HILBERT SPACES OF GENERALIZED FUNCTIONS EXTENDING L^2, (I)

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1. Introduction

By using certain fractional integrals and derivatives it is possible to construct a continuum of Hilbert spaces within the space $L^2(0, \infty)$; these are the spaces $\mathscr{G}_\lambda$ of functions $f(x)$ for which [1] $x^\lambda f^{(\lambda)}(x) \in L^2(0, \infty)$, and they exhibit invariance properties under generalized Fourier transformations. They are described in (**6**) and (**7**).

It is possible also to extend the continuum beyond $L^2(0, \infty)$. This can appropriately be done by completing the spaces $\mathscr{G}_{-\lambda}$ of functions $f(x)$ for which $x^{-\lambda} f^{(-\lambda)}(x) \in L^2(0, \infty)$, $f^{(-\lambda)}$ for $\lambda > 0$ now denoting a λ-order integral of f. These complete spaces $\mathscr{H}_{-\lambda}$ form the subject of the present paper. The elements of the spaces are not always functions, but rather "generalized functions", akin to distributions; however, since the resulting theory is more specialized than that of Schwartz (**8**) and Temple (**9, 10**) and is closer to Love's (**4**), we shall follow Love in calling the elements "sequence-functions". Particular sequence-functions can be identified with ordinary functions: there results a continuum of Hilbert spaces of sequence-functions

$$\mathscr{G}_\mu \subset \mathscr{G}_\lambda \subset L^2(0, \infty) \subset \mathscr{H}_{-\lambda} \subset \mathscr{H}_{-\mu} \quad (\lambda < \mu), \tag{1.1}$$

each space forming in its successors a dense subset. And the invariance property holds throughout the continuum: Watson's generalized Fourier transformations in L^2 can be extended to the larger spaces so that each space is invariant with respect to these transformations, which include in particular the Fourier sine and cosine transformations. In fact, Watson kernels themselves appear as sequence-functions in $\mathscr{H}_{-1}$.

In this paper the spaces $\mathscr{H}$ are described, together with their principal properties (§§ 2—9) and their applicability to Watson transforms (§ 11); the latter are used in constructing a "delta function" for the spaces out of the "discontinuous-integral property" of Watson kernels (§ 10). Finally (§ 12), sequence-functions are compared with distributions.

[1] Here $f^{(\lambda)}(x)$ is not always the derivative of $f(x)$; see § 2, below.

We postpone discussion of continuity, differentiation, integration and multiplication of sequence-functions; and also of certain interesting series developments of sequence-functions in terms of Watson kernels, analogous to Fourier and Schlömilch series.

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2. Definitions

For given complex-valued functions $f(x)$, $f_1(x)$ defined in $0 < x < \infty$ we write

$$||f||_0 = \sqrt{\int_0^\infty |f(t)|^2 dt}, \qquad (f, f_1)_0 = \int_0^\infty f(t)\overline{f_1(t)}dt,$$

and say that $f \in L^2$ when $||f||_0 < \infty$.

We extend the notation of (6) to define for given $f(x)$ (for $x > 0$ and $\lambda > 0$) the functions $f^{(\pm\lambda)}$, $f^{[\pm\lambda]}$ by [2]

$$\text{(2.1)} \qquad \begin{aligned} f(x) &= \Gamma(\lambda)^{-1} \int_x^\infty (t-x)^{\lambda-1} f^{(\lambda)}(t)dt, \\ f(x) &= \Gamma(\lambda)^{-1} x^{-\lambda} \int_0^x (x-t)^{\lambda-1} t^\lambda f^{[\lambda]}(t)dt, \end{aligned}$$

$$\text{(2.2)} \qquad \begin{aligned} f^{(-\lambda)}(x) &= \Gamma(\lambda)^{-1} \int_0^x (x-t)^{\lambda-1} f(t)dt, \\ f^{[-\lambda]}(x) &= \Gamma(\lambda)^{-1} x^\lambda \int_x^\infty (t-x)^{\lambda-1} t^{-\lambda} f(t)dt. \end{aligned}$$

The integrals are presumed to be Lebesgue (absolutely convergent), at least for almost all x. We are principally concerned with $f^{(\lambda)}$ and $f^{(-\lambda)}$, which we call the λ-order derivative and integral of f, respectively; $f^{[\pm\lambda]}$ are "derivative" and "integral" in a less direct sense. The functions within the integrals are determined uniquely: cf. (3), § 5.

From these functions and the above norm we define four function spaces, as follows.

I. $f(x)$ belongs to the space $\mathscr{G}_\lambda$ $(\lambda > 0)$ whenever there exists some function $f^{(\lambda)}(t)$ for which $t^\lambda f^{(\lambda)}(t) \in L^2$; and $\mathscr{G}_0 \equiv L^2$.

II. $f(x)$ belongs to $\mathscr{G}_{-\lambda}$ $(\lambda > 0)$ whenever $f^{(-\lambda)}(x)$ exists almost everywhere (as a Lebesgue integral) and $x^{-\lambda} f^{(-\lambda)}(x) \in L^2$; $\mathscr{G}_{-0} \equiv \mathscr{G}_0 \equiv L^2$.

III. $f(x)$ belongs to $\mathscr{G}_{[\lambda]}$ $(\lambda > 0)$ whenever there exists some function $f^{[\lambda]}(t)$ for which $t^\lambda f^{[\lambda]}(t) \in L^2$; and $\mathscr{G}_{[0]} \equiv L^2$.

[2] Integrals of this type are discussed in Kober (3). We shall use $\Gamma(\lambda)^{-1}$ as an abbreviation for $\{\Gamma(\lambda)\}^{-1}$.

IV. $f(x)$ belongs to $\mathscr{G}_{[-\lambda]}$ $(\lambda > 0)$ whenever $x^{-\lambda} f^{[-\lambda]}(x)$ exists almost everywhere (as a Lebesgue integral) and $x^{-\lambda} f^{[-\lambda]}(x) \in L^2$; $\mathscr{G}_{[-0]} \equiv \mathscr{G}_{[0]} \equiv L^2$.

It is part of these definitions that the appropriate form from (2.1) or (2.2) holds, in each case. For example, if a function g has the property

$$t^l \frac{d^l}{dt^l} g(t) \in L^2, \text{ for some fixed positive integer } l,$$

and we define f by

$$f(x) = \Gamma(l)^{-1} \int_x^\infty (t-x)^{l-1} \frac{d^l}{dt^l} g(t) dt,$$

then

$$\frac{d^l}{dx^l} f(x) = (-1)^l \frac{d^l}{dx^l} g(x) = (-1)^l f^{(l)}(x),$$

so that $f(x)$ and $(-1)^l g(x)$ differ by a polynomial of degree not greater than $l-1$. Here f, but not necessarily g, belongs to $\mathscr{G}_l$: g "belongs to $\mathscr{G}_l$ to within a polynomial of degree $l-1$". Notice also that $f^{(l)}(x)$ is the actual lth derivative multiplied by $(-1)^l$.

Except where the contrary is stated, we shall henceforth assume λ to be real and positive.

