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The Range of the Cesàro Operator Acting on H^∞

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Abstract. In 1993, N. Danikas and A. G. Siskakis showed that the Cesàro operator $\mathcal{C}$ is not bounded on H^∞; that is, $\mathcal{C}(H^\infty) \nsubseteq H^\infty$, but $\mathcal{C}(H^\infty)$ is a subset of $BMOA$. In 1997, M. Essén and J. Xiao gave that $\mathcal{C}(H^\infty) \subsetneqq \mathcal{Q}_p$ for every $0 < p < 1$. In this paper, we characterize positive Borel measures μ such that $\mathcal{C}(H^\infty) \subseteq M(\mathcal{D}_\mu)$ and show that $\mathcal{C}(H^\infty) \subsetneqq M(\mathcal{D}_{\mu_0}) \subsetneqq \bigcap_{0<p<\infty} \mathcal{Q}_p$ by constructing some measures μ_0. Here, $M(\mathcal{D}_\mu)$ denotes the Möbius invariant function space generated by $\mathcal{D}_\mu$, where $\mathcal{D}_\mu$ is a Dirichlet space with superharmonic weight induced by a positive Borel measure μ on the open unit disk. Our conclusions improve results mentioned above.

1 Introduction and Main Results

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $H(\mathbb{D})$ be the space of analytic functions in $\mathbb{D}$. For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ belonging to $H(\mathbb{D})$, the Cesàro operator $\mathcal{C}$ is defined by

$$(\mathcal{C}f)(z) = \sum_{n=0}^{\infty} \Big(\frac{1}{n+1} \sum_{k=0}^{n} a_k \Big) z^n.$$

The study of the Cesàro operator acting on various spaces of analytic functions in $\mathbb{D}$ has attracted a lot of attention (*cf.* [9, 10, 14, 18, 21, 22]).

For $0 < p < \infty$, the Hardy space H^p consists of those functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} \Big(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \Big)^{1/p} < \infty.$$

Denote by H^∞ the space of bounded analytic functions in $\mathbb{D}$. Namely, H^∞ consists of functions $f \in H(\mathbb{D})$ with

$$\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

Every function $f \in H^p$ has non-tangential limits $\widetilde{f}(\zeta)$ for almost every ζ on the unit circle $\partial\mathbb{D}$. See [11] for the theory of Hardy spaces. A. G. Siskakis [21, 22] showed that the Cesàro operator $\mathcal{C}$ is bounded on H^p for $p \ge 1$. J. Miao [19] proved that the same situation holds for $0 < p < 1$.

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Let $BMOA$ be the space of analytic functions on $\mathbb{D}$ whose boundary values are of bounded mean oscillation on $\partial\mathbb{D}$. The space $BMOA$ has its root in the corresponding space in harmonic analysis (see [17]). H^∞ is a subset of $BMOA$. It is well known (*cf.* [4, 15]) that $BMOA$ can be defined as the set of functions $f \in H^1$ satisfying that

$$\|f\|_{BMOA} = |f(0)| + \sup_{w\in\mathbb{D}} \Big(\frac{1}{2\pi} \int_{\partial\mathbb{D}} |\widetilde{f}(\zeta)|^2 \frac{1-|w|^2}{|w-\zeta|^2} dm(\zeta) - |f(w)|^2 \Big)^{1/2}$$

is finite. Here dm is the Lebesgue measure on $\partial\mathbb{D}$.

Let $\mathcal{C}(H^\infty)$ be the range of the Cesàro operator $\mathcal{C}$ acting on H^∞. N. Danikas and A. G. Siskakis [10] showed that $\mathcal{C}(H^\infty) \nsubseteq H^\infty$ but $\mathcal{C}(H^\infty) \subseteq BMOA$. They proved the following interesting result.

