

Energy Methods for the Linearized Field Equations

In the previous chapter, we used energy methods in order to study the Cauchy problem for linear symmetric hyperbolic systems. We now briefly explain how these methods can be adapted to the linearized field equations for causal variational principles as introduced in Chapter 7. These constructions are carried out in detail in [22]; for later developments, see [82, 65]. Here, we do not aim for the largest generality but instead explain the basic ideas in the simplest possible setting. We also note that some of the constructions in this section will be illustrated in Chapter 20 with simple concrete examples.

14.1 Local Foliations by Surface Layers

We consider *causal variational principles in the compact setting* (see Section 6.3). Moreover, for technical simplicity, we again restrict attention to the *smooth setting* by assuming that the Lagrangian is smooth (6.10). Following our procedure for symmetric hyperbolic systems, we want to analyze the initial problem “locally” in an open subset U of spacetime M . In analogy to the time function in a lens-shaped region L (see Section 13.2), we here choose a foliation of a compact subset $L \subset U$ by surface layers. This motivates the following definition.

Definition 14.1.1 *Let $U \subset M$ be an open subset of spacetime and $I := [t_{\min}, t_{\max}]$ a compact interval. Moreover, we let $\eta \in C^\infty(I \times U, \mathbb{R})$ be a function with $0 \leq \eta \leq 1$, which for all $t \in I$ has the following properties:*

- (i) *The function $\theta(t, \cdot) := \partial_t \eta(t, \cdot)$ is nonnegative and compactly supported in U .*
- (ii) *For all $x \in \text{supp } \theta(t, \cdot)$ and all $y \in M \setminus U$, the function $\mathcal{L}(x, y)$ as well as its first and second derivatives vanish.*

*We also write $\eta(t, x)$ as $\eta_t(x)$ and $\theta(t, x)$ as $\theta_t(x)$. We refer to $(\eta_t)_{t \in I}$ as a **local foliation** inside U .*

The situation in mind is shown in Figure 14.1. The parameter t can be thought of as the time of a local observer and will be referred to simply as *time*. The support of the function θ_t is a *surface layer*. The function η_t should be thought of as being equal to one in the past and equal to zero in the future of this surface layer. The condition (i) implies that the set L defined by

$$L := \bigcup_{t \in I} \text{supp } \theta_t, \quad (14.1)$$

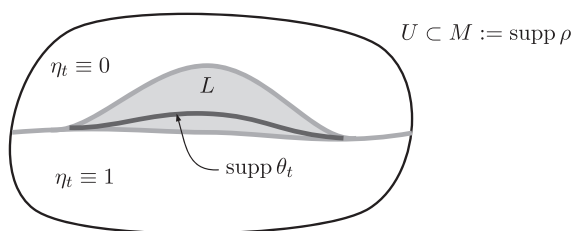


Figure 14.1 A local foliation. From [22].

is compact. It is the region of spacetime described by the local foliation. The condition (ii) has the purpose for ensuring that the dynamics in the region L does not depend on the jets outside U , making it possible to restrict attention to the spacetime region U . Sometimes, we refer to this property that L is $\mathcal{L}$ -localized in U . One way of satisfying (ii) is to simply choose $U = M$. However, in the applications, it may be desirable to “localize” the problem, for example, by choosing U as the domain of a coordinate chart.

Following the procedure for hyperbolic partial differential equations (PDEs), our first goal is to analyze the initial value problem. For the following constructions, it will be useful to combine the functions η_t and θ_t with the measure ρ such as to form new measures: The measure

$$d\rho_t(x) := \theta_t(x) d\rho(x), \quad (14.2)$$

with $t \in I$ is supported in the surface layer at time t . Likewise, the measures

$$\eta_t d\rho \quad \text{and} \quad (1 - \eta_t) d\rho, \quad (14.3)$$

are supported in the past and future, respectively, of the surface layer at time t . For the measures supported in a spacetime strip, we use the notation

$$\eta_{[t_0, t_1]} d\rho \quad \text{with} \quad \eta_{[t_0, t_1]} := \eta_{t_1} - \eta_{t_0} \in C_0^\infty(U), \quad (14.4)$$

where we always choose $t_0, t_1 \in I$ with $t_0 \leq t_1$. Note that the function $\eta_{[t_0, t_1]}$ is supported in L .