3. Preliminary properties of the spaces

Absolute summability is specified in II, IV above in order to justify applications of Fubini's theory and Minkowski's inequalities which we make later. Thus definition II implies that a function of $\mathscr{G}_{-\lambda}$ necessarily belongs to $L(0, X)$ for all finite $X > 0$. In consequence, we have

LEMMA 1. *If $f \in \mathscr{G}_{-\lambda}$, then $f^{(-\alpha)}(x)$ for $\alpha > 0$ exists as a Lebesgue integral almost everywhere.*

Since $f(t) \in L(0, X)$ for $X > 0$, $|f(t)|(y-t)^\alpha$ is summable in $(0, y)$, and

$$\begin{aligned}\int_0^y |f(t)|(y-t)^\alpha dt &= \alpha \int_0^y |f(t)| dt \int_t^y (x-t)^{\alpha-1} dx \\ &= \alpha \int_0^y dx \int_0^x (x-t)^{\alpha-1} |f(t)| dt,\end{aligned}$$

by Tonelli's theorem (**5**, p. 145); the inside integral exists for almost all x, and so therefore does $f^{(-\alpha)}(x)$.

For these Riemann-Liouville integrals there is the iteration formula, valid if $\lambda < \mu$ and if the Lebesgue integral (2.2) for $f^{(-\mu)}$ exists,

$$(3.1)\quad f^{(-\mu)}(x) = \Gamma(\mu-\lambda)^{-1} \int_0^x (x-t)^{\mu-\lambda-1} f^{(-\lambda)}(t) dt = (f^{(-\lambda)})^{(-(\mu-\lambda))}(x).$$

If $f \in \mathscr{G}_{-\lambda}$ the integral converges absolutely, everywhere if $\mu - \lambda > \frac{1}{2}$, or almost everywhere if $0 < \mu - \lambda \leqq \frac{1}{2}$ (cf. (**6**), § 1). For derivatives there is the corresponding formula

$$f^{(\mu)}(x) = (f^{(\lambda)})^{(\mu-\lambda)}(x); \tag{3.2}$$

however, the formula connecting an integral $f^{(-\alpha)}$ and a derivative $f^{(\beta)}$, $\alpha, \beta > 0$, is far more complex. Iteration formulae for the functions $f^{[\pm\lambda]}$ take such forms as

$$\begin{aligned} x^{-2\lambda} f^{[-\mu]}(x) &= \left(x^{-2\lambda} f^{[-\lambda]}(x)\right)^{[-(\mu-\lambda)]}, \\ x^{2\lambda} f^{[\mu]}(x) &= \left(x^{2\lambda} f^{[\lambda]}(x)\right)^{[\mu-\lambda]}. \end{aligned} \tag{3.3}$$

We are interested principally in the spaces $\mathscr{G}_{\lambda}$, $\mathscr{G}_{-\lambda}$. It will become apparent that the properties of $\mathscr{G}_{[\lambda]}$, $\mathscr{G}_{[-\lambda]}$ respectively are similar. In fact (Kober (**3**), p. 207),

$$\mathscr{G}_{\lambda} = \mathscr{G}_{[\lambda]}; \tag{3.4}$$

and we shall show that the closures of $\mathscr{G}_{-\lambda}$ and $\mathscr{G}_{[-\lambda]}$ may be identified. Within each space we can define a scalar product and norm: for $\mathscr{G}_{\lambda}$, $\mathscr{G}_{-\lambda}$ we write

$$\begin{aligned} (f, g)_{\lambda} &= \left(t^{\lambda} f^{(\lambda)}(t), t^{\lambda} g^{(\lambda)}(t)\right)_0, \qquad ||f||_{\lambda} = ||t^{\lambda} f^{(\lambda)}(t)||_0, \\ (f, g)_{-\lambda} &= \left(x^{-\lambda} f^{(-\lambda)}(x), x^{-\lambda} g^{(-\lambda)}(x)\right)_0, \qquad ||f||_{-\lambda} = ||x^{-\lambda} f^{(-\lambda)}(x)||_0, \end{aligned} \tag{3.5}$$

with analogous definitions for the other spaces. With these, $\mathscr{G}_{\lambda}$ becomes a complete and separable Hilbert space; $\mathscr{G}_{-\lambda}$ on the other hand is incomplete (cf. § 10, below). This essential distinction between the two reflects the different placings of the L^2 restriction in definitions I and II: for the former space it is upon the integrand, whereas for the latter it is upon the integral. Similar remarks apply to $\mathscr{G}_{[\pm\lambda]}$.

4. Mellin transforms

The L^2 theory of Mellin transforms is applicable.[3] Let $f(x)$, $x^{\lambda} f^{(\lambda)}(x)$, $x^{\lambda} f^{[\lambda]}(x)$ have Mellin transforms $\mathfrak{F}(s)$, $\mathfrak{F}_{\lambda}(s)$, $\mathfrak{F}_{[\lambda]}(s)$ respectively, where $s = \frac{1}{2} + it$: e.g.

$$\mathfrak{F}(s) = \underset{X\to\infty}{\text{l.i.m.}} \int_{1/X}^{X} f(x) x^{s-1}\, dx, \quad f(x) = \frac{1}{2\pi i} \underset{T\to\infty}{\text{l.i.m.}} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \mathfrak{F}(s) x^{-s} ds.$$

It can be shown that

$$\mathfrak{F}_{\lambda}(s) = \frac{\Gamma(\lambda+s)}{\Gamma(s)} \mathfrak{F}(s), \qquad \mathfrak{F}_{[\lambda]}(s) = \frac{\Gamma(\lambda+1-s)}{\Gamma(1-s)} \mathfrak{F}(s), \tag{4.1}$$

[3] For the theory and notation see Titchmarsh (**11**), pp. 7, 94—5. We assume here that $\Re(s) = \frac{1}{2}$. By "Mellin transforms of L^2" we imply $f(x) \in L^2(0, \infty)$, $\mathfrak{F}(\frac{1}{2} + it) \in L^2(-\infty, \infty)$, etc.

if $f \in \mathscr{G}_\lambda$ (or $\mathscr{G}_{[\lambda]}$); a proof for $\lambda > 0$ follows the method of (**3**), Theorem 5 (a), while for $\lambda > \frac{1}{2}$ the Mellin Parseval relation suffices.

If f is a real function of $\mathscr{G}_\lambda$, the latter relation and (4.1) give

$$||f||_\lambda^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathfrak{F}_\lambda(\tfrac{1}{2}+it)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathfrak{F}_{[\lambda]}(\tfrac{1}{2}+it)|^2 dt = ||f||_{[\lambda]}^2;$$

that is,

$$(4.2) \qquad ||f||_\lambda = ||f||_{[\lambda]}.$$

Statement (3.4) is justified in this manner.

