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Continuity and Realization of multiplicative maps between RKHS and their cyclicity preserving properties

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Abstract. **Motivated by the study of multiplicative linear functionals in reproducing kernel Hilbert space (RKHS) with normalized complete Nevanlinna-Pick kernel, we define and study the multiplicative linear map between two RKHS. We identify the conditions under which such maps are continuous. Additionally, we prove that any unital cyclicity-preserving linear map is multiplicative. Conversely, we also characterize when a multiplicative linear map is unital cyclicity preserving. These results serve as a generalization of the Gleason–Kahane–Żelazko theorem to the setting of multiplicative maps between two RKHS. We present the composition operator as a natural class of examples of multiplicative linear maps on an RKHS. We also prove that every continuous multiplicative linear operator can be realized as a composition operator on various analytic Hilbert spaces over the unit disc $\mathbb{D}$.**

1 Introduction

The study of multiplicative functionals and multiplicative linear maps, also referred as algebra homomorphisms, has been extensively studied in the context of Banach algebra. An important result in this area is the Gleason–Kahane–Żelazko (GKZ) theorem, which characterizes the conditions under which a linear functional is multiplicative.

***Theorem* 1.1** *[10, 13, 25] Let A be a complex unital Banach algebra, and $\Lambda : A \longrightarrow \mathbb{C}$ be a linear functional such that $\Lambda \not\equiv 0$. Then the following statements are equivalent:*

(1) $\Lambda(1) = 1$ *and* $\Lambda(a) \neq 0$ *for all invertible elements* $a \in A$.
(2) Λ *is multiplicative i.e.,* $\Lambda(ab) = \Lambda(a)\Lambda(b)$ *for all* $a, b \in A$.

Also, such multiplicative functionals are automatically continuous. The above theorem has been generalized for algebra homomorphisms or multiplicative linear maps between two commutative Banach algebras, under the assumption that the co-domain is semi-simple.

***Theorem* 1.2** *[21] Let A be a commutative unital Banach algebra, B be a semi-simple commutative unital Banach algebra, and $\Phi : A \longrightarrow B$ be an unital invertibility-preserving linear*

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map. Then Φ *is multiplicative, i.e.,*

$$\Phi(ab) = \Phi(a)\Phi(b) \text{ for all } a, b \in A.$$

Moreover, multiplicative maps between Banach algebras are continuous only when the co-domain is semisimple [19].

In [2], it is proved that in reproducing kernel Hilbert spaces (RKHS) with normalized complete Pick (NCP) kernel, every function can be expressed as the quotient of a multiplier and a cyclic multiplier. And in [12], they proved a factorization result for certain sequences of functions in RKHS with NCP kernel using non-commutative function theory. Building on these two results, in [5], the authors studied the multiplicativity of linear functionals on RKHS and proved their continuity. In particular, they generalized the Gleason–Kahane–Żelazko theorem to a special class of Hilbert spaces, namely RKHS with NCP kernel.

***Theorem* 1.3** *[5] Let* $\mathcal{H}$ *be a reproducing kernel Hilbert space with normalized complete Nevanlinna-Pick kernel, and let* $\Lambda : \mathcal{H} \to \mathbb{C}$ *be a linear functional such that* $\Lambda \not\equiv 0$*. Then the following are equivalent*

(1) $\Lambda(1) = 1$ *and* $\Lambda(f) \neq 0$ *for all cyclic elements* $f \in \mathcal{H}$*.*
(2) $\Lambda(f \cdot g) = \Lambda(f)\Lambda(g)$ *for all* $f, g \in \mathcal{H}$ *such that* $fg \in \mathcal{H}$*.*

A functional satisfying (2) of the above theorem is called a multiplicative functional in RKHS. See Definition 1.5 for multiplicative functional in RKHS. Note that, in the RKHS setting, cyclic elements play the role analogous to that of invertible elements in the Banach algebra setting. There is another version of the GKZ theorem for the Dirichlet space [15]. Also, Kowalski-Słodkowski theorem, which studies when a functional is multiplicative and linear, has also been generalized to the RKHS setting [18].

It is well known that every multiplicative linear functional on a unital commutative Banach algebra is automatically continuous. In a recent study [5], the authors proved that multiplicative linear functionals defined on reproducing kernel Hilbert spaces with normalized complete Nevanlinna–Pick (NCP) kernels are also continuous.

In this article, we take one step further to define and study the multiplicative maps between RKHS. In particular, Section 2 is dedicated to the study of the continuity of multiplicative linear maps between two RKHS. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be RKHS with NCP kernel. In Theorem 2.5, we prove a weaker form of continuity for multiplicative maps between $\mathcal{H}_1$ and $\mathcal{H}_2$. Furthermore, Theorem 2.8 shows that if $T : \mathcal{H}_1 \to \mathcal{H}_2$ is surjective and the norm of $\mathcal{H}_2$ has an algebra structure, then T is automatically continuous. In Section 3, we establish the forward implication of Gleason–Kahane–Żelazko (GKZ) for RKHS. Specifically, Theorem 3.3 proves that every unital cyclicity-preserving linear map T is necessarily multiplicative. Section 4 addresses the converse. In Theorem 4.1, we prove that if the range of a multiplicative map T contains a non-vanishing function, then it is unital and maps cyclic functions to non-vanishing functions. Furthermore, Theorem 4.2 proves such maps are also cyclicity preserving when the maps are surjective and the norm of $\mathcal{H}_2$ has an algebra structure. Together, these results provide a generalization of the Gleason–Kahane–Żelazko theorem to the setting of linear maps between two RKHS.