14.2 Energy Estimates and Hyperbolicity Conditions

For the analysis of the linearized field equations, it is helpful to study the surface layer integrals as introduced in Section 9.5 for our local foliation $(\eta_t)_{t \in I}$. It is useful to “soften” these surface layer integrals by rewriting the integration domains with characteristic functions and replacing the characteristic functions with smooth cutoff functions formed of η_t , that is, symbolically

$$\begin{aligned} \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \cdots &= \int_M d\rho(x) \int_M d\rho(y) \chi_{\Omega}(x) (1 - \chi_{\Omega}(y)) \cdots \\ &\longrightarrow \int_M d\rho(x) \int_M d\rho(y) \eta_t(x) (1 - \eta_t(y)) \cdots \end{aligned} \quad (14.5)$$

We thus define the *softened symplectic form* and the *softened surface layer inner product* by

$$(\mathbf{u}, \mathbf{v})^t = \int_U d\rho(x) \eta_t(x) \int_U d\rho(y) (1 - \eta_t(y)) \\ \times \left(\nabla_{1,\mathbf{u}} \nabla_{1,\mathbf{v}} - \nabla_{2,\mathbf{u}} \nabla_{2,\mathbf{v}} \right) \mathcal{L}(x, y) \quad (14.6)$$

$$\sigma^t(\mathbf{u}, \mathbf{v}) = \int_U d\rho(x) \eta_t(x) \int_U d\rho(y) (1 - \eta_t(y)) \\ \times \left(\nabla_{1,\mathbf{u}} \nabla_{2,\mathbf{v}} - \nabla_{1,\mathbf{v}} \nabla_{2,\mathbf{u}} \right) \mathcal{L}(x, y). \quad (14.7)$$

The quantity $(\mathbf{u}, \mathbf{u})^t$ is of central importance for the following constructions because it will play the role of the energy used in our energy estimates. In preparation of these estimates, we derive an energy identity:

Lemma 14.2.1 (energy identity) *For any jet $\mathbf{u} = (a, u) \in \mathfrak{J}$,*

$$\frac{d}{dt} (\mathbf{u}, \mathbf{u})^t = 2 \int_U \langle \mathbf{u}, \Delta \mathbf{u} \rangle(x) d\rho_t(x) \\ - 2 \int_U \Delta_2[\mathbf{u}, \mathbf{u}] d\rho_t(x) + \mathfrak{s} \int_U a(x)^2 d\rho_t(x), \quad (14.8)$$

where the operator $\Delta_2 : \mathfrak{J} \times \mathfrak{J} \rightarrow \mathfrak{J}^*$ defined by

$$\langle \mathbf{u}, \Delta_2[\mathbf{u}_1, \mathbf{u}_2] \rangle(x) \\ = \frac{1}{2} \nabla_{\mathbf{u}} \left(\int_M (\nabla_{1,\mathbf{u}_1} + \nabla_{2,\mathbf{u}_1}) (\nabla_{1,\mathbf{u}_2} + \nabla_{2,\mathbf{u}_2}) \mathcal{L}(x, y) d\rho(y) - \nabla_{\mathbf{u}_1} \nabla_{\mathbf{u}_2} \mathfrak{s} \right). \quad (14.9)$$

Proof Differentiating (14.6) with respect to t gives

$$\frac{d}{dt} (\mathbf{u}, \mathbf{u})^t = \int_U d\rho(x) \theta_t(x) \int_U d\rho(y) (1 - \eta_t(y)) (\nabla_{1,\mathbf{u}}^2 - \nabla_{2,\mathbf{u}}^2) \mathcal{L}(x, y) \\ - \int_U d\rho(x) \eta_t(x) \int_U d\rho(y) \theta_t(y) (\nabla_{1,\mathbf{u}}^2 - \nabla_{2,\mathbf{u}}^2) \mathcal{L}(x, y) \\ = \int_U d\rho(x) \theta_t(x) \int_U d\rho(y) (\nabla_{1,\mathbf{u}}^2 - \nabla_{2,\mathbf{u}}^2) \mathcal{L}(x, y). \quad (14.10)$$

Next, for all $x \in L$, we may use Definition 14.1.1 (ii) to change the integration range in (8.15) from M to U ,

$$\langle \mathbf{u}, \Delta \mathbf{u} \rangle(x) = \int_U \nabla_{1,\mathbf{u}} (\nabla_{1,\mathbf{u}} + \nabla_{2,\mathbf{u}}) \mathcal{L}(x, y) d\rho(y) - \mathfrak{s} a(x)^2. \quad (14.11)$$

Multiplying by θ_t and integrating, we obtain

$$0 = \int_U \theta_t(x) \langle \mathbf{u}, \Delta \mathbf{u} \rangle(x) d\rho(x) + \mathfrak{s} \int_U \theta_t(x) a(x)^2 d\rho(x) \\ - \int_U d\rho(x) \theta_t(x) \int_U d\rho(y) (\nabla_{1,\mathbf{u}}^2 + \nabla_{1,\mathbf{u}} \nabla_{2,\mathbf{u}}) \mathcal{L}(x, y). \quad (14.12)$$