If instead $f \in \mathscr{G}_{-\lambda}$, then $x^{-\lambda} f^{(-\lambda)}(x)$ necessarily has a Mellin transform $\mathfrak{F}_{-\lambda}(s)$ of L^2, but $f(x)$ may have none. If $\mathfrak{F}(s)$ does exist in some sense, then we should expect

$$(4.3) \qquad \mathfrak{F}_{-\lambda}(s) = \frac{\Gamma(1-s)}{\Gamma(\lambda+1-s)}\mathfrak{F}(s), \quad \mathfrak{F}_{[-\lambda]}(s) = \frac{\Gamma(s)}{\Gamma(\lambda+s)}\mathfrak{F}(s).$$

These formulae are valid in particular if $f \in L^2$ (proof as for (4.1)): then clearly $x^{-\lambda} f^{(-\lambda)}(x) \in \mathscr{G}_{[\lambda]}$, $x^{-\lambda} f^{[-\lambda]}(x) \in \mathscr{G}_\lambda$; moreover, as above we have

$$(4.4) \qquad ||f||_{-\lambda} = ||f||_{[-\lambda]} \quad (f \in L^2).$$

5. Ordering among the spaces

We order the spaces $\mathscr{G}$ by means of inequalities among their norms. It is true by definition that

$$||x^\lambda f^{(\lambda)}(x)||_0 = ||f||_\lambda, \qquad ||x^{-\lambda} f^{(-\lambda)}(x)||_0 = ||f||_{-\lambda};$$

more generally we have

LEMMA 2. *Let* $0 \leqq \lambda < \mu$ *and* $f \not\equiv 0$. *The inequalities*

$$(5.1) \qquad ||f||_\lambda < \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})}||f||_\mu, \qquad ||f||_{-\mu} < \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})}||f||_{-\lambda}$$

are valid whenever the righthand sides exist. The same inequalities hold for the norms of $\mathscr{G}_{[\alpha]}$, $\mathscr{G}_{[-\alpha]}$.

For a proof using an integral-integral form of Minkowski's inequality cf. (**6**), Theorem 3; see also (**2**), Theorems 202, 329.

THEOREM 1. *The functions spaces* $\mathscr{G}$ *form a system satisfying*

$$(5.2) \qquad \mathscr{G}_\mu \subset \mathscr{G}_\lambda \subset L^2 \subset \mathscr{G}_{-\lambda} \subset \mathscr{G}_{-\mu}$$

for $0 < \lambda < \mu$.

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To show for example that $\mathscr{G}_{-\lambda} \subseteqq \mathscr{G}_{-\mu}$, suppose $f \in \mathscr{G}_{-\lambda}$; then $||f||_{-\lambda} < \infty$, and so by (5.1), $||f||_{-\mu} < \infty$, whence $f \in \mathscr{G}_{-\mu}$. As evidence that the spaces are properly contained, it is sufficient to observe the examples

$$\begin{aligned} c(x) &= x^{-\frac{1}{2}\nu} J_\nu(2x^{\frac{1}{2}}), \quad c^{(\lambda)}(x) = x^{-\frac{1}{2}(\lambda+\nu)} J_{\lambda+\nu}(2x^{\frac{1}{2}}), \\ d(x) &= x^{\frac{1}{2}\nu} J_\nu(2x^{\frac{1}{2}}), \quad d^{(-\lambda)}(x) = x^{\frac{1}{2}(\lambda+\nu)} J_{\lambda+\nu}(2x^{\frac{1}{2}}). \end{aligned} \tag{5.3}$$

It may be verified that

$$c \in \mathscr{G}_\lambda \text{ for } 0 < \lambda < \nu - \tfrac{1}{2}, \text{ but not otherwise,}$$
$$d \in \mathscr{G}_{-\lambda} \text{ for } 0 < \nu + \tfrac{1}{2} < \lambda, \text{ but not otherwise.}$$

We add here one further useful result.

LEMMA 3 (Fractional integration by parts). *The formula*

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty f^{(-\lambda)}(x)g^{(\lambda)}(x)dx \tag{5.4}$$

is valid if $f \in L^2$, $g \in \mathscr{G}_\lambda$, $\lambda > 0$. *A similar formula holds with* "$[\pm\lambda]$" *in place* of "$(\pm\lambda)$".

The conditions are those given by Kober (**3**, (3.61)). To prove (5.4) we notice that the lefthand side equals

$$\Gamma(\lambda)^{-1}\int_0^\infty f(x)dx\int_x^\infty (t-x)^{\lambda-1} g^{(\lambda)}(t)dt, = \Gamma(\lambda)^{-1}\int_0^\infty g^{(\lambda)}(t)dt\int_0^t (t-x)^{\lambda-1} f(x)dx,$$

which is the righthand side. This uses Fubini's theorem; the repeated integrals are seen to be absolutely convergent thus:

$$\begin{aligned} &\int_0^\infty |f(x)|dx\int_x^\infty (t-x)^{\lambda-1}|g^{(\lambda)}(t)|dt \\ &\quad \leqq \left\{\int_0^\infty |f(x)|^2dx\right\}^{\frac{1}{2}} \cdot \left\{\int_0^\infty dx\left(x^\lambda\int_1^\infty (u-1)^{\lambda-1}|g^{(\lambda)}(xu)|du\right)^2\right\}^{\frac{1}{2}} \\ &\quad \leqq ||f||_0 \cdot \int_1^\infty (u-1)^{\lambda-1}du\left\{\int_0^\infty |x^\lambda g^{(\lambda)}(xu)|^2dx\right\}^{\frac{1}{2}} < \infty. \end{aligned}$$

An immediate consequence of the lemma and of Schwarz' inequality is

$$|(f, g)_0| \leqq ||f||_{-\lambda} \cdot ||g||_\lambda. \tag{5.5}$$

6. Completion of $\mathscr{G}_{-\lambda}$ to $\mathscr{H}_{-\lambda}$

Let us call a sequence (f_n) of functions of $\mathscr{G}_{-\lambda}$ (λ fixed) a *Cauchy sequence* if

$$||f_n - f_m||_{-\lambda} \to 0 \text{ as } \min(n, m) \to \infty.$$

Following Love, we complete $\mathscr{G}_{-\lambda}$ by means of Cauchy sequences; that is, we construct a *strong completion*. The process is an example of the completion of a metric space: for the general theory see (**1**), pp. 81—7. First the set of sequences is made a vector space with scalar product, by defining the sum

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and scalar multiple by

$$(f_n) + (g_n) = (f_n + g_n), \qquad \alpha(f_n) = (\alpha f_n)$$

and the scalar product and norm by

$$((f_n), (g_n))_{-\lambda} = \lim_{n\to\infty} (f_n, g_n)_{-\lambda}, \quad ||(f_n)||_{-\lambda} = \lim_{n\to\infty} ||f_n||_{-\lambda},$$

these definitions being meaningful. Next, since more than one sequence may converge to the same limit (when this exists), we use the equivalence relation

$$(f_n) \sim (g_n) \text{ when } ||(f_n) - (g_n)||_{-\lambda} = 0$$

to separate the sequences into mutually exclusive equivalence classes. A class F is determined uniquely by any member sequence (f_n); two equivalent sequences determine the same class. We call these classes *sequence-functions*, and denote them generally by small capitals. We write $\mathscr{H}_{-\lambda}$ for the space of sequence-functions.

The sum, scalar multiple, scalar product and norm are then defined in $\mathscr{H}_{-\lambda}$ in the obvious manner: if $(f_n) \in F$, $(g_n) \in G$, we take for $F + G$ and αF the classes determined by the sequences $(f_n) + (g_n)$ and $\alpha(f_n)$ respectively; and write

$$(F, G)_{-\lambda} = ((f_n), (g_n))_{-\lambda} = \lim_{n\to\infty} (f_n, g_n)_{-\lambda},$$
$$||F||_{-\lambda} = ||(f_n)||_{-\lambda} = \lim_{n\to\infty} ||f_n||_{-\lambda}.$$

These definitions are unique, and independent of the particular class members used. The existence of the last limit, for example, follows from

$$(6.1) \qquad \big| ||f_n||_{-\lambda} - ||f_m||_{-\lambda} \big| \leqq ||f_n - f_m||_{-\lambda},$$

and its uniqueness is proved similarly. In addition, we define $F(\alpha x)$ for a positive number α by the sequence $(f_n(\alpha x))$, which is Cauchy, as the identity $||g(\alpha x)||_{-\lambda} = \alpha^{-\frac{1}{2}}||g(x)||_{-\lambda}$ shows.