Theorem A *Suppose $f \in H^\infty$. Then $\mathcal{C}f \in BMOA$ and*

$$\|\mathcal{C}f\|_{BMOA} \le \Big(1 + \frac{\pi}{\sqrt{2}}\Big)\|f\|_\infty.$$

The constant $1 + \frac{\pi}{\sqrt{2}}$ is best possible.

Later, M. Essén and J. Xiao [14] studied the relation between $\mathcal{C}(H^\infty)$ and Möbius invariant $\mathcal{Q}_p$ spaces. Recall that the Möbius group $\mathrm{Aut}(\mathbb{D})$ is the set of one-to-one analytic functions mapping $\mathbb{D}$ onto itself. It is well known that each $\phi \in \mathrm{Aut}(\mathbb{D})$ can be written as

$$\phi(z) = e^{i\theta}\sigma_a(z), \quad \sigma_a(z) = \frac{a-z}{1-\overline{a}z},$$

where θ is real and $a \in \mathbb{D}$. In 1995, R. Aulaskari, J. Xiao, and R. Zhao [3] introduced $\mathcal{Q}_p$ spaces, which have attracted a lot of attention in recent years. For $0 < p < \infty$, the space $\mathcal{Q}_p$ consists of functions $f \in H(\mathbb{D})$ with

$$\|f\|^2_{\mathcal{Q}_p} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \big(1 - |\sigma_a(z)|^2\big)^p dA(z) < \infty,$$

where dA denotes the area measure on $\mathbb{D}$. The space $\mathcal{Q}_p$ is Möbius invariant in the sense that

$$\|f \circ \phi\|_{\mathcal{Q}_p} = \|f\|_{\mathcal{Q}_p}$$

for every $f \in \mathcal{Q}_p$ and $\phi \in \mathrm{Aut}(\mathbb{D})$. Clearly, $\mathcal{Q}_{p_1} \subseteq \mathcal{Q}_{p_2}$ for $0 < p_1 < p_2 < \infty$. It is known that for $1 < p < \infty$, all $\mathcal{Q}_p$ spaces are the same and equal to the Bloch space $\mathcal{B}$ consisting of functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.$$

Also, $\mathcal{Q}_1 = BMOA$ and $\mathcal{Q}_p \subsetneqq BMOA$ when $0 < p < 1$. We refer to J. Xiao's monographs [24, 25] for more results of $\mathcal{Q}_p$ spaces.

M. Essén and J. Xiao [14, Theorem 5.4] gave the relation between $\mathcal{C}(H^\infty)$ and $\mathcal{Q}_p$ spaces as follows.

Theorem B $\mathcal{C}(H^\infty) \subsetneqq \mathcal{Q}_p$ *for* $0 < p < 1$.

In this paper, we investigate $\mathcal{C}(H^\infty)$ further via some Möbius invariant spaces. In particular, we find certain Möbius invariant spaces locating strictly between $\mathcal{C}(H^\infty)$

and $\bigcap_{0<p<\infty} \mathcal{Q}_p$. These Möbius invariant spaces are related to some Dirichlet type spaces induced by superharmonic weights.

S. Richter [20] introduced Dirichlet spaces with harmonic weights. A. Aleman's work [2] initiated a study of Dirichlet spaces with superharmonic weights. Let μ be a positive Borel measure on $\mathbb{D}$. Denote by $\mathcal{D}_\mu$ the space of functions $f \in H(\mathbb{D})$ with

$$\int_{\mathbb{D}} |f'(z)|^2 U_\mu(z) dA(z) < +\infty,$$

where

$$U_\mu(z) = \int_{\mathbb{D}} \log \Big| \frac{1-\overline{w}z}{z-w} \Big| d\mu(w)$$