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Note that cyclicity preserving operators are interesting in their own way, as they naturally appear while identifying functions in $H^2(\mathbb{D}^2)$ that are outer but not cyclic, see [24]. Composition operators are always an example of multiplicative maps. In Section 5, we prove that every continuous multiplicative linear operator from various analytic Hilbert spaces over the unit disc $\mathbb{D}$ to itself can be realised as a composition operator. These spaces includes the Hardy space $H^2(\mathbb{D})$, Dirichlet space $\mathcal{D}$, and weighted Bergman spaces $A^2_\alpha(\mathbb{D})$, among others. Before presenting the main results, we begin with some preliminary concepts.

1.1 Preliminaries

***Definition* 1.1** (Reproducing Kernel Hilbert Space (RKHS)) [20] Let X be a non-empty set, $\mathbb{K}$ be a field, and $\mathcal{F}(X, \mathbb{K})$ be the collection of all functions from X to $\mathbb{K}$. We call $\mathcal{H} \subseteq \mathcal{F}(X, \mathbb{K})$ a RKHS on X if

(1) $\mathcal{H}$ is a Hilbert space.
(2) For every $x \in X$, the evaluation functional $E_x : \mathcal{H} \to \mathbb{K}$ defined by $E_x(f) = f(x)$ is continuous.

***Definition* 1.2** (Kernel function) A function $K : X \times X \to \mathbb{K}$ is said to be kernel function if

(1) K is Hermitian, that is $K(x, y) = \overline{K(y, x)}$
(2) K is a positive semi-definite function.

According to the theory of RKHS by Aronszajn in [3], every RKHS is associated with a unique kernel function $K : X \times X \to \mathbb{K}$, such that the set $\{K(\cdot, y), y \in X\}$ is dense in space. Also satisfying, for every $f \in \mathcal{H}$ and $y \in X$, the reproducing property holds (i.e)

$$f(y) = \langle f, K(\cdot, y)\rangle,$$

We call K as the reproducing kernel for RKHS. Conversely, given any kernel function K with an appropriate norm, there exists a unique RKHS with K as its reproducing kernel with the reproducing property.

***Definition* 1.3** (Normalized Kernel) A kernel function $K : X \times X \to \mathbb{K}$, is said to be normalized, if there exists $x_0 \in X$ such that,

$$K(x, x_0) = 1, \text{ for all } x \in X.$$

If $\mathcal{H}$ is an RKHS with normalized kernel K, then the function $K(\cdot, x_0) = k_{x_0} \in \mathcal{H}$, and serves as the constant function 1 in $\mathcal{H}$.

***Definition* 1.4** (Complete Nevanlinna-Pick kernels) [1] A reproducing kernel K of an RKHS on X is said to be a Complete Nevanlinna-Pick kernel if

(1) $K(x, y) \neq 0$, for all $x, y \in X$
(2) There exists $x_0 \in X$ such that $F(x, y) = 1 - \frac{K(x, x_0)K(x_0, y)}{K(x, y)K(x_0, x_0)}$ is a positive semi-definite function on $X \times X$

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Unlike Banach algebras, Hilbert spaces do not have a multiplicative structure. However, in the case of reproducing kernel Hilbert space (RKHS), each element is a function from the underlying set X to the field $\mathbb{K}$. Therefore, pointwise multiplication "$\cdot$" of functions is a well-defined operation. However, RKHS is not generally closed under pointwise multiplication, so it does not form an algebra under this operation. Consequently, the multiplicativity condition for linear functionals over RKHS $\mathcal{H}$ is only considered for those pairs $f, g \in \mathcal{H}$ for which the product $f \cdot g \in \mathcal{H}$. And if the kernel is normalized at $x_0 \in X$, the function k_{x_0} will serve as the multiplicative identity under pointwise multiplication, whenever defined.

***Definition* 1.5** (Multiplicative functional in RKHS) Let $\mathcal{H}$ be an RKHS, a linear functional $\Lambda : \mathcal{H} \to \mathbb{C}$ is said to be a multiplicative functional on $\mathcal{H}$ if, for all $f, g \in \mathcal{H}$ such that $f \cdot g \in \mathcal{H}$, the multiplicativity condition holds (i.e.) $\Lambda(f \cdot g) = \Lambda(f)\Lambda(g)$

Motivated by the notion of algebra homomorphisms between Banach algebras, we define and study multiplicative maps between RKHS.

***Definition* 1.6** (Multiplicative maps between RKHS) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two RKHS. The linear map $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ is said to be a multiplicative map if, for all $f, g \in \mathcal{H}_1$ such that $f \cdot g \in \mathcal{H}_1$,

$$\Phi(f \cdot g) = \Phi(f) \cdot \Phi(g).$$

It is important to note that, in the case of a multiplicative functional Λ, the product $\Lambda(f)\Lambda(g) \in \mathbb{K}$, since $\mathbb{K}$ is an algebra. However, in the case of multiplicative maps between RKHS, the co-domain need not be an algebra. Therefore, a map $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ is said to be multiplicative, if $\forall f, g \in \mathcal{H}_1$ such that $f \cdot g \in \mathcal{H}_1$, the pointwise product $\Phi(f) \cdot \Phi(g) \in \mathcal{H}_2$, and the above equality holds.