Among the Cauchy sequences in $\mathscr{G}_{-\lambda}$ are the *principal sequences*, $f, f, f, \cdots$ formed by repetition of a single element of $\mathscr{G}_{-\lambda}$; these we distinguish by the notation $\{f\}$. A sequence-function possessing a principal sequence is called *principal*. It is easily seen that *a sequence-function F is principal if and only if all its sequences converge to an element of $\mathscr{G}_{-\lambda}$*. Certainly if it possesses one such sequence, converging to $f \in \mathscr{G}_{-\lambda}$, it also possesses the principal sequence $\{f\}$, and any other sequence (f'_n) of F converges to the same limit, since by the definition of equivalence,

$$\lim_{n\to\infty} ||f'_n - f||_{-\lambda} = ||(f'_n) - \{f\}||_{-\lambda} = 0.$$

We shall identify such a principal sequence-function with its limit, and thus write

$$\mathscr{G}_{-\lambda} \subseteqq \mathscr{H}_{-\lambda} \quad (\lambda > 0), \qquad \mathscr{G}_0 \equiv \mathscr{H}_0.$$

It is then permissible to say, when $(f_n) \in F$, that $||f_n - F||_{-\lambda} \to 0$ (the norm being that of $\mathscr{H}_{-\lambda}$); for if also $(f'_n) \in F$,

$$\begin{aligned} ||f_n - F||_{-\lambda} &= ||\{f_n\} - (f'_m)||_{-\lambda} \\ &= \lim_{m\to\infty} ||f_n - f'_m||_{-\lambda} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

This limit in $\mathscr{H}_{-\lambda}$ will also be shown as

$$F = \lim_{n\to\infty\,(-\lambda)} f_n$$

(in particular, $\lim_{(-0)}$ is the limit in mean square, "l.i.m."). If $||f_n||_{-\lambda}$ converges to 0, $||f_n - 0||_{-\lambda} \to 0$, then $(f_n) \sim \{0\}$; $O = \{0\}$ is a principal sequence-function, and $||F||_{-\lambda} = 0$ if and only if $F = O$.

We know from the general theory that $\mathscr{H}_{-\lambda}$ is complete and contains $\mathscr{G}_{-\lambda}$ as a dense subset. The scalar product and norm (which fulfil the usual requirements) make $\mathscr{H}_{-\lambda}$ a Hilbert space. The space $\mathscr{G}_{[-\lambda]}$ can likewise be completed to a Hilbert space $\mathscr{H}_{[-\lambda]}$; later we put $\mathscr{H}_{-\lambda} = \mathscr{H}_{[-\lambda]}$.

By writing $f = f_n - f_m$ in Lemma 2, we see that a sequence which is Cauchy in $\mathscr{G}_{-\lambda}$ is also Cauchy in $\mathscr{G}_{-\mu}$, for $\mu > \lambda$: hence we identify the sequence-functions defined by it in $\mathscr{H}_{-\lambda}$ and $\mathscr{H}_{-\mu}$, and write

$$\mathscr{H}_{-\lambda} \subset \mathscr{H}_{-\mu}.$$

(It may be verified that the elements of $\mathscr{G}_{-\mu}$ which are not in $\mathscr{G}_{-\lambda}$ define principal sequence-functions in $\mathscr{H}_{-\mu}$ which do not belong to $\mathscr{H}_{-\lambda}$; thus $\mathscr{H}_{-\lambda}$ is properly contained.) In particular, if F of $\mathscr{H}_{-\lambda}$ has a defining sequence converging in L^2, we say $F \in L^2$. The *order* of a sequence-function F is the greatest lowest bound of numbers ω for which $||g_n - g_m||_{-\omega} \to 0$ for some sequence $(g_n) \in F$.

7. Isometries among the spaces

By the definition of $\mathscr{H}_{-\lambda}$, for every sequence-function $F = (f_n)$ the sequence $(x^{-\lambda} f_n^{(-\lambda)}(x))$ converges in L^2 and so defines a principal sequence-function, which it is appropriate to write

$$x^{-\lambda} F^{(-\lambda)}(x) = \lim_{n\to\infty\,(0)} x^{-\lambda} f_n^{(-\lambda)}(x). \tag{7.1}$$

This defines for almost all positive x a (not necessarily sequence-) function $F^{(-\lambda)}$, which coincides with the λ-integral of F when $F \in \mathscr{G}_{-\lambda}$. (A sequence-function $F(x)$, like a distribution, does not necessarily possess numerical values for all x. The class of those which are "numerical" in this sense clearly

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contains the principal sequence-functions; so that $x^{-\lambda}F^{(-\lambda)}(x)$ is numerical.)

Equation (7.1) implies that $\mathscr{H}_{-\lambda}$ can be mapped onto L^2 and $\mathscr{G}_\lambda$. We prove

THEOREM 2. *The relations*

$$x^{-\lambda}F^{(-\lambda)}(x) = \phi(x) = x^\lambda g^\lambda(x)$$

among elements $F \in \mathscr{H}_{-\lambda}$, $\phi \in L^2$, $g \in \mathscr{G}_\lambda$ *determine isometries*

$$X_\lambda : \mathscr{H}_{-\lambda} \cong L^2, \qquad Y_\lambda : L^2 \cong \mathscr{G}_\lambda, \qquad Z_\lambda : \mathscr{H}_{-\lambda} \cong \mathscr{G}_\lambda$$

among the spaces by

$$\phi = X_\lambda F, \qquad g = Y_\lambda \phi, \qquad g = Z_\lambda F:$$

that is, linear biunique correspondences for which

$$(F_1, F_2)_{-\lambda} = (\phi_1, \phi_2)_0 = (g_1, g_2)_\lambda.$$

The isometric nature of Y_λ is implied in the definition of $\mathscr{G}_\lambda$. We remarked above that X_λ maps uniquely; it remains to show that X_λ^{-1} is also unique, i.e. that ϕ determines F uniquely. For this purpose we assume the identity of $\mathscr{G}_\lambda$ and $\mathscr{G}_{[\lambda]}$.