is a superharmonic function on $\mathbb{D}$. For the study of $\mathcal{D}_\mu$ spaces, we assume that $\int_{\mathbb{D}}(1-|z|^2)d\mu(z) < \infty$. Otherwise, the space $\mathcal{D}_\mu$ contains only constant functions. $\mathcal{D}_\mu$ spaces are always subsets of the Hardy space H^2 (cf. [2, 12]). Let $d\mu_p(z) = -\Delta[(1-|z|^2)^p]dA(z)$, $z \in \mathbb{D}$, $p \in (0,1)$, where Δ is the Laplace operator. By [1], the space $\mathcal{D}_{\mu_p}$ is equal to the radial Dirichlet type space $\mathcal{D}_p$ consisting of functions $f \in H(\mathbb{D})$ with

$$\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^p dA(z) < \infty.$$

The classical Dirichlet space $\mathcal{D}$ is the set of functions $f \in H(\mathbb{D})$ satisfying the formula above with $p = 0$. By [5, Corollary 5.6], there exists a positive Borel measure μ such that $\mathcal{D}_\mu$ is not equal to any generalized radial Dirichlet type space. We know from [5, Lemma 5.1] that every $\mathcal{D}_\mu$ space can also be defined as the class of functions $f \in H(\mathbb{D})$ for which

$$\|f\|^2_{\mathcal{D}_\mu} = \int_{\mathbb{D}} |f'(z)|^2 V_\mu(z) dA(z) < +\infty,$$

where

$$V_\mu(z) = \int_{\mathbb{D}} (1-|\sigma_z(w)|^2) d\mu(w).$$

Recently, G. Bao, J. Mashreghi, S. Pouliasis, and H. Wulan [8] investigated $M(\mathcal{D}_\mu)$, the Möbius invariant space generated by the space $\mathcal{D}_\mu$. Namely $M(\mathcal{D}_\mu)$ consists of functions $f \in H(\mathbb{D})$ with

$$\|f\|_{M(\mathcal{D}_\mu)} = \sup_{\phi \in \mathrm{Aut}(\mathbb{D})} \|f \circ \phi - f(\phi(0))\|_{\mathcal{D}_\mu} < \infty.$$

Equivalently,

$$\|f\|^2_{M(\mathcal{D}_\mu)} = \sup_{a\in\mathbb{D}, \lambda \in \partial\mathbb{D}} \int_{\mathbb{D}} |f'(w)|^2 V_\mu\big(\lambda \sigma_a(w)\big) dA(w).$$

We will say that $M(\mathcal{D}_\mu)$ is trivial if $M(\mathcal{D}_\mu)$ contains only constant functions. For example, set $d\nu(z) = (1-|z|)^{-3} dA(z)$. Then $M(\mathcal{D}_\nu)$ is trivial. In fact, it is known from [8] that if $M(\mathcal{D}_\mu)$ is not trivial, then $\mathcal{D} \subseteq M(\mathcal{D}_\mu) \subseteq BMOA$. Furthermore, $M(\mathcal{D}_\mu) = BMOA$ if and only if $\mu(\mathbb{D}) < \infty$. Let $d\mu_p(z) = -\Delta[(1-|z|^2)^p]dA(z)$ as before. Then $M(\mathcal{D}_{\mu_p}) = \mathcal{Q}_p$ when $0 < p < 1$.

For an increasing function $K\colon (0,1] \to [0,\infty)$, let $\mathcal{Q}_K$ be the space of functions $f \in H(\mathbb{D})$ for which

$$\|f\|^2_{\mathcal{Q}_K} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K\big(1-|\sigma_a(z)|^2\big) dA(z) < \infty.$$

If $K(t) = t^p$, then $\mathcal{Q}_K$ is the space $\mathcal{Q}_p$. See the monograph [23] for $\mathcal{Q}_K$ spaces. From [8, p. 5], if $K \in C^2(0,1]$ is increasing and concave on $(0,1]$ with $\lim_{t\to 0} K(t) = 0$, then $\mathcal{Q}_K = M(\mathcal{D}_\nu)$, where

$$d\nu(w) = -\Delta\big(K(1-|w|^2)\big)dA(w), \quad w \in \mathbb{D}.$$

The aim of this paper is to consider the relation between $\mathcal{C}(H^\infty)$ and $M(\mathcal{D}_\mu)$. In particular, we construct measures μ such that $\mathcal{C}(H^\infty) \subsetneqq M(\mathcal{D}_\mu) \subsetneqq \bigcap_{0<p<\infty} \mathcal{Q}_p$, which improves Theorems A and B.