***Definition* 1.7** (Multiplier algebra) Let $\mathcal{H}$ be an RKHS over a non-empty set X. A function $g : X \to \mathbb{C}$ is said to be a multiplier element if for all $f \in \mathcal{H}$, the pointwise product $g \cdot f \in \mathcal{H}$. The collection of all multiplier elements on $\mathcal{H}$ is denoted by the multiplier algebra $\mathcal{M}$.

For each $g \in \mathcal{M}$, the associated multiplication operator $M_g : \mathcal{H} \to \mathcal{H}$, defined by $M_g(f) = g \cdot f$, is bounded. The space $\mathcal{M}$ with norm $||g||_{\mathcal{M}} = ||M_g||_{op}$ forms a commutative Banach algebra under pointwise multiplication.

Let $\mathcal{H}$ be an RKHS with multiplier algebra $\mathcal{M}$. For any element $f \in \mathcal{H}$, the closed $\mathcal{M}$ invariant subspace generated by f is defined as $[f] = \overline{\mathcal{M}f}$, where $\mathcal{M}f = \{g \cdot f : g \in \mathcal{M}\}$, and the closure is taken over Hilbert space topology.

***Definition* 1.8** (Cyclic function) Let $\mathcal{H}$ be an RKHS with multiplier algebra $\mathcal{M}$. An element $f \in \mathcal{H}$ is said to be cyclic, if $[f] = \mathcal{H}$. That is, for every $g \in \mathcal{H}$, there exists a sequence $(h_n) \subseteq \mathcal{M}$, satisfying $h_n \cdot f \to g$. Note that every cyclic function is non-vanishing.

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Let us see some examples of multiplicative operators between RKHS to itself.

Example 1.4 In the Hardy Hilbert space $H^2(\mathbb{D})$, define the operator

$$T : H^2(\mathbb{D}) \to H^2(\mathbb{D})$$

$$T\Big(\sum_{n=0}^{\infty} a_n z^n\Big) = \sum_{n=0}^{\infty} a_n z^{2n}.$$

$$\text{Let } f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

$$\text{Then } (f \cdot g)(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k} z^n, \textbf{ and } T(f \cdot g)(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k} z^{2n}.$$

$$(T(f) \cdot T(g))(z) = \sum_{n=0}^{\infty} a_n z^{2n} \cdot \sum_{n=0}^{\infty} b_n z^{2n} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k} z^{2n}$$

Here $T(f \cdot g)(z) = (T(f) \cdot T(g))(z)$, when $f \cdot g \in H^2(\mathbb{D})$. Therefore, T is a multiplicative linear map.

Note that to check the multiplicativity of T, if $f \cdot g \in H^2(\mathbb{D})$, then it is required that $Tf \cdot Tg \in H^2(\mathbb{D})$ and the equality should hold. However, if $f \cdot g \notin H^2(\mathbb{D})$, the multiplicativity condition need not be verified for such pairs of functions. In the above example, if we replace 2 with any $k \in \mathbb{N}$ in the definition of T, the resulting map T is also multiplicative. That is, the map $T\Big(\sum_{n=0}^{\infty} a_n z^n\Big) = \sum_{n=0}^{\infty} a_n z^{kn}$, for any $k \in \mathbb{N}$ is multiplicative

Example 1.5 Let $A^2(\mathbb{D})$ be the Bergman space over open unit disc $\mathbb{D}$. Let $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic map. Then the map $C_\phi : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ defined as $C_\phi(f)(z) = f(\phi(z))$, for all $z \in \mathbb{D}$ is well-defined and continuous [4]. By the definition, the composition operator C_ϕ is also multiplicative linear map.

We can also define a composition operator that is multiplicative between general RKHS as follows:

Example 1.6 Let $\mathcal{H}$ be an RKHS over X and $\phi : X \to X$. Then define the composition operator

$$C_\phi : \mathcal{H} \to \mathcal{H}$$

$$C_\phi(g)(x) = g(\phi(x)), \text{ where } g \in \mathcal{H}, x \in X$$

By definition, if the composition operator C_ϕ is well-defined, then it is a linear and multiplicative operator.

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Note that in the case of Hardy-Hilbert space $H^2(\mathbb{D})$, if ϕ is an analytic self-map on the unit disc $\mathbb{D}$, then the associated composition operator C_ϕ is bounded. Please refer [6] for the study of composition operator on analytic function spaces.

The above example gives a class of multiplicative linear operators over RKHS. Notably, the Perron-Frobenius operator, which is widely used in dynamical systems and machine learning, belongs to this class.

2 Continuity of Multiplicative maps

Every multiplicative linear functional from a complex Banach algebra is continuous. Furthermore, the continuity of multiplicative maps between Banach algebras are also studied.

***Theorem* 2.1** *[22, 19] Let A and B be Banach algebras such that B is commutative and semi-simple. Then, every multiplicative linear map $T : A \longrightarrow B$ is continuous.*

***Theorem* 2.2** *[19] Let A and B be Banach algebras such that B is semi-simple. Then, any surjective multiplicative linear map $T : A \longrightarrow B$ is continuous.*

Recently, in [5], the authors generalized continuity to multiplicative linear functionals on RKHS with NCP kernel.

***Theorem* 2.3** *[5] Let $\mathcal{H}$ be an RKHS with a normalized complete Nevanlinna-Pick (NCP) kernel, and $\Lambda : \mathcal{H} \longrightarrow \mathbb{C}$ be a multiplicative functional. Then, Λ is continuous.*

In this section, we study the continuity of the multiplicative maps between RKHS with NCP kernels.