It is known [4] that $\mathscr{G}_\lambda$ is dense in L^2; thus we can find a sequence (ϕ_n), $\phi_n \in \mathscr{G}_\lambda$ for all n, for which $||\phi - \phi_n||_0 \to 0$ as $n \to \infty$. Now each ϕ_n can be written, by (3.4),

$$\phi_n(x) = \Gamma(\lambda)^{-1}x^{-\lambda}\int_0^x (x-t)^{\lambda-1}t^\lambda \phi_n^{[\lambda]}(t)dt.$$

Put $t^\lambda\phi_n^{[\lambda]}(t) = f_n(t)$, so that $\phi_n(x) = x^{-\lambda}f_n^{(-\lambda)}(x)$ and $f_n \in L^2$. Since $||\phi - \phi_n||_0 \to 0$,

$$||f_n - f_m||_{-\lambda} = ||\phi_n - \phi_m||_0 \to 0 \quad \text{as } n, m \to \infty,$$

and (f_n) defines a sequence-function F, for which

$$||\phi(x) - x^{-\lambda}f_n^{(-\lambda)}(x)||_0 \to 0;$$

that is, $\phi(x) \equiv x^{-\lambda}F^{(-\lambda)}(x)$. Thus every ϕ of L^2 is the image under X_λ of an F of $\mathscr{H}_{-\lambda}$. The equalities among the scalar products are obvious. Therefore if F_1 and F_2 both determine ϕ, $||F_1 - F_2||_{-\lambda} = 0$ and $F_1 = F_2$: X_λ^{-1} is unique. Since $Z_\lambda = Y_\lambda X_\lambda$, Z_λ and Z_λ^{-1} are likewise isometric.

We notice that if $f \in L^2$, $\phi(x) = x^{-\lambda}f^{(-\lambda)}(x)$ belongs to $\mathscr{G}_{[\lambda]}$; i.e. X_λ maps the subset L^2 onto $\mathscr{G}_{[\lambda]}$ when taken as an endomorphism in $\mathscr{H}_{-\lambda}$. But X_λ and Y_λ do not coincide in L^2, for

$$X_\lambda f(x) = x^{-\lambda}f^{(-\lambda)}(x), \qquad Y_\lambda f(x) = x^{-\lambda}f^{[-\lambda]}(x).$$

Consider the analogous correspondences defined in $\mathscr{H}_{[-\lambda]}$ by

$$x^{-\lambda}F^{[-\lambda]}(x) = \phi(x) = x^\lambda g^{[\lambda]}(x).$$

[4] Kober (3), Theorem 6(b).

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Examination of the integral forms of § 2 shows that these can be written $\phi = Y_\lambda F$, $g = X_\lambda \phi$, giving an extension of Y_λ from L^2 to $\mathscr{H}_{[-\lambda]}$:

$$X_\lambda : L^2 \cong \mathscr{G}_{[\lambda]}, \qquad Y_\lambda : \mathscr{H}_{[-\lambda]} \cong L^2.$$

By the previous argument we can show that Y_λ^{-1} maps L^2 onto $\mathscr{H}_{[-\lambda]}$ uniquely. Then $Y_\lambda^{-1} X_\lambda$ determines a mapping of $\mathscr{H}_{-\lambda}$ onto $\mathscr{H}_{[-\lambda]}$ which is again an isometry, and maps subset L^2 onto L^2.

It is now easily seen that $\mathscr{H}_{-\lambda}$ is a separable Hilbert space, and that L^2 is dense in $\mathscr{H}_{-\lambda}$. The first is a consequence of the isometry X_λ^{-1} and the separability of the Hilbert space L^2. The second follows by noting that X_λ^{-1} maps $\mathscr{G}_\lambda$ onto L^2, and that $\mathscr{G}_\lambda$ is dense in L^2.

A corollary is that $\mathscr{G}_\mu$ is dense in $\mathscr{H}_{-\lambda}$; for F in $\mathscr{H}_{-\lambda}$ can be approximated in $\mathscr{H}_{-\lambda}$ by a sequence (f_n) from L^2, and each f_n can be approximated in L^2 by a sequence $(g_{n,m})$ from $\mathscr{G}_\mu$. Since convergence in L^2 implies convergence in $\mathscr{H}_{-\lambda}$ (cf. Lemma 2), a subsequence of $(g_{n,m})$ can be chosen converging in $\mathscr{H}_{-\lambda}$ to F. The property of denseness allows an arbitrary sequence-function to be specified by a sequence of functions all of L^2—"(f_n), $f_n \in L^2$"—or even of $\mathscr{G}_\mu$. This is the principal means by which we extend operations from L^2 to $\mathscr{H}_{-\lambda}$. For many purposes it is preferable to regard $\mathscr{H}_{-\lambda}$ as a completion of L^2 rather than of $\mathscr{G}_{-\lambda}$. We summarize these results in

THEOREM 3. *For a fixed positive* λ, $\mathscr{H}_{-\lambda}$ *is a complete and separable Hilbert space; and for* $0 < \lambda < \mu$,

$$\mathscr{G}_\mu \subset \mathscr{G}_\lambda \subset L^2 \subset \mathscr{H}_{-\lambda} \subset \mathscr{H}_{-\mu}, \tag{7.2}$$

each space being dense in every succeeding space (with the norm of the latter).

Let us now identify $\mathscr{H}_{-\lambda}$ and $\mathscr{H}_{[-\lambda]}$ other than by the isometry $Y_\lambda^{-1} X_\lambda$, as follows. Let $F \in \mathscr{H}_{-\lambda}$, and define F by a sequence (f_n), $f_n \in L^2$. Then by (4.4),

$$||f_n - f_m||_{[-\lambda]} = ||f_n - f_m||_{-\lambda} \to 0 \quad \text{as } n, m \to \infty,$$

and (f_n) defines a sequence-function $\tilde{F}$ in $\mathscr{H}_{[-\lambda]}$. Also $||f_n||_{[-\lambda]} = ||f_n||_{-\lambda}$ for all n, and since by (6.1) the limit of each side exists, we find on taking the limit that $||\tilde{F}||_{[-\lambda]} = ||F||_{-\lambda}$. If $F \in L^2$ then

$$||F - f_n||_{[-\lambda]} = ||F - f_n||_{-\lambda} \to 0$$

and $\tilde{F} = F$. We identify $\tilde{F}$ and F for the whole spaces and write

$$\mathscr{H}_{-\lambda} = \mathscr{H}_{[-\lambda]}. \tag{7.3}$$

The relation $||F||_{-\lambda} = ||F||_{[-\lambda]}$ is now valid throughout.

In (7.1) the λ-order integral of a sequence-function was defined. It is possible to define integrals of any other order, but discussion of these and of derivatives is deferred to a later paper. We turn to the integral $(F, g)_0$, defined for $F \in \mathscr{H}_{-\lambda}$.

8. The scalar product between $\mathscr{H}_{-\lambda}$ and $\mathscr{G}_{\lambda}$

The scalar product $(f, g)_0$, which of course is distinct from the scalar products of the Hilbert spaces $\mathscr{H}_{-\lambda}$ and $\mathscr{G}_{\lambda}$, can be given an extended meaning if one member belongs to the first space and the other to the second; in this way the notion of weak convergence can be introduced. Let F be an arbitrary sequence-function of $\mathscr{H}_{-\lambda}$ defined by (f_n), $f_n \in L^2$: the limit

$$\lim_{n\to\infty} (f_n, g)_0 = \lim_{n\to\infty} \left(x^{-\lambda} f_n^{(-\lambda)}(x), x^{\lambda} g^{(\lambda)}(x)\right)_0 = \int_0^\infty x^{-\lambda} F^{(-\lambda)}(x) x^{\lambda} \bar{g}^{(\lambda)}(x) dx$$

exists if $g \in \mathscr{G}_{\lambda}$, by virtue of Kober's formula (Lemma 3) and the properties of mean convergence. We take this limit as a definition of the scalar product:

$$(8.1) \qquad (F, g)_0 = \int_0^\infty F(x)\bar{g}(x)dx = \lim_{n\to\infty} (f_n, g)_0 = \int_0^\infty x^{-\lambda} F^{(-\lambda)}(x) x^{\lambda} \bar{g}^{(\lambda)}(x) dx.$$

The definition coincides with the ordinary one if $F \in L^2$, and is consistent when $F \in \mathscr{H}_{-\kappa}$, $0 < \kappa < \lambda$.