Theorem 1.1 *Let μ be a positive Borel measure on $\mathbb{D}$. Then the following conditions are equivalent.*

(i) $\mathcal{C}(H^\infty) \subseteq M(\mathcal{D}_\mu)$.
(ii) $\log(1-z) \in M(\mathcal{D}_\mu)$.
(iii)

$$\sup_{\lambda\in\partial\mathbb{D}} \int_{\mathbb{D}} \frac{V_\mu(z)}{|1-\lambda z|^2} dA(z) < \infty. \tag{1.1}$$

(iv)

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{V_\mu(z)}{|1-az|^2} dA(z) < \infty. \tag{1.2}$$

Theorem 1.2 *Let*

$$d\mu_\alpha(z) = -\Delta\left(\frac{1}{\left(\log \frac{e^{1+\alpha}}{1-|z|^2}\right)^\alpha}\right) dA(z), \quad \alpha > 0, \quad z \in \mathbb{D}.$$

(i) *If $0 < \alpha \le 1$, then $M(\mathcal{D}_{\mu_\alpha})$ is not trivial and $\mathcal{C}(H^\infty) \not\subseteq M(\mathcal{D}_{\mu_\alpha})$.*
(ii) *If $\alpha > 1$, then $\mathcal{C}(H^\infty) \subsetneqq M(\mathcal{D}_{\mu_\alpha}) \subsetneqq \bigcap_{0<p<\infty} \mathcal{Q}_p$.*

Throughout this paper, the symbol $A \approx B$ means that $A \lesssim B \lesssim A$. We say that $A \lesssim B$ if there exists a constant C such that $A \le CB$.

2 Some Preliminary Results

In this section, we collect some results that will be used to prove Theorem 1.1 or Theorem 1.2.

Theorem C ([8, Theorem 3.3]) *Let μ be a positive Borel measure on $\mathbb{D}$. Then the following conditions are equivalent.*

(i) *$M(\mathcal{D}_\mu)$ is not trivial.*
(ii) $\mathcal{D} \subseteq M(\mathcal{D}_\mu)$.
(iii) $\mathcal{D} \subsetneqq \mathcal{D}_\mu$.
(iv) *$(1-|z|^2)d\mu(z)$ is a Carleson measure on $\mathbb{D}$,* i.e., $\sup_{w\in\mathbb{D}} V_\mu(w) < \infty$.

Lemma D ([6, Lemma 2.2]) *Let ν be a positive Borel measure on $\mathbb{D}$. Then*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{1-|z|^2}{|1-\overline{a}z|^2} d\nu(z) = \sup_{\zeta\in\partial\mathbb{D}} \int_{\mathbb{D}} \frac{1-|z|^2}{|1-\overline{\zeta}z|^2} d\nu(z).$$

Lemma E ([16, Theorem 1.7]) *Let $z \in \mathbb{D}$ and let β be any real number. Then*

$$\int_0^{2\pi} \frac{d\theta}{|1 - ze^{-i\theta}|^{1+\beta}} \approx \begin{cases} 1 & \text{if } \beta < 0, \\ \log \frac{1}{1-|z|^2} & \text{if } \beta = 0, \\ \frac{1}{(1-|z|^2)^\beta} & \text{if } \beta > 0, \end{cases}$$

as $|z| \to 1^-$.

The following result can be found in [8, p. 5].