We multiply this equation by two and add (14.10). This gives

$$\begin{aligned} \frac{d}{dt} (\mathbf{u}, \mathbf{u})^t = & - \int_U d\rho(x) \theta_t(x) \int_U d\rho(y) (\nabla_{1,\mathbf{u}} + \nabla_{2,\mathbf{u}})^2 \mathcal{L}(x, y) \\ & + 2 \int_U \theta_t(x) \langle \mathbf{u}, \Delta \mathbf{u} \rangle(x) d\rho(x) + 2\mathfrak{s} \int_U \theta_t(x) a(x)^2 d\rho(x). \end{aligned} \quad (14.13)$$

Using the property in Definition 14.1.1 (ii), in the y -integral, we may replace the integration range U by M , making it possible to apply (14.9). Rewriting the obtained integrals using the notation (14.2) gives (14.8). $\square$

In order to make use of this energy identity, we need to impose a condition, which we call hyperbolicity condition. This notion can be understood as follows. As explained in Chapter 13, in the theory of hyperbolic partial differential equations, the hyperbolicity of the equations gives rise to a positive energy (see $E(\lambda)$ in (13.38)). The positivity of this energy was a consequence of the structure of the equations (more precisely, for symmetric linear hyperbolic systems, it is a consequence of the positivity statement in Definition 13.1.1 (ii)). The positivity of the energy is crucial for the analysis of hyperbolic equations because it gives both uniqueness (see Section 13.2) and existence of weak solutions (see Section 13.3). With this in mind, our strategy is to express the hyperbolic nature of the linearized field equations by imposing a positivity condition for our “energy” $(\mathbf{u}, \mathbf{u})^t$. As we shall see, this so-called hyperbolicity condition is precisely what is needed in order to obtain the existence and uniqueness of solutions. For Dirac systems in Minkowski space, the hyperbolicity conditions can be verified by direct computation (for details, see [50]). With this in mind, our hyperbolicity conditions are physically sensible. But in most situations, imposing the hyperbolicity conditions for all jets in $\mathfrak{J}$ is a too strong assumption. Instead, these conditions will or can be satisfied only on a suitably chosen subspace of jets, which we denote by

$$\mathfrak{J}^{\text{vary}} \subset \mathfrak{J}. \quad (14.14)$$

Clearly, the smaller the jet space $\mathfrak{J}^{\text{vary}}$ is chosen, the easier it is to satisfy (14.18). The drawback is that the Cauchy problem will be solved in a weaker sense.

In order to define the hyperbolicity conditions, for all $x \in M$, we choose the subspace of the tangent space spanned by the test jets,

$$\Gamma_x := \{u(x) \mid u \in \Gamma^{\text{test}}\} \subset T_x \mathcal{F}. \quad (14.15)$$

We introduce a Riemannian metric g_x on Γ_x . The choice of the Riemannian metric is arbitrary; the resulting freedom can be used in order to satisfy the hyperbolicity conditions mentioned later (note, however, that for causal fermion systems, a canonical Riemannian metric is obtained from the Hilbert–Schmidt scalar product; see [60, 67]). This Riemannian metric also induces a pointwise scalar product on the jets. Namely, setting

$$\mathfrak{J}_x := \mathbb{R} \oplus \Gamma_x, \quad (14.16)$$

we obtain the scalar product on $\mathfrak{J}_x$

$$\langle \cdot, \cdot \rangle_x : \mathfrak{J}_x \times \mathfrak{J}_x \rightarrow \mathbb{R}, \quad \langle \mathbf{u}, \tilde{\mathbf{u}} \rangle_x := a(x) \tilde{a}(x) + g_x(u(x), \tilde{u}(x)), \quad (14.17)$$

(where we again denote the scalar and vector components of the jet by $\mathbf{u} = (a, u)$). We denote the corresponding norm by $\|\cdot\|_x$.

Definition 14.2.2 *The local foliation $(\eta_t)_{t \in I}$ inside U satisfies the **hyperbolicity condition** if there is a constant $C > 0$ such that for all $t \in I$,*

$$(\mathbf{u}, \mathbf{u})^t \geq \frac{1}{C^2} \int_U \left(\|\mathbf{u}(x)\|_x^2 + |\Delta_2[\mathbf{u}, \mathbf{u}]| \right) d\rho_t(x) \quad \text{for all } \mathbf{u} \in \mathfrak{J}^{\text{vary}}. \quad (14.18)$$

A compact set $L \subset M$ is a **lens-shaped region** inside U if there is a local foliation $(\eta_t)_{t \in I}$ inside U satisfying (14.1), which satisfies the hyperbolicity conditions.

We point out that these hyperbolicity conditions also pose constraints for the choice of the functions η_t ; these constraints can be understood as replacing the condition in the theory of hyperbolic PDEs that the initial data surface be spacelike. In general situations, the inequality (14.18) is not obvious and must be arranged and verified in the applications. More specifically, one can use the freedom in choosing the jet space $\mathfrak{J}^{\text{vary}}$, the Riemannian metric in the scalar product (14.17) and the functions η_t in Definition 14.1.1 in order to ensure that (14.18) holds.