9. Orthonormal sets in $\mathscr{H}_{-\lambda}$

Any (complete) orthonormal set $\{\alpha_n(x)\}$ of L^2 determines a (complete) orthonormal set $\{\Phi_n(x)\}$ in $\mathscr{H}_{-\lambda}$ by

$$\alpha_n(x) = X_\lambda \Phi_n(x) = x^{-\lambda} \Phi_n^{(-\lambda)}(x),$$

for which

$$(\alpha_n, \alpha_m)_0 = (\Phi_n, \Phi_m)_{-\lambda} = \delta_{n,m}.$$

A particular orthogonal set is found by writing

$$x^{-\lambda} \Phi_n^{(-\lambda)}(x) = e^{-\frac{1}{2}x} x^{\lambda} L_n^{(2\lambda)}(x).$$

The theory is an inversion of that for $\mathscr{G}_{\lambda}$, given in (**6**), § 6. Using the same notation [5] we find that

$$\Phi_n(x) = \frac{\Gamma(n+2\lambda+1)}{\Gamma(n+1)} K_n^{\lambda}(x), \qquad (\Phi_n, \Phi_m)_{-\lambda} = \frac{\Gamma(n+2\lambda+1)}{\Gamma(n+1)} \delta_{n,m},$$

and can prove

THEOREM 4. *If $F \in \mathscr{H}_{-\lambda}$ for some λ in $\frac{1}{2} < \lambda < 1$, then there is an expansion*

$$F(x) = \lim_{N\to\infty\,(-\lambda)} \sum_{n=0}^{N} a_n K_n^{\lambda}(x),$$

the coefficients being given by

$$a_n = 2^{\lambda} \int_0^\infty F(x) e^{-\frac{1}{2}x} M_n^{\lambda}(x) dx.$$

(In $\mathscr{G}_{\lambda}$ the roles of K_n^{λ} and M_n^{λ} were reversed.)

[5] In (**6**) the factor $\{\Gamma(\lambda)\}^{-1}$ in the expression for $f(x)$ in Theorem 15 should be deleted.

10. A delta sequence-function

It has not yet been verified that $\mathscr{G}_{-\lambda}$ is incomplete and actually distinct from $\mathscr{H}_{-\lambda}$. We proceed to do this (at least for $\lambda \geqq 1$) by constructing a Cauchy sequence in $\mathscr{G}_{-\lambda}$ whose limit is a Dirac delta sequence-function.

Let k be a generalized Fourier kernel of Watson type, generating in L^2 the involutory transformation [6]

$$g(x) = \underset{n\to\infty}{\text{l.i.m.}} \int_0^n k(xt)f(t)dt, \tag{10.1}$$

and write

$$\delta_n(x, \xi) = \int_0^n k(xt)k(\xi t)dt, \tag{10.2}$$

supposing $\xi > 0$. We prove

THEOREM 5. *For $\lambda \geqq 1$ and for suitable kernel k, we have*

$$\underset{n\to\infty}{\text{l.i.m.}}\, x^{-\lambda}\delta_n^{(-\lambda)}(x, \xi) = \begin{cases} 0 & (0 < x < \xi), \\ \Gamma(\lambda)^{-1}(x-\xi)^{\lambda-1}x^{-\lambda} & (x > \xi), \end{cases} \tag{10.3}$$

and the sequence (δ_n) determines a sequence-function Δ_ξ in $\mathscr{H}_{-\lambda}$ with the property

$$\int_0^\infty \Delta_\xi(x)\phi(x)dx = \phi(\xi) \text{ for all } \phi \in \mathscr{G}_\lambda. \tag{10.4}$$

Sufficient conditions on k for these formulae to hold are: (i) *$k(x)$ continuous in $0 \leqq x < \infty$*, (ii) *$k^{[-1]}(x) = o(x^{\frac{1}{2}})$ as $x \to \infty$.*

If (i), then $k(x)$ is bounded for finite x, and we may assume: (iii) $k(x) \in L^2(0, X)$, $k(x)x^{-\frac{1}{2}} \in L(0, X)$ for all positive X.

The superfix $^{(-\lambda)}$ refers for δ_n to the first variable, so that

$$\delta_n^{(-\lambda)}(x, \xi) = \Gamma(\lambda)^{-1}\int_0^x (x-t)^{\lambda-1}dt\int_0^n k(tu)k(\xi u)du.$$

The order of integration may be reversed, by Fubini's theorem and (iii). If after inversion we make a change of variable $ut = xv$ in the inside integral and then reinvert (by (iii), again), we have

$$\begin{aligned} \Gamma(\lambda)x^{-\lambda}\delta_n^{(-\lambda)}(x, \xi) &= \int_0^n k(xv)dv\left\{\int_v^{\to\infty} - \int_n^{\to\infty}\right\}(u-v)^{\lambda-1}u^{-\lambda}k(\xi u)du \\ &= A(x, n) - R(x, n). \end{aligned} \tag{10.5}$$

Now if (ii) holds, the double-integral formula for k-transforms shows that $\Gamma(\lambda)^{-1}A(x, n)$ converges in mean to the righthand side of (10.3). For, by a change of variable,

[6] Cf. (**11**), ch. VIII and (**1**), ch. V. For simplicity we suppose that $k(x) \in L^2(0, X)$ for all finite $X > 0$, so that the integral in (10.1) does not involve a mean limit at $t = 0$.

$$A(x, n) = \int_0^n k(xv)dv \int_\xi^{\to\infty} (t-\xi)^{\lambda-1} t^{-\lambda} k(vt)dt;$$

let $\Gamma(\lambda)^{-1} f_\xi(x)$ denote the righthand side of (10.3): for $\lambda > \frac{1}{2}$, $f_\xi(t)$ belongs to $L^2(0, \infty)$ both as a function of ξ and as a function of t. Its k-transform is

$$g_\xi(v) = \underset{m\to\infty}{\text{l.i.m.}^{(v)}} \int_\xi^m (t-\xi)^{\lambda-1} t^{-\lambda} k(vt)dt,$$

and

$$\begin{aligned} f_\xi(x) &= \underset{n\to\infty}{\text{l.i.m.}^{(x)}} \int_0^n k(xv)dv \underset{m\to\infty}{\text{l.i.m.}^{(v)}} \int_\xi^m (t-\xi)^{\lambda-1} t^{-\lambda} k(vt)dt \\ &= \underset{n\to\infty}{\text{l.i.m.}^{(x)}} A(x, n) \end{aligned}$$