Lemma F *Suppose $K \in C^2(0,1]$ is increasing and concave on $(0,1]$ with $\lim_{t\to 0} K(t) = 0$. Then $\mathcal{Q}_K = M(\mathcal{D}_\nu)$, where*

$$d\nu(w) = -\Delta\Big(K(1 - |w|^2)\Big)dA(w), \quad w \in \mathbb{D}.$$

We also need the following result on $\mathcal{Q}_K$ spaces, which is from [13, Theorem 2.6]. See [8, p. 10] for the corresponding result on $M(\mathcal{D}_\mu)$ spaces.

Theorem G *Suppose K_1 and K_2 are increasing and positive functions on $(0,1]$. Let $K_1(r)/K_2(r) \to 0$ as $r \to 0$ and let $\mathcal{Q}_{K_2} \neq \mathcal{B}$. Then $\mathcal{Q}_{K_2} \subsetneqq \mathcal{Q}_{K_1}$.*

3 Proof of Theorem 1.1

By [10, p. 295], if $f \in H(\mathbb{D})$, then $\mathcal{C}f$ also belongs to $H(\mathbb{D})$ and $\mathcal{C}f$ can be written as

$$(\mathcal{C}f)(z) = \frac{1}{z}\int_0^z \frac{f(\zeta)}{1-\zeta}d\zeta, \quad z \in \mathbb{D}.$$

For convenience, set

$$\frac{1}{2}\,\mathbb{D} = \Big\{z \in \mathbb{D} : 0 < |z| < \frac{1}{2}\Big\}.$$

(i)⇒(ii). Let $\mathcal{C}(H^\infty) \subseteq M(\mathcal{D}_\mu)$. Clearly, the function

$$(\mathcal{C}1)(z) = \frac{1}{z}\log\frac{1}{1-z}$$

belongs to $M(\mathcal{D}_\mu)$. By Theorem C,

$$\sup_{w\in\mathbb{D}} V_\mu(w) < \infty.$$

Consequently,

$$\begin{aligned} \infty &> \sup_{a\in\mathbb{D},\lambda\in\partial\mathbb{D}} \int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}} |(\mathcal{C}1)'(z)|^2 V_\mu\big(\lambda\sigma_a(z)\big)dA(z) \\ &= \sup_{a\in\mathbb{D},\lambda\in\partial\mathbb{D}} \int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}} \left|\frac{1}{z(1-z)} - \frac{1}{z^2}\log\frac{1}{1-z}\right|^2 V_\mu\big(\lambda\sigma_a(z)\big)dA(z). \end{aligned}$$

Note that

$$\sup_{a\in\mathbb{D},\lambda\in\partial\mathbb{D}} \int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}} \left|\frac{1}{z^2}\log\frac{1}{1-z}\right|^2 V_\mu(\lambda\sigma_a(z))dA(z) \lesssim$$
$$\sup_{w\in\mathbb{D}} V_\mu(w)\int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}} \left|\log\frac{1}{1-z}\right|^2 dA(z) < \infty.$$

Thus,

$$\sup_{a\in\mathbb{D},\lambda\in\partial\mathbb{D}}\int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}}\left|\frac{1}{1-z}\right|^2 V_\mu(\lambda\sigma_a(z))dA(z)<\infty,$$

which gives that $\log(1-z)\in M(\mathcal{D}_\mu)$.

(ii)⇒(iii). Suppose $\log(1-z)\in M(\mathcal{D}_\mu)$. Then

$$\sup_{a\in\mathbb{D},\lambda\in\partial\mathbb{D}}\int_{\mathbb{D}}\left|\frac{1}{1-\lambda\sigma_a(z)}\right|^2\frac{(1-|a|^2)^2}{|1-\overline{a}z|^4}V_\mu(z)dA(z)<\infty,$$

which yields

$$\sup_{\lambda\in\partial\mathbb{D}}\int_{\mathbb{D}}\frac{V_\mu(z)}{|1-\lambda z|^2}dA(z)<\infty.$$

(iii)⇔(iv). Set

$$d\nu(z)=\frac{V_\mu(z)}{1-|z|^2}dA(z),\quad z\in\mathbb{D},$$

in Lemma D. Then the desired result follows.

