Gamma_i) \right) \\
&\quad - \frac{G}{2} \sum_j \sqrt{\Omega}_i (a_{nj} + b_{nj}) \left(\sum_i \sqrt{\Omega}_i (\Gamma_i^\dagger - \Gamma_i) \right) \\
&\quad = W_n \sum_i a_{ni} \Gamma_i^\dagger + W_n \sum_i b_{ni} \Gamma_i.
\end{aligned}$$

That is,

$$\begin{aligned}
&\sum_i \left\{ 2E_i a_{ni} + \frac{1}{2} G \left(\sum_j \sqrt{\Omega}_j f_j (b_{nj} - a_{nj}) \right) \sqrt{\Omega}_i f_i \right. \\
&\left. - \frac{1}{2} G \left(\sum_j \sqrt{\Omega}_j (a_{nj} + b_{nj}) \right) \sqrt{\Omega}_i \right\} \Gamma_i^\dagger \\
&+ \sum_i \left\{ -2E_i a_{ni} + \frac{1}{2} G \left(\sum_j \sqrt{\Omega}_j f_j (b_{nj} - a_{nj}) \right) \sqrt{\Omega}_i f_i \right. \\
&\left. + \frac{1}{2} G \left(\sum_j \sqrt{\Omega}_j (a_{nj} + b_{nj}) \right) \sqrt{\Omega}_i \right\} \Gamma_i \\
= {} & W_n \sum_i a_{ni} \Gamma_i^\dagger + W_n \sum_i b_{ni} \Gamma_i.
\end{aligned}$$

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This relation leads to

$$2E_i a_{ni} + \frac{1}{2}G\left(\sum_j \sqrt{\Omega}_j f_j(b_{nj} - a_{nj})\right)\sqrt{\Omega}_i f_i - \frac{1}{2}G\left(\sum_j \sqrt{\Omega}_j(a_{nj} + b_{nj})\right)\sqrt{\Omega}_i = W_n a_{ni}$$

and

$$-2E_i b_{ni} + \frac{1}{2}G\left(\sum_j \sqrt{\Omega}_j f_j(b_{nj} - a_{nj})\right)\sqrt{\Omega}_i f_i + \frac{1}{2}G\left(\sum_j \sqrt{\Omega}_j(a_{nj} + b_{nj})\right)\sqrt{\Omega}_i = W_n b_{ni}.$$

Defining the quantities

$$\Lambda_{1n} = -\frac{1}{2}G\left(\sum_j \sqrt{\Omega}_j f_j(b_{nj} - a_{nj})\right),$$

and (J.23)

$$\Lambda_{2n} = \frac{1}{2}G\left(\sum_j \sqrt{\Omega}_j(a_{nj} + b_{nj})\right),$$

one can rewrite the above equations as

$$\begin{aligned} 2E_i a_{ni} - \Lambda_{1n}\sqrt{\Omega}_i f_i - \Lambda_{2n}\sqrt{\Omega}_i &= W_n a_{ni}, \\ -2E_i b_{ni} - \Lambda_{1n}\sqrt{\Omega}_i f_i + \Lambda_{2n}\sqrt{\Omega}_i &= W_n b_{ni}, \end{aligned}$$

leading to

$$\begin{aligned} (2E_i - W_n)a_{ni} &= \Lambda_{1n}\sqrt{\Omega}_i f_i + \Lambda_{2n}\sqrt{\Omega}_i, \\ (2E_i + W_n)b_{ni} &= -\Lambda_{1n}\sqrt{\Omega}_i f_i + \Lambda_{2n}\sqrt{\Omega}_i, \end{aligned}$$

from which the RPA amplitudes

$$\begin{aligned} a_{ni} &= \frac{\Lambda_{1n} f_i + \Lambda_{2n}}{(2E_i - W_n)}\sqrt{\Omega}_i, \\ b_{ni} &= \frac{-\Lambda_{1n} f_i + \Lambda_{2n}}{(2E_i + W_n)}\sqrt{\Omega}_i, \end{aligned} \tag{J.24}$$

are determined.