(since we imply in (ii) that the integral for $g_\xi(v)$ converges pointwise, and hence the limit and limit in mean are equal almost everywhere). Thus (10.3) will follow if

$$\underset{n\to\infty}{\text{l.i.m.}}\, R(x, n) = 0. \tag{10.6}$$

To prove this we require

LEMMA 4. *For $r \geqq 1$ write*

$$k_r(x) = \int_0^x t^{r-1} k(t)dt, \quad K_r(x) = \int_x^\infty t^{-r} k(t)dt,$$

so that $k_1(x) = k^{(-1)}(x)$, $K_1(x) = x^{-1}k^{[-1]}(x)$. Then if $K_1(x) = o(x^{-\frac{1}{2}})$ as $x \to \infty$, also $K_r(x) = o(x^{-r+\frac{1}{2}})$ uniformly with respect to $r \geqq 1$. Also

$$||x^{-r} k_r(x)||_0 < 2||k||_{-1}. \tag{10.7}$$

To prove (10.7) we use the identity

$$k_r(x) = x^{r-1} k_1(x) - (r-1) \int_0^x t^{r-2} k_1(t)dt, \tag{10.8}$$

which gives

$$\begin{aligned} \left\{ \int_0^\infty |x^{-r} k_r(x)|^2 dx \right\}^{\frac{1}{2}} &\leqq \left\{ \int_0^\infty |x^{-1} k_1(x)|^2 dx \right\}^{\frac{1}{2}} \\ &\quad + (r-1) \left\{ \int_0^\infty |x^{-1} \int_0^1 u^{r-2} k_1(xu)du|^2 dx \right\}^{\frac{1}{2}} \\ &\leqq ||x^{-1} k_1(x)||_0 + (r-1) \int_0^1 u^{r-2} du \left\{ \int_0^\infty |x^{-1} k_1(xu)|^2 dx \right\}^{\frac{1}{2}} \\ &= ||x^{-1} k^{(-1)}(x)||_0 \left\{ 1 + \frac{r-1}{r-\frac{1}{2}} \right\}. \end{aligned}$$

The result follows. The first part of the lemma is deducible from an equation similar to (10.8); the order relation is uniform for $r \geqq 1$.

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We return to the proof that

$$R(x, n) = \int_0^n k(xv)dv \int_n^{\to\infty} (u - v)^{\lambda-1} u^{-\lambda} k(\xi u)du$$

has mean limit zero. Supposing λ non-integral, expand $(u - v)^{\lambda-1}$ as a series in v/u and integrate term by term: the process may be justified in terms of two successive integrations of uniformly convergent series of continuous functions over finite ranges, if (i) and (ii) hold, by virtue of the lemma. The details are left to the reader: we find

$$R(x, n) = \sum_{r=0}^{\infty} (-1)^r \binom{\lambda - 1}{r} x^{-r-1} k_{r+1}(nx) \cdot \xi^r K_{r+1}(n\xi),$$

the series being finite ((i) unnecessary, (iii) retained) if λ is a positive integer. Then by Minkowski's inequality,

$$\left\{\int_0^\infty |R(x, n)|^2 dx\right\}^{\frac{1}{2}} \leqq \sum_{r=0}^{\infty} \left| \binom{\lambda - 1}{r} K_{r+1}(n\xi) \right| \xi^r \cdot \left\{\int_0^\infty |x^{-r-1} k_{r+1}(nx)|^2 dx\right\}^{\frac{1}{2}}.$$

Now suppose $K_1(y) = o(y^{-\frac{1}{2}})$ as $y \to \infty$, and $k \in \mathscr{G}_{-1}$. The lemma gives

$$\begin{aligned} ||R(x, n)||_0 &= \sum_{r=0}^{\infty} \left| \binom{\lambda - 1}{r} \right| \cdot o((n\xi)^{-r-\frac{1}{2}}) \xi^r \cdot n^{r+\frac{1}{2}} 2||k||_{-1} \\ &= o(1)\xi^{-\frac{1}{2}} ||k||_{-1} \sum_{r=0}^{\infty} \left| \binom{\lambda - 1}{r} \right|, \quad \text{as } n \to \infty, \end{aligned}$$

the series converging for $\lambda > 1$ or reducing to one term if $\lambda = 1$. Now $k \in \mathscr{G}_{-1}$ by hypothesis on k (put $f(t) = 1$ $(0 < t < 1)$, 0 $(t > 1)$ in (10.1)). Hence $R(x, n)$ converges in mean to zero under the present conditions, and (10.3) is established.

It remains to prove (10.4). Let $\phi \in \mathscr{G}_\lambda$ and use (8.1); for all finite n, $\delta_n(x, \xi)$ is the k-transform of a function of L^2, by (iii), so that $\delta_n \in L^2$, and we have

$$\begin{aligned} \int_0^\infty \Delta_\xi(x)\phi(x)dx &= \int_0^\infty x^{-\lambda} \Delta_\xi^{(-\lambda)}(x) x^\lambda \phi^{(\lambda)}(x)dx \\ &= \Gamma(\lambda)^{-1} \int_\xi^\infty (x - \xi)^{\lambda-1} \phi^{(\lambda)}(x)dx = \phi(\xi). \end{aligned}$$

This completes the proof of Theorem 5. The conditions (i) and (ii) are not severe, and allow most Watson kernels.

Theorem 5 is interesting as an attenuated form of the "discontinuous integral property" which characterizes Watson kernels; cf. (**11**), (8.2.2) and § 8.14.

11. Generalized Fourier transformations in $\mathscr{H}_{-\lambda}$

In the present context a Watson kernel (without the restrictions of § 10) is defined as a sequence-function κ of $\mathscr{H}_{-1}$, with the property

(11.1) $$x^{-1}K^{(-1)}(x) = \frac{1}{2\pi i} \underset{T\to\infty}{\text{l.i.m.}} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{\mathfrak{K}(s)}{1-s} x^{-s}\, ds,$$

for some function $\mathfrak{K}$ for which

(11.2) $$\mathfrak{K}(\tfrac{1}{2}+it)\mathfrak{K}(\tfrac{1}{2}-it) = 1, \; |\mathfrak{K}(\tfrac{1}{2}+it)| = 1.$$

A generalized Fourier transformation (or Watson transformation) T then takes the form

(11.3) $$x^{-1}g^{(-1)}(x) = \int_0^\infty f(t)\,(xt)^{-1}K^{(-1)}(xt)dt, \qquad g = Tf,$$

for $f, g \in L^2$; see (**11**), ch. VIII. We show how the spaces $\mathscr{H}_{-\lambda}$ form the natural means of extending these transformations beyond L^2.

Among the Mellin transforms, the relations (11.2) and

$$\mathfrak{G}(s) = \mathfrak{F}(1-s)\mathfrak{K}(s)$$

(a necessary and sufficient consequence of (11.3)) imply the relations (cf. (4.3))

$$\mathfrak{G}_{-\lambda}(s) = \mathfrak{F}_{-\lambda}(1-s)\Omega_\lambda(s),$$
$$\Omega_\lambda(\tfrac{1}{2}+it)\Omega_\lambda(\tfrac{1}{2}-it) = 1, \qquad |\Omega_\lambda(\tfrac{1}{2}+it)| = 1,$$

where the function

$$\Omega_\lambda(s) = \frac{\Gamma(1-s)\Gamma(\lambda+s)}{\Gamma(s)\Gamma(1+\lambda-s)}\mathfrak{K}(s)$$

determines in the manner of (11.1) a kernel Ω_λ. Thus to T there corresponds a transformation