(iv)⇒(i). Suppose condition (1.2) holds. Then

$$\begin{aligned}\sup_{\phi\in\mathrm{Aut}(\mathbb{D})}\int_{\mathbb{D}}|\phi'(z)|^2V_\mu(z)dA(z)&=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|a|^2)^2}{|1-\overline{a}z|^4}V_\mu(z)dA(z)\\&\lesssim\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{V_\mu(z)}{|1-\overline{a}z|^2}dA(z)<\infty.\end{aligned}$$

This means that the identity function belongs to $M(\mathcal{D}_\mu)$. By Theorem C, we get that $\sup_{w\in\mathbb{D}}V_\mu(w)<\infty$.

Let $f\in H^\infty$. Write $g=\mathcal{C}f$ for convenience. Clearly, to show $g\in M(\mathcal{D}_\mu)$, it suffices to prove that

$$\sup_{\phi\in\mathrm{Aut}(\mathbb{D})}\int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}}|g'(z)|^2V_\mu(\phi(z))dA(z)<\infty. \tag{3.1}$$

Since

$$g(z)=\frac{1}{z}\int_0^z\frac{f(\zeta)}{1-\zeta}d\zeta,\quad z\in\mathbb{D},$$

we see that

$$g'(z)=\frac{f(z)}{z(1-z)}-\frac{g(z)}{z}$$

and

$$\begin{aligned}|g(z)|&\le\int_0^1\frac{|f(zt)|}{|1-zt|}dt\le\|f\|_\infty\int_0^1\frac{1}{1-|z|t}dt\\&=\|f\|_\infty\frac{1}{|z|}\log\frac{1}{1-|z|}.\end{aligned}$$

Thus, for any $\phi \in \mathrm{Aut}(\mathbb{D})$, we deduce that

$$\begin{aligned}
&\int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}} |g'(z)|^2 V_\mu\big(\phi(z)\big)dA(z)\\
&\quad\lesssim \int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}} \frac{|f(z)|^2}{|1-z|^2} V_\mu\big(\phi(z)\big)dA(z) + \int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}} |g(z)|^2 V_\mu\big(\phi(z)\big)dA(z)\\
&\quad\lesssim \|f\|_\infty^2 \int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}} \frac{1}{|1-z|^2} V_\mu\big(\phi(z)\big)dA(z)\\
&\qquad + \|f\|_\infty^2 \sup_{w\in\mathbb{D}} V_\mu(w) \int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}} \Big(\log\frac{1}{1-|z|}\Big)^2 dA(z)\\
&\quad\lesssim \|f\|_\infty^2 \int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}} \frac{1}{|1-z|^2} V_\mu\big(\phi(z)\big)dA(z) + \|f\|_\infty^2.
\end{aligned}$$

Consequently, (3.1) holds if

$$\sup_{\phi\in\mathrm{Aut}(\mathbb{D})} \int_{\mathbb{D}} \frac{1}{|1-z|^2} V_\mu(\phi(z))dA(z) < \infty;$$

that is,

$$I =: \sup_{a\in\mathbb{D},\lambda\in\partial\mathbb{D}} \int_{\mathbb{D}} \frac{1}{|1-\lambda\sigma_a(z)|^2} \frac{(1-|a|^2)^2}{|1-\overline{a}z|^4} V_\mu(z)dA(z) < \infty. \tag{3.2}$$

Since

$$1-\lambda\sigma_a(z) = (1-\lambda a)\frac{1+\frac{\lambda-\overline{a}}{1-\lambda a}z}{1-\overline{a}z},$$

we obtain

$$I = \sup_{a\in\mathbb{D},\lambda\in\partial\mathbb{D}} \frac{(1-|a|^2)^2}{|1-\lambda a|^2} \int_{\mathbb{D}} \frac{V_\mu(z)}{|1+\frac{\lambda-\overline{a}}{1-\lambda a}z|^2|1-\overline{a}z|^2} dA(z).$$