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Replacing the amplitudes in equations (J.23) one gets

$$\Lambda_{1n} = -\tfrac{1}{2}G\sum_i \sqrt{\Omega_i}\, f_i \sqrt{\Omega_i}\left\{\frac{-\Lambda_{1n} f_i + \Lambda_{2n}}{2E_i + W_n} - \frac{\Lambda_{1n} f_i + \Lambda_{2n}}{2E_i - W_n}\right\}$$

$$= -\tfrac{1}{2}G\sum_i \Omega_i f_i \left\{\frac{(-\Lambda_{1n} f_i + \Lambda_{2n})(2E_i - W_n) - (\Lambda_{1n} f_i + \Lambda_{2n})(2E_i + W_n)}{(4E_i^2 - W_n^2)}\right\}$$

$$= -\tfrac{1}{2}G\sum_i \Omega_i f_i$$

$$\times\left\{\frac{-\Lambda_{1n} f_i 2E_i + \Lambda_{1n} f_i W_n + \Lambda_{2n} 2E_i - \Lambda_{2n} W_n - \Lambda_{1n} f_i 2E_i - \Lambda_{1n} f_i W_n - \Lambda_{2n} 2E_i - \Lambda_{2n} W_n}{(4E_i^2 - W_n^2)}\right\}$$

$$= -\tfrac{1}{2}G\sum_i \frac{\Omega_i f_i}{(4E_i^2 - W_n^2)}\{-4\Lambda_{1n} f_i E_i - 2\Lambda_{2n} W_n\}$$

$$= G\sum_i \frac{\Omega_i f_i}{(4E_i^2 - W_n^2)}\{2E_i f_i \Lambda_{1n} + W_n \Lambda_{2n}\}$$

$$= \left(G\sum_i \frac{\Omega_i f_i^2 2E_i}{(4E_i^2 - W_n^2)}\right)\Lambda_{1n} + \left(G\sum_i \frac{\Omega_i f_i W_n}{(4E_i^2 - W_n^2)}\right)\Lambda_{2n},$$

$$\left(\sum_i \frac{\Omega_i f_i^2 2E_i}{(4E_i^2 - W_n^2)} - \frac{1}{G}\right)\Lambda_{1n} + \left(W_n \sum_i \frac{\Omega_i f_i}{(4E_i^2 - W_n^2)}\right)\Lambda_{2n} = 0 \tag{J.25}$$

and

$$\Lambda_{2n} = \tfrac{1}{2}G\sum_i \sqrt{\Omega_i}\sqrt{\Omega_i}\left\{\frac{\Lambda_{1n} f_i + \Lambda_{2n}}{2E_i - W_n} + \frac{-\Lambda_{1n} f_i + \Lambda_{2n}}{2E_i + W_n}\right\}$$

$$= \tfrac{1}{2}G\sum_i \Omega_i$$

$$\times\left\{\frac{\Lambda_{1n} f_i (2E_i + W_n) + \Lambda_{2n}(2E_i + W_n) - \Lambda_{1n} f_i (2E_i - W_n) + \Lambda_{2n}(2E_i - W_n)}{(4E_i^2 - W_n^2)}\right\}$$

$$= \tfrac{1}{2}G\sum_i \frac{\Omega_i}{(4E_i^2 - W_n^2)}\{\Lambda_{1n} f_i 2W_n + 4E_i \Lambda_{2n}\}$$

$$= \left(G\sum_i \frac{\Omega_i f_i W_n}{(4E_i^2 - W_n^2)}\right)\Lambda_{1n} + \left(G\sum_i \frac{\Omega_i 2E_i}{(4E_i^2 - W_n^2)}\right)\Lambda_{2n}$$

$$\left(W_n \sum_i \frac{\Omega_i f_i}{(4E_i^2 - W_n^2)}\right)\Lambda_{1n} + \left(\sum_i \frac{\Omega_i 2E_i}{(4E_i^2 - W_n^2)} - \frac{1}{G}\right)\Lambda_{2n} = 0. \tag{J.26}$$

In order that the system of equations (J.25), (J.26) has a solution we set the determinant of the coefficients to be zero, i.e.

$$\begin{vmatrix} \left(\sum_i \frac{\Omega_i f_i^2 2E_i}{(4E_i^2 - W_n^2)} - \frac{1}{G}\right) & W_n\left(\sum_i \frac{\Omega_i f_i}{(4E_i^2 - W_n^2)}\right) \\ W_n\left(\sum_i \frac{\Omega_i f_i}{(4E_i^2 - W_n^2)}\right) & \left(\sum_i \frac{\Omega_i 2E_i}{(4E_i^2 - W_n^2)} - \frac{1}{G}\right) \end{vmatrix} = 0. \tag{J.27}$$

Taking into account that we have previously solved the BCS equations, (in particular $\frac{2}{G} = \sum_i \frac{\Omega_i}{E_i}$), the element $\left(\sum_i \frac{\Omega_i 2E_i}{(4E_i^2 - W_n^2)} - \frac{1}{G}\right)$ of the determinant can be written as

$$\left(\sum_i \frac{\Omega_i 2E_i}{(4E_i^2-W_n^2)} - \frac{1}{G}\right) = \sum_i \frac{\Omega_i 2E_i}{(4E_i^2-W_n^2)} - \sum_i \frac{\Omega_i}{2E_i}$$

$$= \sum_i \frac{\Omega_i(4E_i^2 - 4E_i^2 + W_n^2)}{2E_i(4E_i^2-W_n^2)} = W_n^2 \sum_i \frac{\Omega_i}{2E_i(4E_i^2-W_n^2)},$$

i.e.