(11.4) $$x^{-1}[x^{-\lambda}g^{(-\lambda)}(x)]^{(-1)} = \int_0^\infty t^{-\lambda}f^{(-\lambda)}(t)\cdot(xt)^{-1}\Omega_\lambda^{(-1)}(xt)dt,$$

and for this the Parseval relation has the form

(11.5) $$||g||_{-\lambda} = ||f||_{-\lambda}.$$

To extend (11.3) to a transformation in $\mathscr{H}_{-\lambda}$, let F be an arbitrary sequence-function of $\mathscr{H}_{-\lambda}$ defined by (f_n) $f_n \in L^2$; and let $g_n = Tf_n$ be the K-transform of f_n. Then (g_n) is also a Cauchy sequence, since by (11.5),

$$||g_n - g_m||_{-\lambda} = ||f_n - f_m||_{-\lambda} \to 0 \text{ as } n, m \to \infty,$$

and so (g_n) defines a sequence-function G of $\mathscr{H}_{-\lambda}$. This is determined uniquely by F, for if (f'_n) also defines F and $g'_n = Tf'_n$, (11.5) shows similarly that $(g'_n) \sim (g_n)$. We take G to be the transform of F and extend T to $\mathscr{H}_{-\lambda}$ by writing $G = T_{-\lambda}F$. So constructed, $T_{-\lambda}$ is a self-adjoint unitary transformation in $\mathscr{H}_{-\lambda}$ which coincides with T in L^2; and $T_{-\kappa} \subset T_{-\lambda}$ for $0 \leqq \kappa < \lambda$.

The transformation is closed and continuous. Asymmetric Watson transformations are extended in the same manner (they are not unitary).

12. Regular convergence and bounded linear functionals

We conclude by comparing $\mathscr{H}_{-\lambda}$ with Schwartz' distribution spaces. Although $\mathscr{H}_{-\lambda}$ has been constructed by using strong convergence, it is in fact equally definable by the kind of weak (regular) convergence used by Temple in his simplified exposition of distributions, and so as the space of continuous linear functionals on a space of test functions. We use the product of § 8.

Following Temple, we call a sequence (f_n) of L^2 functions "regular in $\mathscr{H}_{-\lambda}$" if

(i) $(\phi, f_n - f_m)_0 \to 0$ as $n, m \to \infty$, for all $\phi \in \mathscr{G}_\lambda$, so that the functional $F(\phi) = \lim (\phi, f_n)_0$ exists; and

(ii) F is continuous, in the sense that

$$F(\phi_\nu) \to 0 \quad \text{if } ||\phi_\nu||_\lambda \to 0.$$

That is, we take $\mathscr{G}_\lambda$ for the space of test functions.

Now (ii) is a consequence of (i) (cf. Schwartz (**8**), pp. 69, 72). For (i) and Lemma 3 imply

$$(12.1) \qquad (\psi(x), x^{-\lambda} f_n^{(-\lambda)}(x) - x^{-\lambda} f_m^{(-\lambda)}(x))_0 \to 0, \text{ for all } \psi \in L^2,$$

i.e. $(x^{-\lambda} f_n^{(-\lambda)}(x))$ is a weakly converging sequence in L^2 and is therefore bounded, by the Banach-Steinhaus theorem ((**12**), p. 155):

$$||f_n||_{-\lambda} = ||x^{-\lambda} f_n^{(-\lambda)}(x)||_0 \leqq C \text{ for all } n.$$

Thus

$$(12.2) \quad |F(\phi_\nu)| = \lim_{n\to\infty} |(\phi_\nu, f_n)_0| \leqq ||\phi_\nu||_\lambda \cdot \limsup_{n\to\infty} ||f_n||_{-\lambda} \leqq C||\phi_\nu||_\lambda,$$

and (ii) follows; also F is a bounded linear functional on $\mathscr{G}_\lambda$. The proof holds equally if the f_n are sequence-functions, since (5.5) clearly implies

$$(12.3) \qquad |(\phi, F)_0| \leqq ||\phi||_\lambda \cdot ||F||_{-\lambda}, \quad \phi \in \mathscr{G}_\lambda, \quad F \in \mathscr{H}_{-\lambda}.$$

This inequality, with $F = F_n - F_m$, shows that any Cauchy sequence in $\mathscr{H}_{-\lambda}$ is regular; conversely,

THEOREM 6. *$\mathscr{H}_{-\lambda}$ is regularly complete: a regular sequence (F_n) converges regularly to a sequence-function of $\mathscr{H}_{-\lambda}$; i.e. for some F, $(\phi, F_n - F)_0 \to 0$ for all $\phi \in \mathscr{G}_\lambda$.*

PROOF. Since as before $(x^{-\lambda} F_n^{(-\lambda)}(x))$ converges weakly in L^2, which itself is weakly complete ((**12**), p. 156), the sequence converges weakly to a function of L^2 which, by Theorem 2, may be written as $x^{-\lambda}F^{(-\lambda)}(x)$, for some $F \in \mathscr{H}_{-\lambda}$. It follows that

$$(\phi, F_n)_0 = \left(x^\lambda \phi^{(\lambda)}(x), x^{-\lambda} F_n^{(-\lambda)}(x)\right)_0 \to \left(x^\lambda \phi^{(\lambda)}(x), x^{-\lambda} F^{(-\lambda)}(x)\right)_0 = (\phi, F)_0.$$

Let $\mathscr{G}_\lambda^*$ denote the conjugate Hilbert space of $\mathscr{G}_\lambda$, with norm $||F||^*$, whose members F are the bounded linear functionals on $\mathscr{G}_\lambda$.

THEOREM 7. *Any scalar product* $(\phi, F)_0$, $F \in \mathscr{H}_{-\lambda}$, *defines a bounded linear functional on* $\mathscr{G}_\lambda$; *and conversely, any bounded linear functional on* $\mathscr{G}_\lambda$ *can be written in the form*

$$F(\phi) = (\phi, F)_0$$

for some unique $F \in \mathscr{H}_{-\lambda}$; *and* $||F||^* = ||F||_{-\lambda}$.

The proof of the first part is similar to (12.2).

Let F be an arbitrary bounded linear functional of $\mathscr{G}_\lambda^*$. By the Fréchet-Riesz theorem ((**12**), p. 138) for the Hilbert space $\mathscr{G}_\lambda$, there exists a unique $f \in \mathscr{G}_\lambda$ such that

$$F(\phi) = (\phi, f)_\lambda \text{ for all } \phi \in \mathscr{G}_\lambda,$$

and $||F||^* = ||f||_\lambda$. By Theorem 2 there exists $F, = Z_\lambda^{-1} f$, for which $x^\lambda f^{(\lambda)}(x) = x^{-\lambda} F^{(-\lambda)}(x)$. Then

$$F(\phi) = (\phi, f)_\lambda = (\phi, F)_0$$

and

$$||F||^* = ||f||_\lambda = ||F||_{-\lambda},$$

which was to be proved. Thus $\mathscr{G}_\lambda^*$ is precisely the set of functionals $(\phi, F)_0$, $F \in \mathscr{H}_{-\lambda}$. We can set up an isometry between $\mathscr{G}_\lambda^*$ and $\mathscr{H}_{-\lambda}$ by writing $F \leftrightarrow F$ whenever $F(\phi) = (\phi, F)_0$. In this sense $\mathscr{H}_{-\lambda}$ is isometric with the space of bounded linear functionals on $\mathscr{G}_\lambda$.

Thus Cauchy sequences and regular sequences determine the same set of functionals, namely all the bounded linear functionals on $\mathscr{G}_\lambda$.

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