We begin by observing that every RKHS is semi-simple.

***Definition* 2.1** (Semi-simple Banach algebra) Let A be a commutative Banach algebra and $\Delta(A)$ denote the collection of all non-zero multiplicative linear functionals. Then A is semi-simple if

$$\bigcap_{f \in \Delta(A)} Ker(f) = \{0\}$$

We observe that every RKHS is semi-simple. That is, if $\Delta(H)$ denotes the multiplicative linear functional on H, then all point evaluations $\Lambda_x \in \Delta(H)$, for all $x \in X$, and

$$\bigcap_{\Lambda \in \Delta(H)} Ker(\Lambda) \subseteq \bigcap_{x \in X} Ker(\Lambda_x) = \{0\}$$

Let $\mathcal{M}$ denote the multiplier algebra of RKHS. Then for each point $x \in X$, the evaluation functional restricted to multiplier algebra, $\Lambda_x \backslash_{\mathcal{M}} : \mathcal{M} \rightarrow \mathbb{C}$ is multiplicative and linear. Also, the intersection of the kernels of all such functionals is zero. This implies $\mathcal{M}$ is semi-simple Banach algebra. Additionally, if the reproducing kernel is normalized, then $\mathcal{M}$ is also unital.

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***Lemma* 2.4** *Let $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ be a multiplicative linear map between two RKHS $\mathcal{H}_1$ and $\mathcal{H}_2$ with NCP kernel, and let $\Lambda : \mathcal{H}_2 \to \mathbb{C}$ be a multiplicative linear functional. Then $\Lambda \circ \Phi : \mathcal{H}_1 \to \mathbb{C}$ is a multiplicative linear functional.*

The proof of the above lemma follows from the same argument as that for multiplicative linear maps between Banach algebras. The above theorem implies that the composition of a multiplicative linear functional with a multiplicative linear map gives another multiplicative linear functional. This observation helps to reduce the problem of the maps between Hilbert spaces to a linear functional from the Hilbert space.

***Theorem* 2.5** *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two RKHS with NCP kernel over the non-empty sets X_1 and X_2, respectively, and $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$, be a multiplicative linear map. Then $\forall\, f_n, f \in \mathcal{H}_1$ satisfying $f_n \to f$, $\Phi(f_n)$ converges to $\Phi(f)$ pointwise, (i.e.) $\Phi(f_n)(y) \to \Phi(f)(y)$, for all $y \in X_2$.*

Proof For every $\Lambda : \mathcal{H}_2 \to \mathbb{C}$ multiplicative linear functional, from Lemma 2.4, we get that the map $\Lambda \circ \Phi$ is a multiplicative linear functional. By Theorem 2.3, the map $\Lambda \circ \Phi$ is continuous in $\mathcal{H}$. That is,

$$|\Lambda \circ \Phi(f_n + a) - \Lambda \circ \Phi(a)| \to 0, \quad as\ f_n \to 0$$

As Λ is linear,

$$|\Lambda[\Phi(f_n + a) - \Phi(a)]| \to 0, \text{ as } f_n \to 0,$$

This happens for all multiplicative linear functionals. In particular, for all point evaluations Λ_y, where $y \in X_2$. That is,

$$|\Lambda_y[\Phi(f_n + a) - \Phi(a)]| \to 0, \text{ as } f_n \to 0$$
$$|(\Phi(f_n + a) - \Phi(a))(y)| \to 0, \text{ as } f_n \to 0$$

That is, $\Phi(f_n)$ converges to $\Phi(f)$ pointwise for all $y \in X_2$. ∎

Note that, if the above convergence happens in the Hilbert space norm, then the map Φ is continuous.

Both the examples 1.4 and 1.6 are continuous multiplicative operators. Previously, we proved that if Φ is a multiplicative linear map between RKHS with NCP kernel, and if $f_n \to f$ in the Hilbert space norm, then $\Phi(f_n) \to \Phi(f)$ pointwise. This naturally raises the question of under what conditions this convergence will also happen in the Hilbert space norm? For that, we will use a characterization of Jury and Martin.

***Theorem* 2.6** *[12] Let $\mathcal{H}$ be an RKHS with NCP kernel, and let $\mathcal{M}$ be its multiplier algebra. If (f_n) is a sequence in $\mathcal{H}$ such that $\sum_{n=0}^{\infty} ||f_n||^2_{\mathcal{H}_1} < \infty$, then there exists a sequence $(h_n) \in \mathcal{M}$ and a cyclic function $g \in \mathcal{H}_1$ such that:*

(1) $f_n = h_n \cdot g$ *for all n,*

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(2) $\sum_{n=0}^{\infty} ||f_n||^2_{\mathcal{H}} = ||g||^2_{\mathcal{H}}$,

(3) $\sum_{n=0}^{\infty} ||h_n f||^2_{\mathcal{H}_1} \leq ||f||^2_{\mathcal{H}}$ *for all* $f \in \mathcal{H}$*, and in particular* $||h_n||_{\mathcal{M}} \leq 1$*, for all n.*

***Theorem* 2.7** *Let* $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ *be a linear map between two RKHS with NCP kernel, satisfying* $\forall f, g \in \mathcal{H}_1$ *with* $f \cdot g \in \mathcal{H}_1$*, the pointwise product* $\Phi(f) \cdot \Phi(g) \in \mathcal{H}_2$*. And if* Φ *is surjective, then* Φ *maps the multiplicative elements* $\mathcal{M}_1$ *of* $\mathcal{H}_1$ *to the multiplicative elements* $\mathcal{M}_2$ *of* $\mathcal{H}_2$*.*