We now explain how the above hyperbolicity condition can be used to derive energy estimates. We let L be a lens-shaped region inside U with the local foliation $(\eta_t)_{t \in I}$. We denote the norm corresponding to the jet scalar product by $\|\mathbf{u}\|^t := \sqrt{(\mathbf{u}, \mathbf{u})^t}$. We begin with a simple estimate of the energy identity in Lemma 14.2.1.

Lemma 14.2.3 *Assume that the hyperbolicity condition of Definition 14.2.2 holds. Then, for every $t \in I$ and all $\mathbf{u} \in \mathfrak{J}$,*

$$\frac{d}{dt} \|\mathbf{u}\|^t \leq C \|\Delta \mathbf{u}\|_{L^2(U, d\rho_t)} + c \|\mathbf{u}\|^t, \quad (14.19)$$

with

$$c := C^2 + \frac{C^2 \mathfrak{s}}{2}. \quad (14.20)$$

Proof Applying (14.18) in (14.8), we obtain

$$\begin{aligned} \frac{d}{dt} (\mathbf{u}, \mathbf{u})^t &\leq 2 \int_U \langle \mathbf{u}, \Delta \mathbf{u} \rangle_x d\rho_t(x) - 2 \int_U \Delta_2[\mathbf{u}, \mathbf{u}] d\rho_t(x) + \mathfrak{s} \int_U b(x)^2 d\rho_t(x) \\ &\leq 2 \int_U \langle \mathbf{u}, \Delta \mathbf{u} \rangle_x d\rho_t(x) + (2C^2 + C^2 \mathfrak{s}) (\mathbf{u}, \mathbf{u})^t \\ &\leq 2 \|\mathbf{u}\|_{L^2(U, d\rho_t)} \|\Delta \mathbf{u}\|_{L^2(U, d\rho_t)} + 2c (\mathbf{u}, \mathbf{u})^t \\ &\leq 2C \|\mathbf{u}\|^t \|\Delta \mathbf{u}\|_{L^2(U, d\rho_t)} + 2c (\mathbf{u}, \mathbf{u})^t, \end{aligned} \quad (14.21)$$

where in the last line, we applied (14.18). Using the relation

$$\partial_t \|\mathbf{u}\|^t = \partial_t (\mathbf{u}, \mathbf{u})^t / (2\|\mathbf{u}\|^t), \quad (14.22)$$

gives the result. $\square$

Applying Grönwall-type estimates (see, e.g., [1, Lemma 1.15 in Section VII.1] or the proof of Proposition 14.2.4), the inequality (14.19) shows that $\|\mathbf{u}\|^t$ grows at most exponentially in time, provided that $\Delta\mathbf{u}$ decays in time sufficiently fast. We here make this statement precise by estimates in Hilbert spaces of jets with zero initial values. In the lens-shaped region L , we work with the L^2 -scalar product

$$\langle \mathbf{u}, \mathbf{v} \rangle_{L^2(L)} := \int_L \langle \mathbf{u}(x), \mathbf{v}(x) \rangle_x \eta_I(x) \, d\rho(x), \quad (14.23)$$

which, according to (14.2) and (14.4), can also be written in terms of a time integral,

$$\langle \mathbf{u}, \mathbf{v} \rangle_{L^2(L)} = \int_{t_0}^{t_{\max}} \langle \mathbf{u}, \mathbf{v} \rangle_{L^2(U, d\rho_t)} \, dt. \quad (14.24)$$

The corresponding norm is denoted by $\|\cdot\|_{L^2(L)}$.

Proposition 14.2.4 (energy estimate) *Assume that the hyperbolicity condition of Definition 14.2.2 holds. Then, choosing*

$$\Gamma = 2C e^{2c(t_{\max}-t_0)} (t_{\max} - t_0), \quad (14.25)$$

the following estimate holds,

$$\|\mathbf{u}\|_{L^2(L)} \leq \Gamma \|\Delta\mathbf{u}\|_{L^2(L)} \quad \text{for all } \mathbf{u} \in \mathfrak{J} \text{ with } \|\mathbf{u}\|^{t_0} = 0. \quad (14.26)$$

Proof We write the energy estimate of Lemma 14.2.3 as

$$\frac{d}{dt} (e^{-2ct} (\mathbf{u}, \mathbf{u})^t) \leq 2 e^{-2ct} C \|\mathbf{u}\|^t \|\Delta\mathbf{u}\|_{L^2(U, d\rho_t)}. \quad (14.27)$$