Set

$$\eta = \frac{\lambda-\overline{a}}{1-\lambda a}.$$

Then $|\eta| = 1$. By the change of variables, one gets

$$\begin{aligned}
I &= \sup_{a\in\mathbb{D},\lambda\in\partial\mathbb{D}} \frac{(1-|a|^2)^2}{|1-\lambda a|^2} \int_{\mathbb{D}} \frac{V_\mu(\overline{\eta}\zeta)}{|1+\zeta|^2|1-\overline{a\eta}\zeta|^2} dA(\zeta)\\
&= \sup_{a\in\mathbb{D},\lambda\in\partial\mathbb{D}} \int_{\mathbb{D}} V_\mu(\overline{\eta}\zeta) \left|\frac{1}{1+\zeta} + \frac{\overline{a\eta}}{1-\overline{a\eta}\zeta}\right|^2 dA(\zeta)\\
&\lesssim \sup_{\eta\in\partial\mathbb{D}} \int_{\mathbb{D}} \frac{V_\mu(\overline{\eta}\zeta)}{|1+\zeta|^2} dA(\zeta) + \sup_{a\in\mathbb{D},\lambda\in\partial\mathbb{D}} \int_{\mathbb{D}} \frac{V_\mu(\overline{\eta}\zeta)}{|1-\overline{a\eta}\zeta|^2} dA(\zeta)\\
&\approx \sup_{\eta\in\partial\mathbb{D}} \int_{\mathbb{D}} \frac{V_\mu(z)}{|1+\eta z|^2} dA(z) + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{V_\mu(z)}{|1-\overline{a}z|^2} dA(z).
\end{aligned}$$

Combining this with the validity of conditions (1.1) and (1.2), we obtain that (3.2) holds. Hence, (3.1) also holds. The proof of Theorem 1.1 is complete. ∎

Corollary 3.1 *Suppose $K \in C^2(0,1]$ is increasing and concave on $(0,1]$ with $\lim_{t\to 0} K(t) = 0$. Then the following conditions are equivalent:*

(i) $\mathcal{C}(H^\infty) \subseteq \mathcal{Q}_K$;

(ii) $\log(1-z) \in \mathcal{Q}_K$;

(iii) $\int_0^1 \frac{K(t)}{t} dt < \infty$.

Proof Let K satisfy the hypothesis in this corollary. By Lemma F, $\mathcal{Q}_K = M(\mathcal{D}_\nu)$, where

$$d\nu(w) = -\Delta\big(K(1-|w|^2)\big)dA(w), \quad w \in \mathbb{D}.$$

In fact, Green's Theorem (*cf.* [1, p. 99]) yields

$$K(1-|z|^2) = U_\nu(z), \quad z \in \mathbb{D}.$$

From Theorem 1.1, we know that both (i) and (ii) are equivalent to

$$\sup_{\lambda\in\partial\mathbb{D}} \int_{\mathbb{D}} \frac{V_\nu(z)}{|1-\lambda z|^2} dA(z) < \infty. \tag{3.3}$$

It follows from [7, p. 693] that (3.3) holds if and only if

$$\sup_{\lambda\in\partial\mathbb{D}} \int_{\mathbb{D}} \frac{U_\nu(z)}{|1-\lambda z|^2} dA(z) < \infty.$$

By Lemma E (see also [27, Lemma 3.10]),

$$\begin{aligned}
&\sup_{\lambda\in\partial\mathbb{D}} \int_{\mathbb{D}} \frac{K(1-|z|^2)}{|1-\lambda z|^2} dA(z) \\
&\quad = \frac{1}{\pi} \sup_{\lambda\in\partial\mathbb{D}} \int_0^1 K(1-r^2) r\,dr \int_0^{2\pi} \frac{1}{|1-\lambda r e^{i\theta}|^2} d\theta \\
&\quad \approx \int_0^1 \frac{K(1-r^2)r}{1-r^2} dr \approx \int_0^1 \frac{K(t)}{t} dt.
\end{aligned}$$