Proof Let Φ be a surjective map, then for all $\tilde{h} \in \mathcal{H}_2$, there exists a $h \in \mathcal{H}_1$, such that $\Phi(h) = \tilde{h}$. Let $m \in \mathcal{M}_1$, then $m \cdot h \in \mathcal{H}_1$ and $\Phi(m) \cdot \tilde{h} = \Phi(m) \cdot \Phi(h)$. By hypothesis $\Phi(m) \cdot \tilde{h} \in \mathcal{H}_2$. This implies that $\Phi(m)$ is multiplicative. ∎

***Theorem* 2.8** *Let* $\mathcal{H}_1, \mathcal{H}_2$ *be two RKHS with NCP kernel. Let* $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ *be a surjective multiplicative linear map, with the norm* $||.||_{\mathcal{H}_2}$ *being an algebra norm. Then* Φ *is continuous.*

Proof Suppose $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ is not continuous. Then there exists a sequence $f_n \subset \mathcal{H}_1$, such that $\sum_{n=0}^{\infty} ||f_n||^2_{\mathcal{H}_1} < \infty$, but $||\Phi(f_n)||_{\mathcal{H}_2} \to \infty$. By Theorem 2.6, $f_n = h_n \cdot g$, where h_n belongs to multiplier algebra $\mathcal{M}_1$ and $g \in \mathcal{H}_1$ is cyclic. But

$$||\Phi(f_n)||_{\mathcal{H}_2} \to \infty$$

$$||\Phi(h_n) \cdot \Phi(g)||_{\mathcal{H}_2} \to \infty$$

The norm on $\mathcal{H}_2$ is an algebra norm, this implies $||\Phi(h_n)|| \to \infty$.

By Theorem 2.7, $\Phi\backslash_{\mathcal{M}_1} : \mathcal{M}_1 \to \mathcal{M}_2$, is a multiplicative map between two Multiplier algebras. By Theorem 2.1, $\Phi\backslash_{\mathcal{M}_1}$ is continuous. Since $||h_n||_{\mathcal{M}_1}$ is bounded, $||\Phi(h_n)||_{\mathcal{M}_2}$ is bounded. Since the norm in multiplier algebra is stronger, $||\Phi(h_n)||_{\mathcal{H}_2}$ is bounded, which is a contradiction. This implies Φ is continuous. ∎

Note that the above Theorem is a generalization of Theorem 2.2 to RKHS setting. We would like to emphasize that the algebra norm condition is required only for those pairs $f, g \in \mathcal{H}_1$, for which the product $f \cdot g \in \mathcal{H}_1$. Reproducing kernel Hilbert algebra (RKHA) satisfies the norm condition mentioned above. That is, RKHA forms a Banach algebra under pointwise multiplication of functions. For further details on reproducing kernel Hilbert algebras, we refer [8, 17].

3 Multiplicativity of cyclicity preserving maps

By Theorem 1.2, every unital invertibility preserving linear map from an unital commutative Banach algebra to an unital commutative semi-simple Banach algebra is multiplicative. In this section, we generalize the forward implication of Theorem 1.3

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to RKHS setting (i.e.), we study the multiplicativity of unital invertibility preserving linear maps between two RKHS. As demonstrated in Theorem 1.3, the role of invertible elements in Banach algebra is replaced by cyclic elements in the RKHS setting.

***Theorem* 3.1** *Let $\mathcal{H}_1, \mathcal{H}_2$ be two RKHS with NCP kernel, and let $\Phi : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a unital cyclicity preserving linear map , and if $\Lambda : \mathcal{H}_2 \longrightarrow \mathbb{C}$ is a multiplicative linear functional, then $\Lambda \circ \Phi$ is multiplicative linear functional.*

Proof Since both Φ and Λ are linear, implies the composition $\Lambda \circ \Phi$ is a linear functional. For any cyclic element $c \in \mathcal{H}_1$, by the hypothesis $\Phi(c)$ is cyclic in $\mathcal{H}_2$. Since Λ is a multiplicative linear functional, by Theorem 1.3 $\Lambda \circ \Phi(c) \neq 0$. Additionally, $\Lambda \circ \Phi(1_{\mathcal{H}_1}) = \Lambda(\Phi(1_{\mathcal{H}_1})) = \Lambda(1_{\mathcal{H}_2}) = 1$, because Λ is multiplicative. Using the same Theorem 1.3, it follows that $\Lambda \circ \Phi$ is a multiplicative linear functional. ■

***Theorem* 3.2** *Let $\mathcal{H}_1, \mathcal{H}_2$ be two RKHS with NCP kernel, and $\Phi : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a linear maps satisfying $\forall\ f, g \in \mathcal{H}_1$ with $f \cdot g \in \mathcal{H}_1$, the pointwise product $\Phi(f) \cdot \Phi(g) \in \mathcal{H}_2$. If $\Lambda \circ \Phi$ is a multiplicative linear functional, for all multiplicative linear functionals $\Lambda : \mathcal{H}_2 \longrightarrow \mathbb{C}$, then Φ is multiplicative.*