$$\left(\sum_i \frac{\Omega_i 2E_i}{(4E_i^2 - W_n^2)} - \frac{1}{G}\right) = W_n^2 \sum_i \frac{\Omega_i}{2E_i(4E_i^2 - W_n^2)}. \tag{J.28}$$

In the same way

$$\left(\sum_i \frac{\Omega_i 2E_i f_i^2}{(4E_i^2-W_n^2)} - \frac{1}{G}\right) = \sum_i \frac{\Omega_i f_i^2 2E_i}{(4E_i^2-W_n^2)} - \sum_i \frac{\Omega_i}{2E_i}$$

$$= \sum_i \frac{\Omega_i(f_i^2 4E_i^2 - 4E_i^2 + W_n^2)}{2E_i(4E_i^2-W_n^2)} = \sum_i \frac{4E_i^2(f_i^2-1)+W_n^2}{2E_i(4E_i^2-W_n^2)} \Omega_i.$$

We shall now rewrite the expression

$$4E_i^2(f_i^2 - 1) = 4E_i^2((U_i^2 - V_i^2)^2 - 1) = 4E_i^2(U_i^4 + V_i^4 - 2U_i^2 V_i^2 - 1),$$

making use of the BCS relations

$$(U_i^2 + V_i^2)^2 = 1, \quad U_i^4 + V_i^4 + 2U_i^2 V_i^2 = 1,$$

$$U_i^4 + V_i^4 - 2U_i^2 V_i^2 = 1 - 4U_i^2 V_i^2,$$
$$U_i^4 + V_i^4 - 2U_i^2 V_i^2 - 1 = -4U_i^2 V_i^2 = (f_i^2 - 1).$$

Because

$$2U_i V_i = \frac{\Delta}{E_\nu}$$

one can write

$$(f_i^2 - 1) = -\frac{\Delta^2}{E_\nu^2},$$

and thus

$$4E_i^2(f_i^2 - 1) = -4\Delta^2.$$

Consequently,

$$\left(\sum_i \frac{\Omega_i 2E_i f_i^2}{(4E_i^2 - W_n^2)} - \frac{1}{G}\right) = (W_n^2 - 4\Delta^2) \sum_i \frac{\Omega_i}{2E_i(4E_i^2 - W_n^2)}. \tag{J.29}$$

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Making use of equations (J.28) and (J.29) the determinant (J.27) can be written as

$$\begin{vmatrix} (W_n^2 - 4\Delta^2)\sum_i \frac{\Omega_i}{2E_i(4E_i^2-W_n^2)} & W_n\left(\sum_i \frac{\Omega_i f_i}{(4E_i^2-W_n^2)}\right) \\ W_n\left(\sum_i \frac{\Omega_i f_i}{(4E_i^2-W_n^2)}\right) & W_n^2 \sum_i \frac{\Omega_i}{2E_i(4E_i^2-W_n^2)} \end{vmatrix} = 0, \tag{J.30}$$

$$W_n^2\left[(W_n^2 - 4\Delta^2)\left(\sum_i \frac{\Omega_i}{2E_i(4E_i^2 - W_n^2)}\right)^2 - \left(\sum_i \frac{\Omega_i f_i}{(4E_i^2 - W_n^2)}\right)^2\right] = 0. \tag{J.31}$$

Introducing

$$\mathcal{Y}_n^2 = W_n^2 - 4\Delta^2,$$

one can write

$$4E_i^2 - W_n^2 = 4(\epsilon_i - \lambda)^2 + 4\Delta^2 - W_n^2 = 4(\epsilon_i - \lambda)^2 - \mathcal{Y}_n^2.$$