Integrating over t from t_0 to some $t \in I$ and using the hyperbolicity condition (14.18), we obtain

$$\begin{aligned} e^{-2ct} (\mathbf{u}, \mathbf{u})^t &= \int_{t_0}^t \frac{d}{dt'} (e^{-2ct'} (\mathbf{u}, \mathbf{u})^{t'}) \, dt' \\ &\leq 2C \int_{t_0}^t e^{-2ct'} \|\mathbf{u}\|^{t'} \|\Delta\mathbf{u}\|_{L^2(U, d\rho_{t'})} \, dt'. \end{aligned} \quad (14.28)$$

Multiplying by e^{2ct} gives the inequality

$$\begin{aligned} (\mathbf{u}, \mathbf{u})^t &\leq 2C \int_{t_0}^t e^{2c(t-t')} \|\mathbf{u}\|^{t'} \|\Delta\mathbf{u}\|_{L^2(U, d\rho_{t'})} \, dt' \\ &\leq 2C e^{2c(t_{\max}-t_0)} \int_{t_0}^{t_{\max}} \|\mathbf{u}\|^{t'} \|\Delta\mathbf{u}\|_{L^2(U, d\rho_{t'})} \, dt' \\ &\leq 2C e^{2c(t_{\max}-t_0)} \|\Delta\mathbf{u}\|_{L^2(L)} \left(\int_{t_0}^{t_{\max}} (\mathbf{u}, \mathbf{u})^{t'} \, dt' \right)^{\frac{1}{2}}, \end{aligned} \quad (14.29)$$

where in the last step, we used the Schwarz inequality and (14.24). Integrating once again over t from t_0 to $t_{\max}$ gives

$$\left(\int_{t_0}^{t_{\max}} (\mathbf{u}, \mathbf{u})^t \, dt \right)^{\frac{1}{2}} \leq 2C e^{2c(t_{\max}-t_0)} (t_{\max} - t_0) \|\Delta\mathbf{u}\|_{L^2(L)}. \quad (14.30)$$

Finally, we apply the hyperbolicity condition (14.18) in (14.24),

$$\|v\|_{L^2(L)} = \left(\int_{t_0}^{t_{\max}} \|u\|_{L^2(U, d\rho_t)}^2 dt \right)^{\frac{1}{2}} \leq C \left(\int_{t_0}^{t_{\max}} (u, u)^t dt \right)^{\frac{1}{2}}. \quad (14.31)$$

Combining this inequality with (14.30) gives the result. $\square$

14.3 Uniqueness of Strong Solutions

Based on the above energy estimates, we can now prove uniqueness of strong solutions of the Cauchy problem. The method is quite similar to that employed in Section 13.2 for symmetric hyperbolic systems. In preparation of formulating the Cauchy problem, we need to introduce jets that vanish at the initial time $t_{\min}$. To this end, we demand that the jet vanishes in the surface layer and that the corresponding softened surface layer integrals (14.6) and (14.7) vanish,

$$\begin{aligned} \mathfrak{J}_{t_{\min}} &:= \{u \in \mathfrak{J} \mid \eta_{t_{\min}} u \equiv 0 \quad \text{and} \\ &\quad (u, v)^{t_{\min}} = 0 = \sigma^{t_{\min}}(u, v) \text{ for all } v \in \mathfrak{J}\}. \end{aligned} \quad (14.32)$$

Similarly, we define the space of jets that vanish at final time $t_{\max}$ by

$$\begin{aligned} \overline{\mathfrak{J}}_U^{t_{\max}} &:= \{u \in \mathfrak{J} \mid (1 - \eta_{t_{\max}}) u \equiv 0 \quad \text{and} \\ &\quad (u, v)^{t_{\max}} = 0 = \sigma^{t_{\max}}(u, v) \text{ for all } v \in \mathfrak{J}\}. \end{aligned} \quad (14.33)$$

A *strong solution* of the Cauchy problem is a jet $u \in \mathfrak{J}_U$, which satisfies the equations

$$\Delta u = w \quad \text{in } L \quad \text{and} \quad u - u_0 \in \mathfrak{J}_{t_{\min}}, \quad (14.34)$$

where $u_0 \in \mathfrak{J}$ is the initial data and w is the *inhomogeneity*. More precisely, as explained after (8.16), the jet $w(x)$ can be regarded as a dual jet. Here, having already introduced a scalar product on the jets at every spacetime point (see (14.17)), we can identify dual jets with jets. With this in mind, the inhomogeneity simply is a jet $w \in \mathfrak{J}_U$.