Proof Given $\Lambda \circ \Phi$ is multiplicative functional for all multiplicative functional Λ and for all $f, g \in \mathcal{H}_1$ such that $f \cdot g \in \mathcal{H}_1$,

$$\Lambda \circ \Phi(f \cdot g) = (\Lambda \circ \Phi(f))(\Lambda \circ \Phi(g))$$

By linearity of the map Λ,

$$\Lambda(\Phi(f \cdot g) - \Phi(f) \cdot \Phi(g)) = 0$$

This is true for all multiplicative linear functionals $\Lambda : \mathcal{H}_2 \longrightarrow \mathbb{C}$. In particular, the point evaluation functional $\Lambda_y : \mathcal{H}_2 \longrightarrow \mathbb{C}$ is a multiplicative linear functional for all $y \in X_2$. This implies

$$\begin{aligned} \Lambda_y(\Phi(f \cdot g) - \Phi(f) \cdot \Phi(g)) &= 0, \text{for all } y \in X_2 \\ (\Phi(f \cdot g) - \Phi(f) \cdot \Phi(g))(y) &= 0 \text{ for all } y \in X_2 \\ \Phi(f \cdot g) - \Phi(f) \cdot \Phi(g) &\equiv 0 \\ \Phi(f \cdot g) &\equiv \Phi(f) \cdot \Phi(g) \end{aligned}$$

That is Φ is multiplicative. ■

***Theorem* 3.3** *Let $\mathcal{H}_1, \mathcal{H}_2$ be two RKHS with NCP kernel, and $\Phi : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a linear map satisfying, for all $f, g \in \mathcal{H}_1$, such that $f \cdot g \in \mathcal{H}_1$, $\Phi(f).\Phi(g) \in \mathcal{H}_2$. If Φ is an unital cyclicity preserving linear map, then Φ is multiplicative.*

Proof By Theorem 3.1, for all multiplicative linear functional $\Lambda : \mathcal{H}_2 \longrightarrow \mathbb{C}$, $\Lambda \circ \Phi$ is multiplicative. Then, from Theorem 3.2, we get Φ is multiplicative linear map. ■

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We will see examples of maps that satisfy the first condition of the preceding theorem namely, for all $f, g \in \mathcal{H}_1$ such that $f \cdot g \in \mathcal{H}_1$, it should hold that $\Phi(f) \cdot \Phi(g) \in \mathcal{H}_2$. It is important to note that every multiplicative map necessarily satisfies this condition.

On the Hardy-Hilbert space $H^2(\mathbb{D})$, the right shift operator M_z, the left shift operator M_z^*, and the truncation operator T_n are not multiplicative maps. But, all three operators satisfy the above condition, i.e., for all $f, g \in H^2(\mathbb{D})$ such that $f \cdot g \in H^2(\mathbb{D})$, it holds that $M_z(f) \cdot M_z(g)$, $M_z^*(f) \cdot M_z^*(g)$, $T_n(f) \cdot T_n(g) \in \mathcal{H}_2$. Note that $M_z^2(f \cdot g) = M_z(f) \cdot M_z(g)$ holds exclusively for the right shift operator M_z.

4 Cyclicity preserving properties of Multiplicative maps

In the Banach algebra setting, the unital invertibility preserving properties of algebra homomorphism have been extensively studied in [7]. Similarly, cyclicity preserving operators have been studied on the Hilbert spaces of analytic functions on $\mathbb{C}^n$ [23]. In this section, we generalize the backward implication of Theorem 1.3 to RKHS setting (i.e.), we explore the conditions under which a multiplicative linear map between RKHS is unital and cyclicity preserving.

***Definition* 4.1** (Strongly non-zero map) Let $\mathcal{H}_1, \mathcal{H}_2$ be two RKHS over the non-empty sets X_1 and X_2, respectively. The map $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ is said to be a strongly non-zero map if Range(Φ) contains at least one non-vanishing function. That is there exists a function $\tilde{f} \in$ Range(Φ), such that $\tilde{f}(x) \neq 0$, for all $x \in X_2$.

***Theorem* 4.1** *Let $\mathcal{H}_1, \mathcal{H}_2$ be two RKHS with NCP kernel, and $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ be a strongly non-zero multiplicative linear map. Then $\Phi(1_{\mathcal{H}_1}) = 1$ and $\Phi(c)$ is non-vanishing for all cyclic functions $c \in \mathcal{H}_1$.*

Proof Given Φ is a strongly nonzero map, then there exists a non-vanishing function $\tilde{f} \in$ Range $\Phi \subset \mathcal{H}_2$, (i.e.) there exists $f \in \mathcal{H}_1$ such that $\Phi(f) = \tilde{f}$. Let c be a cyclic element of $\mathcal{H}_1$, then for $f \in \mathcal{H}_1$, there exists a sequence $h_n \in \mathcal{M}_1$, where $\mathcal{M}_1$ is the multiplier algebra of $\mathcal{H}_1$, such that $h_n \cdot c \to f$. By Theorem 2.5,

$$\Phi(h_n \cdot c)(y) \to \Phi(f)(y), \text{ for all } y \in X_2,$$
$$\Phi(h_n)(y)\Phi(c)(y) \to \tilde{f}(y), \text{ where } \Phi(f) = \tilde{f},$$

Here $\tilde{f}$ is a non-vanishing function, which implies $\Phi(c)$ is non-vanishing for all cyclic elements $c \in \mathcal{H}_2$. Then $(\Phi(c))^{-1}$ exist as a well-defined function from X_2 to $\mathbb{C}$. This implies

$$\Phi(c) = \Phi(c) \cdot \Phi(1_{\mathcal{H}_1}),$$
$$1_{\mathcal{H}_2} = \Phi(1_{\mathcal{H}_1}).$$

Hence Φ is unital and $\Phi(c)$ is non-vanishing for all cyclic element $c \in \mathcal{H}_2$. ■

We can observe that every cyclic function in RKHS is non-zero for every point in the underlying set X. In the preceding theorem, we proved that every multiplicative map

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Φ is unital-preserving and maps cyclic elements to non-vanishing functions. We now explore the question: under what conditions Φ will be cyclicity preserving?