Thus, we obtain the desired result. ∎

4 Proof of Theorem 1.2

For $\alpha > 0$, set

$$K_\alpha(t) = \frac{1}{(\log \frac{e^{1+\alpha}}{t})^\alpha}, \quad t \in (0,1].$$

Then $K_\alpha \in C^2(0,1]$ and K_α is increasing and concave on $(0,1]$ with

$$\lim_{t\to 0} K_\alpha(t) = 0. \tag{4.1}$$

Note that

$$d\mu_\alpha(z) = -\Delta\Big(\frac{1}{\big(\log \frac{e^{1+\alpha}}{1-|z|^2}\big)^\alpha}\Big) dA(z), \quad z \in \mathbb{D}.$$

By Lemma F, $M(\mathcal{D}_{\mu_\alpha}) = \mathcal{Q}_{K_\alpha}$.

(i) For $0 < \alpha \le 1$,

$$\int_0^1 \frac{K_\alpha(t)}{t} dt = \infty.$$

By Corollary 3.1, $\mathcal{C}(H^\infty) \nsubseteq \mathcal{Q}_{K_\alpha}$. Because of (4.1), it follows from Theorem G that $\mathcal{D} \subsetneqq \mathcal{Q}_{K_\alpha}$ and hence $\mathcal{Q}_{K_\alpha}$ is not trivial.

(ii) For $\alpha > 1$,

$$\int_0^1 \frac{K_\alpha(t)}{t}\,dt < \infty.$$

By Corollary 3.1 again, $\mathcal{C}(H^\infty) \subseteq \mathcal{Q}_{K_\alpha}$. Note that

$$\lim_{t\to 0} \frac{K_{\alpha_1}(t)}{K_{\alpha_2}(t)} = 0, \quad \alpha_1 > \alpha_2 > 1,$$

and $M(\mathcal{D}_\mu) \subseteq BMOA \subsetneqq \mathcal{B}$ for all positive Borel measures μ. From Theorem G, one gets that $\mathcal{Q}_{K_{\alpha_2}} \subsetneqq \mathcal{Q}_{K_{\alpha_1}}$. Thus, $\mathcal{C}(H^\infty) \subsetneqq \mathcal{Q}_{K_\alpha}$. If $p > 0$, then

$$\lim_{t\to 0} \frac{t^p}{K_\alpha(t)} = 0.$$

By Theorem G again, $\mathcal{Q}_{K_\alpha} \subseteq \bigcap_{0<p<\infty} \mathcal{Q}_p$. Note that $\mathcal{Q}_{K_{\alpha_2}} \subsetneqq \mathcal{Q}_{K_{\alpha_1}}$ for $\alpha_1 > \alpha_2 > 1$. Thus, $\mathcal{Q}_{K_\alpha} \subsetneqq \bigcap_{0<p<\infty} \mathcal{Q}_p$. The proof of Theorem 1.2 is complete.

5 Final Remark

In the theory of Möbius invariant $\mathcal{Q}_p$ spaces, the fact that $\log(1-z) \in \mathcal{Q}_p$, $0 < p < 1$, plays certain role in the proofs of some important results (*cf.* [24–26]). As mentioned in Section 1, the space $\mathcal{Q}_p$, $0 < p < 1$, is a special case of $M(\mathcal{D}_\mu)$ spaces. Theorems 1.1 and 1.2 in this paper yield that there exist nontrivial $M(\mathcal{D}_\mu)$ spaces such that $\log(1-z) \notin M(\mathcal{D}_\mu)$. It is interesting to develop further the theory of this kind of $M(\mathcal{D}_\mu)$ spaces.

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