Proposition 14.3.1 (uniqueness of strong solutions) *Let $(\eta_t)_{t \in I}$ be a local foliation inside U which satisfies the hyperbolicity conditions (see Definitions 14.1.1 and 14.2.2). Then, the Cauchy problem (14.34) with $u_0, w \in \mathfrak{J}_U$ has at most one solution u in L .*

Proof Let u be the difference of two solutions. Then, u is a solution of the homogeneous equation with zero initial data. Applying Lemma 14.2.3, we obtain

$$\left| \frac{d}{dt} \|u\|^t \right| \leq c \|u\|^t \quad \text{and thus} \quad \frac{d}{dt} (e^{-ct} \|u\|^t) \leq 0. \quad (14.35)$$

It follows that $\|u\|^t$ vanishes for all t in the respective interval. Using (14.18), we conclude that u vanishes identically in L . This gives the result. $\square$

As explained in Section 13.2 for symmetric hyperbolic systems, this uniqueness statement also gives information on the speed of propagation and the resulting causal structure. For details, we refer to [22, 23].

14.4 Existence of Weak Solutions

Our existence proof is inspired by the method invented by K. O. Friedrichs for symmetric hyperbolic systems in [91] as outlined in Section 13.3. Our first step is to formulate the linearized field equations weakly. To this end, we need to “integrate by parts” with the help of the following lemma.

Lemma 14.4.1 (Green’s formula) *For all $u, v \in \mathfrak{J}$,*

$$\sigma^{t_{\max}}(u, v) - \sigma^{t_{\min}}(u, v) = \langle u, \Delta v \rangle_{L^2(L)} - \langle \Delta u, v \rangle_{L^2(L)}. \quad (14.36)$$

Proof Using the definition of the L^2 -scalar product in (14.23) and the definition of the linearized field operator (8.15), we obtain

$$\begin{aligned} \langle u, \Delta v \rangle_{L^2(L)} - \langle \Delta u, v \rangle_{L^2(L)} &= \int_U \left(\langle u, \Delta v \rangle - \langle \Delta u, v \rangle \right) \eta_I \, d\rho \\ &= \int_U d\rho(x) \eta_I(x) \nabla_u \left(\int_M (\nabla_{1,v} + \nabla_{2,v}) \mathcal{L}(x, y) \, d\rho(y) - \nabla_v \mathfrak{s} \right) \\ &\quad - \int_U d\rho(x) \eta_I(x) \nabla_v \left(\int_M (\nabla_{1,u} + \nabla_{2,u}) \mathcal{L}(x, y) \, d\rho(y) - \nabla_u \mathfrak{s} \right). \end{aligned} \quad (14.37)$$

Here, the spacetime point x is in L . Using Definition 14.1.1 (ii), we get a contribution to the integrals only if $y \in U$. Therefore, we may replace the integration range M by U . We thus obtain

$$\begin{aligned} \langle u, \Delta v \rangle_{L^2(L)} - \langle \Delta u, v \rangle_{L^2(L)} &= \int_U d\rho(x) \eta_I(x) \int_U d\rho(y) (\nabla_{1,u} \nabla_{2,v} - \nabla_{2,u} \nabla_{1,v}) \mathcal{L}(x, y), \end{aligned} \quad (14.38)$$

where we used that, following our convention (8.1), the second derivatives of the Lagrangian are symmetric. Using the definition (14.4) as well as the anti-symmetry of the integrand, the term (14.38) can be rewritten as

$$\begin{aligned} &\int_U d\rho(x) \eta_I(x) \int_U d\rho(y) (\nabla_{1,u} \nabla_{2,v} - \nabla_{2,u} \nabla_{1,v}) \mathcal{L}(x, y) \\ &= \int_U d\rho(x) \int_U d\rho(y) \eta_t(x) (\nabla_{1,u} \nabla_{2,v} - \nabla_{2,u} \nabla_{1,v}) \mathcal{L}(x, y) \Big|_{t_0}^{t_{\max}} \\ &= \int_U d\rho(x) \int_U d\rho(y) \left(\eta_t(x) - \eta_t(x) \eta_t(y) \right) (\nabla_{1,u} \nabla_{2,v} - \nabla_{2,u} \nabla_{1,v}) \mathcal{L}(x, y) \Big|_{t_0}^{t_{\max}} \\ &= \int_U d\rho(x) \int_U d\rho(y) \eta_t(x) (1 - \eta_t(y)) (\nabla_{1,u} \nabla_{2,v} - \nabla_{2,u} \nabla_{1,v}) \mathcal{L}(x, y) \Big|_{t_0}^{t_{\max}} \\ &= \sigma^{t_{\max}}(u, v) - \sigma^{t_{\min}}(u, v). \end{aligned} \quad (14.39)$$

This gives the result. $\square$

Assume that u is a strong solution to the Cauchy problem (14.34). As usual, replacing u by $u - u_0$ and v by $v - \Delta u_0 \in \mathfrak{J}$, it suffices to consider the Cauchy problem for zero initial data, that is,

$$\Delta u = v \quad \text{in } U \quad \text{and} \quad u \in \mathfrak{J}_{U, t_{\min}}. \quad (14.40)$$