***Theorem* 4.2** *Let $\mathcal{H}_1, \mathcal{H}_2$ be two RKHS with NCP kernel, such that the norm of $\mathcal{H}_2$ is algebra norm, and if $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ be a strongly non-zero multiplicative linear map and Φ is surjective, then Φ is cyclicity preserving.*

Proof Let c be a cyclic element in $\mathcal{H}_1$. Since Φ is surjective, for any $\tilde{h} \in \mathcal{H}_2$ there exists $h \in \mathcal{H}_1$, such that $\Phi(h) = \tilde{h}$. By the cyclicity of c, for $h \in \mathcal{H}_1$, there exists a sequence $m_n \subset \mathcal{M}_1$, such that $m_n \cdot c \to h$. By Theorem 2.5, Φ is continuous. This implies $\Phi(m_n) \cdot \Phi(c) \to \Phi(h) = \tilde{h}$. By Theorem 2.7, $\Phi(m_n) \in \mathcal{M}_2$. By definition of cyclicity, $\Phi(c)$ is cyclic in $\mathcal{H}_2$. ■

5 Realization of continuous multiplicative operators

The realization of multiplicative linear functionals on RKHS has been studied previously. If $\mathcal{H}$ is an RKHS over a non-empty set X, then for every $x \in X$ the point-evaluation functional Λ_x is multiplicative. In [2], the authors proved the converse in the case of maximal domain, (i.e) if X is maximal domain for $\mathcal{H}$, then every multiplicative linear functional can be realized as a point-evaluation functionals. In [23] the same result is proved for more general spaces.

Note that in the last section, we proved that multiplicative maps between RKHS with NCP maps the unit elements of the domain to the unit elements of the co-domain, and map cyclic functions to non-vanishing functions. The map also preserves cyclicity, if the norm of co-domain has an algebraic structure. In [16], it is proved that every linear map from the Hardy space $H^p(\mathbb{D})$ to the space of holomorphic functions on $\mathbb{D}$, that maps cyclic functions to non-vanishing functions, is necessarily a weighted composition operator. They also generalized the result to more general spaces over $\mathbb{D}$. In [23], the authors proved that cyclicity-preserving maps are weighted composition operators over more general spaces, which need not be Hilbert spaces.

In this section, we will realize multiplicative operators between Hilbert spaces. In Section 2, we presented composition operators as a natural class of multiplicative operators. Now the natural question is whether all multiplicative operators are, in fact, composition operators. While this is not true in general, we provide a partial answer. Over Hardy-Hilbert space $H^2(\mathbb{D})$, Dirichlet space $\mathcal{D}$, weighted Bergman space $A^2_\alpha(\mathbb{D})$, and many more, every continuous multiplicative operator is indeed a composition operator.

***Theorem* 5.1** *Let $M : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be a continuous multiplicative operator on the Hardy-Hilbert space. Then there exists an analytic self-map $\phi : \mathbb{D} \to \mathbb{D}$, such that $M = C_\phi$, where C_ϕ is the composition operator induced by ϕ.*

Proof Let $M : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be a continuous multiplicative linear operator. If $f \in H^2(\mathbb{D})$, then it admits the power series expansion $f = \sum a_n z^n$, and the co-effecients $\{a_n\} \in \ell^2$, where z denotes the identity function on $\mathbb{D}$. Since M is a

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continuous multiplicative linear map, we have

$$M(f)(x) = M(\sum_{n=0}^{\infty} a_n z^n)(x)$$
$$= \sum_{n=0}^{\infty} a_n (M(z)(x))^n, \text{ for all } x \in \mathbb{D}$$

That is for any function $\sum a_n z^n \in H^2(\mathbb{D})$, the image $\sum a_n M(z)^n$ should belong to $H^2(\mathbb{D})$, and $z \in H^2(\mathbb{D})$, this implies $M(z) \in H^2(\mathbb{D})$. We prove that the Range$M(z)$ is a subset of the open unit disc $\mathbb{D}$. Suppose not, there exists $x \in \mathbb{D}$, such that $M(z)(x) = re^{i\theta}$, where $r \geq 1$. We define the function

$$f_{r,\theta} = \sum_{n=0}^{\infty} \frac{1}{r^n} \frac{e^{-in\theta}}{n} z^n$$
$$||f_{r,\theta}|| = \sum_{n=0}^{\infty} \frac{1}{r^{2n} n^2}$$

Here $||f_{r,\theta}|| < \infty$, for all $r \geq 1$, this implies $f_{r,\theta} \in H^2(\mathbb{D})$. Since M is a well-defined operator, $M(f_{r,\theta}) \in H^2(\mathbb{D})$. But for that $x \in \mathbb{D}$, $M(f_{r,\theta})(x) = \sum \frac{1}{n}$, which is not defined. This contradicts the fact that M is well-defined. Now, we define the map $\phi : \mathbb{D} \to \mathbb{D}$ as $\phi(x) = M(z)(x)$, for $x \in \mathbb{D}$. Therefore, as defined above, every continuous multiplicative linear map can be seen as a composition operator C_ϕ for above defined ϕ. ∎