Consequently, equation (J.31) becomes

$$\mathcal{Y}_n^2\left(\sum_i \frac{\Omega_i}{2E_i(4(\epsilon_i - \lambda)^2 - \mathcal{Y}_n^2)}\right)^2 = \left(\sum_i \frac{\Omega_i f_i}{4(\epsilon_i - \lambda)^2 - \mathcal{Y}_n^2}\right)^2,$$

which, making use of the relation

$$f_i = U_i^2 - V_i^2 = \frac{\epsilon_i - \lambda}{E_i},$$

leads to

$$\mathcal{Y}_n \sum_i \frac{\Omega_i}{2E_i(4(\epsilon_i - \lambda)^2 - \mathcal{Y}_n^2)} = \sum_i \frac{\Omega_i(\epsilon_i - \lambda)}{(4(\epsilon_i - \lambda)^2 - \mathcal{Y}_n^2)E_i},$$

and finally to

$$\sum_i \frac{(\mathcal{Y}_n - 2(\epsilon_i - \lambda))\Omega_i}{2E_i\left\{(2(\epsilon_i - \lambda) + \mathcal{Y}_n)(2(\epsilon_i - \lambda) - \mathcal{Y}_n)\right\}} = 0.$$

Consequently,

$$\sum_i \frac{\Omega_i}{E_i(\mathcal{Y}_n + 2(\epsilon_i - \lambda))} = 0. \tag{J.32}$$

From equation (J.26)

$$\frac{\Lambda_{2n}}{\Lambda_{1n}} = -\frac{W_n \sum_i \frac{\Omega_i f_i}{(4E_i^2 - W_n^2)}}{\sum_i \frac{\Omega_i 2E_i}{(4E_i^2 - W_n^2)} - \frac{1}{G}}$$

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and equation (J.28)

$$\frac{\Lambda_{2n}}{\Lambda_{1n}} = -\frac{W_n \sum_i \frac{\Omega_i f_i}{(4E_i^2 - W_n^2)}}{W_n^2 \sum_i \frac{\Omega_i}{2E_i(4E_i^2 - W_n^2)}},$$

one obtains

$$\frac{\Lambda_{2n}}{\Lambda_{1n}} = -\frac{\sum_i \frac{\Omega_i f_i}{(4E_i^2 - W_n^2)}}{W_n \sum_i \frac{\Omega_i}{2E_i(4E_i^2 - W_n^2)}}. \tag{J.33}$$

Making use of the normalization condition,

$$\begin{aligned} 1 &= \sum_i (a_{ni}^2 - b_{ni}^2) \\ &= \sum_i \left\{ \left(\frac{\Lambda_{1n} f_i + \Lambda_{2n}}{2E_i - W_n} \sqrt{\Omega_i} \right)^2 - \left(\frac{-\Lambda_{1n} f_i + \Lambda_{2n}}{2E_i + W_n} \sqrt{\Omega_i} \right)^2 \right\} \\ &= \sum_i \Omega_i \left\{ \frac{\Lambda_{1n}^2 f_i^2 + 2\Lambda_{1n}\Lambda_{2n} f_i + \Lambda_{2n}^2}{(2E_i - W_n)^2} - \frac{\Lambda_{1n}^2 f_i^2 - 2\Lambda_{1n}\Lambda_{2n} f_i + \Lambda_{2n}^2}{(2E_i + W_n)^2} \right\} \\ &= \sum_i \frac{f_i^2 W_n 8E_i \Lambda_{1n}^2 + 4 f_i (4E_i^2 + W_n^2)\Lambda_{1n}\Lambda_{2n} + 8E_i W_n \Lambda_{2n}^2}{(4E_i^2 - W_n^2)^2} \Omega_i, \end{aligned}$$

which leads to

$$\begin{aligned} \frac{1}{\Lambda_{1n}^2} = 4 \Bigg[& \left(\sum_i \frac{f_i^2 2E_i W_n \Omega_i}{(4E_i^2 - W_n^2)^2} \right) + \left(\sum_i \frac{f_i (4E_i^2 + W_n^2)\Omega_i}{(4E_i^2 - W_n^2)^2} \right) \left(\frac{\Lambda_{2n}}{\Lambda_{1n}} \right) \\ & + \left(\sum_i \frac{2E_i W_n \Omega_i}{(4E_i^2 - W_n^2)^2} \right) \left(\frac{\Lambda_{2n}}{\Lambda_{1n}} \right)^2 \Bigg]. \end{aligned}$$

That is,

$$\begin{aligned} \Lambda_{1n} = \frac{1}{2} \Bigg[& W_n \left(\sum_i \frac{f_i^2 2E_i \Omega_i}{(4E_i^2 - W_n^2)^2} \right) + \left(\sum_i \frac{f_i (4E_i^2 + W_n^2)\Omega_i}{(4E_i^2 - W_n^2)^2} \right) \left(\frac{\Lambda_{2n}}{\Lambda_{1n}} \right) \\ & + W_n \left(\sum_i \frac{2E_i \Omega_i}{(4E_i^2 - W_n^2)^2} \right) \left(\frac{\Lambda_{2n}}{\Lambda_{1n}} \right)^2 \Bigg]^{-1/2}. \end{aligned} \tag{J.34}$$

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