Then, applying the above Green's formula, we obtain for any $\mathbf{v} \in \mathfrak{J}$,

$$\langle \mathbf{v}, \mathfrak{w} \rangle_{L^2(L)} = \langle \mathbf{v}, \Delta \mathbf{u} \rangle_{L^2(L)} = \langle \Delta \mathbf{v}, \mathbf{u} \rangle_{L^2(L)} - \sigma^{t_{\max}}(\mathbf{v}, \mathbf{u}) + \sigma^{t_{\min}}(\mathbf{v}, \mathbf{u}). \quad (14.41)$$

Having implemented the vanishing initial data by the condition $\mathbf{u} \in \mathfrak{J}_{t_0}$, the symplectic form vanishes at time $t_{\min}$. In order to also get rid of the boundary values at time $t_{\max}$, we restrict attention to test jets that vanish at $t_{\max}$. This leads us to the following definition:

Definition 14.4.2 *A jet $\mathbf{u} \in L^2(L)$ is a **weak solution** of the Cauchy problem $\Delta \mathbf{u} = \mathfrak{w}$ with zero initial data if*

$$\langle \Delta \mathbf{v}, \mathbf{u} \rangle_{L^2(L)} = \langle \mathbf{v}, \mathfrak{w} \rangle_{L^2(L)} \quad \text{for all } \mathbf{v} \in \overline{\mathfrak{J}}^{t_{\max}}. \quad (14.42)$$

Clearly, the energy estimate of Proposition 14.2.4 also holds if we exchange the roles of $t_{\max}$ and $t_{\min}$, that is,

$$\|\mathbf{u}\|_{L^2(L)} \leq \Gamma \|\Delta \mathbf{u}\|_{L^2(L)} \quad \text{for all } \mathbf{u} \in \overline{\mathfrak{J}}^{t_{\max}}, \quad (14.43)$$

(where the constant Γ is again given by (14.25)).

We introduce the positive semi-definite bilinear form

$$\langle \cdot, \cdot \rangle : \overline{\mathfrak{J}}_U^{t_{\max}} \times \overline{\mathfrak{J}}_U^{t_{\max}} \rightarrow \mathbb{R}, \quad \langle \mathbf{u}, \mathbf{v} \rangle = \langle \Delta \mathbf{u}, \Delta \mathbf{v} \rangle_{L^2(L)}. \quad (14.44)$$

Dividing out the null space and forming the completion, we obtain a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. The corresponding norm is denoted by $\|\cdot\|$.

We now consider the linear functional $\langle \mathfrak{w}, \cdot \rangle_{L^2(L)}$ on $\overline{\mathfrak{J}}_U^{t_{\max}}$. Applying the Schwarz inequality and (14.43), we obtain

$$|\langle \mathfrak{w}, \mathbf{u} \rangle_{L^2(L)}| \leq \|\mathfrak{w}\|_{L^2(L)} \|\mathbf{u}\|_{L^2(L)} \leq \Gamma \|\mathfrak{w}\|_{L^2(L)} \|\mathbf{u}\|, \quad (14.45)$$

proving that the linear functional $\langle \mathfrak{w}, \cdot \rangle_{L^2(L)}$ on $\overline{\mathfrak{J}}^{t_{\max}}$ is bounded on $\mathcal{H}$. Therefore, it can be extended uniquely to a bounded linear functional on all of $\mathcal{H}$. Moreover, by the Fréchet–Riesz theorem, there is a unique vector $U \in \mathcal{H}$ with

$$\langle \mathfrak{w}, \mathbf{v} \rangle_{L^2(L)} = \langle U, \mathbf{v} \rangle = \langle \Delta U, \Delta \mathbf{v} \rangle_{L^2(L)} \quad \text{for all } \mathbf{v} \in \overline{\mathfrak{J}}_U^{t_{\max}}. \quad (14.46)$$

Hence, $\mathbf{u} := \Delta U \in L^2(L)$ is the desired weak solution. We point out that in the above estimates, the inhomogeneity $\mathfrak{w}$ enters only via its L^2 -norm, making it possible to generalize our methods to $\mathfrak{w} \in L^2(L)$. We have obtained the following result:

Theorem 14.4.3 *Assume that $(\eta_t)_{t \in I}$ is a local foliation satisfying the hyperbolicity conditions (see Definitions 14.1.1 and 14.2.2). Then, for every $\mathfrak{w} \in L^2(L)$, there is a weak solution $\mathbf{u} \in L^2(L)$ of the Cauchy problem (14.42).*

We remark that the construction of weak solutions is the starting point for the more detailed analysis of linearized fields as carried out in [22, 82, 23, 65]. One task is to deal with the uniqueness problem for weak solutions (see Exercise 14.3). Another issue is to construct global solutions (for various methods for doing so, see [22, Section 4], [82, Section 6.3], [23, Section 3.3] and [65]). Once global advanced and retarded solutions have been obtained for a general class of

inhomogeneities, one can also introduce corresponding Green's operators (see [22, Section 5], [65, Section 4] or Exercise 14.4).