Note that the above theorem is also true for Bergman space, Dirichlet space, and many other spaces. For Bergman space $\mathcal{A}^2(\mathbb{D})$, define the function

$$f_{r,\theta} = \sum_{n=0}^{\infty} \frac{1}{r^n} e^{-in\theta} z^n \in \mathcal{A}^2(\mathbb{D}).$$

And for Dirichlet space $\mathcal{D}$, define the function

$$f_{r,\theta} = \sum_{n=0}^{\infty} \frac{1}{r^n} \frac{e^{-in\theta}}{n \log n} z^n \in \mathcal{D}.$$

***Theorem* 5.2** *Let $\mathcal{H}$ be an analytic Hilbert space over $\mathbb{D}$, and $\{h_n z^n\}$ be its orthonormal basis of $\mathcal{H}$, where $h_n \in \mathbb{C}$. If $|h_n| \geq \frac{1}{\sqrt{n}}$, then every continuous multiplicative linear operator $M : \mathcal{H} \to \mathcal{H}$, can be characterized as a composition operator C_ϕ, for some $\phi : \mathbb{D} \to \mathbb{D}$.*

Proof Let $\{h_n z^n\}$ be the orthonormal basis of $\mathcal{H}$, then a power series $f = \sum_{n=0}^{\infty} a_n z^n$ belongs to $\mathcal{H}$, if $||f|| = \sum_{n=0}^{\infty} \left|\frac{a_n}{h_n}\right|^2 < \infty$.

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Suppose if there exists $x \in X$, such that $(Mz)(x) = re^{i\theta}$, where $r \geq 1$, then define

$$\begin{aligned} f_{r,\theta} &= \sum_{n=0}^{\infty} \frac{e^{-i\theta}}{r^n n \log n} z^n \\ ||f_{r,\theta}|| &= \sum \frac{1}{r^{2n} n^2 \log^2 n |h_n|^2} \\ &\leq \sum \frac{n}{r^{2n} n^2 \log^2 n} \end{aligned}$$

this is a convergent series for any $r \geq 1$, implies $f_{r,\theta} \in \mathcal{H}$. But $M(f_{r,\theta}(x)) = \sum \frac{1}{n \log n}$ is not defined. This contradicts the fact that M is a well-defined operator. So range$(M) \subseteq \mathbb{D}$. Now, the proof follows as in Theorem 5.1. ■

Additionally, the condition $\forall\, n \in \mathbb{N}$, $|h_n| \geq \frac{1}{\sqrt{n}}$, can be weakened by assuming the inequality eventually.

It is worth noting that many well-known Hilbert spaces satisfy the above condition of orthonormal basis. For instance, in Hardy-Hilbert space $H^2(\mathbb{D})$, refer [14] the co-effecient $h_n = 1$. In the Dirichlet space $\mathcal{D}$, refer [9] the co-effecient $h_n = \frac{1}{\sqrt{n}}$. And in the weighted Bergman space $A^2_\alpha(\mathbb{D})$, refer [11] the co-effecient $h_n = \sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}}$. In each of these cases, the sequence h_n satisfies the hypothesis of Theorem 5.2.

In Section 2, we presented composition operators as examples of multiplicative operators. Now, we have proved that every continuous multiplicative operator belongs to the class of composition operators.

Also, note that our line of proof will not work for analytic Hilbert space $\mathcal{H}$ with $|h_n| \leq \frac{1}{n^{1+\epsilon}}$, for any $\epsilon > 0$. This is because, if $f = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}$, then

$$\sum_{n=0}^{\infty} \left|\frac{a_n}{h_n}\right|^2 < \infty,$$

Since this is a convergent series,

$$\begin{aligned} \left|\frac{a_n}{h_n}\right|^2 &< 1, \\ |a_n| \leq |h_n| &\leq \frac{1}{n^{1+\epsilon}}, \end{aligned}$$

For such a_n, $\sum |a_n|$ will be convergent. This implies that, for $r \geq 1$, we cannot find $f_{r,\theta} = \sum_{n=0}^{\infty} \frac{1}{r^n} a_n$, such that $\sum_{n=0}^{\infty} \left|\frac{a_n}{h_n}\right|^2 < \infty$, and $\sum_{n=0}^{\infty} |a_n|$ is divergent.

Note that for any $n \in \mathbb{N}$, and $\epsilon > 0$, the inequality $\frac{1}{\sqrt{n}} < \frac{1}{(n)^{1+\epsilon}}$ holds. We proved that for analytic Hilbert space with $h_n \leq \frac{1}{\sqrt{n}}$, every continuous multiplicative linear operator can be characterized as a composition operator C_ϕ, for some $\phi : \mathbb{D} \to \mathbb{D}$. Also, we showed that our line of proof will not work for $h_n \geq \frac{1}{(n)^{1+\epsilon}}$. But the question is still open for h_n between $\frac{1}{\sqrt{n}}$ and $\frac{1}{n}$.

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6 Conclusion

Our study establishes some analogies between semi-simple Banach algebras and RKHS with normalized complete Pick kernels through multiplicative linear map. Other Banach algebra results of multiplicative functionals, such as the Gelfand-Naimark construction, in the RKHS setting are still open for exploration.

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