14.5 Exercises

Exercise 14.1 (*Differentiated form of conservation laws*) Conservation laws for causal variational principles are formulated in terms of surface layer integrals (see, e.g., Theorem 9.2.3, Theorem 9.3.2, Proposition 9.3.1 or Lemma 14.4.1). For the proofs, we rewrote the surface layer integrals as double volume integrals, making use of anti-symmetry properties (see, e.g., the proof of Theorem 9.3.2). Alternatively, one can prove the conservation laws by computing the time derivatives. The goal of this exercise is to illustrate this method (for more details, see, e.g., [57, Section 2.6]).

- (a) Given a local foliation $(\eta_t)_{t \in I}$, we consider the softened symplectic form given by (14.7). For $\mathbf{u}, \mathbf{v} \in \mathfrak{J}$, compute

$$\frac{d}{dt} \sigma^t(\mathbf{u}, \mathbf{v}), \quad (14.47)$$

in a similar style as in the proof of Lemma 14.2.1. Use this formula to give an alternative proof of Lemma 14.4.1.

- (b) Given a local foliation $(\eta_t)_{t \in I}$, formulate a softened version of the surface layer integral I_{k+1}^Ω in Theorem 9.26. Differentiate with respect to the time parameter t to obtain an alternative proof of Theorem 9.26.

Exercise 14.2 (*Counter example to hyperbolicity conditions*) The hyperbolicity conditions in Definition 14.2.2 were formulated only for jets in a subspace $\mathfrak{J}^{\text{vary}} \subset \mathfrak{J}$. The goal of this exercise is to explain why it would not be sensible to impose the inequality (14.18) for all $\mathbf{u} \in \mathfrak{J}$. To this end, consider for simplicity the unsoftened surface layer integral (9.58). Show that there is a jet $\mathbf{u} \in C_0^0(M, \mathbb{R}) \oplus C_0^0(M, T\mathcal{F})$ with $(\mathbf{u}, \mathbf{u})^t < 0$. *Hint:* Choose points $x \in \Omega$ and $y \in M \setminus \Omega$ for which $\mathcal{L}(x, y) \neq 0$. Choose $\mathbf{u}$ as a scalar jet, which is supported in a small neighborhood of x and y .

Exercise 14.3 (*Nonuniqueness of weak solutions*) As illustrated in the previous exercise (Exercise 14.2), in order to satisfy the hyperbolicity conditions, the jet space $\mathfrak{J}^{\text{vary}}$ must not be chosen too large. In particular, in typical examples, the jet space $\mathfrak{J}^{\text{vary}}$ is not dense in $L^2(L)$. This leads to a nonuniqueness issue for weak solutions, which will be illustrated in this exercise.

- (a) Given an inhomogeneity $\mathbf{w} \in L^2(L)$, to which extent are weak solutions unique? Specify the jet space of all homogeneous solutions.
- (b) On the other hand, the construction before Theorem 14.4.3 gives a unique solution $\mathbf{u} = \Delta U$. How does this result fit together with the nonuniqueness in (a)? In which sense is the solution $\mathbf{u} = \Delta U$ distinguished?

Hint: Similar questions are analyzed in [22, Section 3].

Exercise 14.4 (*Causal Green's operators for the linearized field equations*) In order to avoid the issue of how to “glue together” local solutions obtained in different lens-shaped regions such as to obtain global solutions, one can simplify the setting by assuming that the spacetime M admits a global foliation $(\eta_t)_{t \in \mathbb{R}}$.

- (a) How can Definition 14.1.1 be modified in order to describe a global foliation? What is the resulting global analog of Theorem 14.4.3? *Hint:* It might be instructive to compare your definition with Definition 6.4 in [82].
- (b) Suppose that we know that for any compactly supported jet $\mathfrak{w} \in L_0^2(M)$, there is a distinguished global weak solution $\mathfrak{v} \in L_{\text{loc}}^2(M)$, that is,

$$\langle \Delta \mathfrak{v}, \mathfrak{u} \rangle_{L^2(M)} = \langle \mathfrak{v}, \mathfrak{w} \rangle_{L^2(M)} \quad \text{for all } \mathfrak{v} \in \mathfrak{J}_0. \quad (14.48)$$

Then, the operator $S : \mathfrak{w} \rightarrow -\mathfrak{v}$ is referred to as the Green's operator. How can one distinguish between the advanced Green's operator $S^\vee$ and the retarded Green's operator $S^\wedge$. Show that their difference $G := S^\wedge - S^\vee$ maps to homogeneous weak solutions. *Hint:* More details on Green's operators and their causal properties can be found in [22, Section 5], [23, Section 3] and [65, Section 4].