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ULTRAPOWERS OF BANACH ALGEBRAS AND MODULES

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Abstract. The Arens products are the standard way of extending the product from a Banach algebra $\mathcal{A}$ to its bidual $\mathcal{A}''$. Ultrapowers provide another method which is more symmetric, but one that in general will only give a bilinear map, which may not be associative. We show that if $\mathcal{A}$ is Arens regular, then there is at least one way to use an ultrapower to recover the Arens product, a result previously known for C*-algebras. Our main tool is a principle of local reflexivity result for modules and algebras.

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1. Introduction. Given a Banach algebra $\mathcal{A}$, there are two canonical ways to extend the products on $\mathcal{A}$ to products on the bidual $\mathcal{A}''$, called the Arens products (see [**2**]). Another, less common, way to view the bidual of a Banach space is as a quotient of an ultrapower of that space (see [**13**]). As the ultrapower of a Banach algebra is again a Banach algebra, this suggests another way of defining a 'product' on $\mathcal{A}''$.

We recall below the notion of an ultrapower of $\mathcal{A}$ with respect to an ultrafilter $\mathcal{U}$, written $(\mathcal{A})_{\mathcal{U}}$. By weak*-compactness of the ball of $\mathcal{A}''$, we can always define a norm-decreasing map

$$\sigma_{\mathcal{U}} : (\mathcal{A})_{\mathcal{U}} \to \mathcal{A}''; \quad \langle \sigma_{\mathcal{U}}((a_i)), \mu \rangle = \lim_{i \to \mathcal{U}} \langle \mu, a_i \rangle \qquad (\mu \in \mathcal{A}', (a_i) \in (\mathcal{A})_{\mathcal{U}}).$$

It follows from the principle of local reflexivity (see [**13**, Proposition 6.7] or Theorem 2.2) that for a suitable $\mathcal{U}$, that is an isometry $K : \mathcal{A}'' \to (\mathcal{A})_{\mathcal{U}}$ such that $\sigma_{\mathcal{U}} \circ K$ is the identity on $\mathcal{A}''$. Using this, we may define a bilinear map

$$\star = \underset{K,\mathcal{U}}{\star} : \mathcal{A}'' \times \mathcal{A}'' \to \mathcal{A}''; \quad \Phi \star \Psi = \sigma_{\mathcal{U}}(K(\Phi)K(\Psi)) \qquad (\Phi, \Psi \in \mathcal{A}'').$$

This idea was explored by Godefroy and Iochum in [**12**] and Iochum and Loupias in [**14**]. In general this might only lead to a bilinear map (which can fail to be associative). However, for C*-algebras, we always recover the Arens products.

In this paper, we shall explore these ideas further, and also consider related ideas for Banach modules. The Banach space tool which relates an ultrapower of a Banach space E to the bidual E'' is the principle of local reflexivity. It should hence come as no surprise that our line of attack is to prove various strengthenings of the principle of local reflexivity for Banach algebras and modules. For example, we show that when $\mathcal{A}$ is Arens regular, that is, the two Arens products agree on $\mathcal{A}''$, then there is at least one way to use an ultrapower of $\mathcal{A}$ to induce the same product on $\mathcal{A}''$. As a result, we get a 'symmetric' definition of the product on $\mathcal{A}''$.

We introduce some notation below, and summarise the results of [**12**] and [**14**]. We then study the principle of local reflexivity, and prove versions for Banach modules and Banach algebras which allow us to draw conclusions about ultrapowers. We investigate when $\star$ can actually be chosen to be an algebra homomorphism, and how these ideas interact with dual Banach algebras. Finally, we show that $\star$ can be badly behaved even for C*-algebras if the map K is not chosen to be an isometry.

1.1. Notations and basic concepts. Let E be a Banach space. We write E' for the dual space of E, and for $x \in E$ and $\mu \in E'$, we write $\langle \mu, x \rangle$ for $\mu(x)$. Recall the canonical map $\kappa_E : E \to E''$ defined by $\langle \kappa_E(x), \mu \rangle = \langle \mu, x \rangle$ for $x \in E$ and $\mu \in E'$. When κ_E is an isomorphism, we say that E is *reflexive*.

Recall the notions of filter and ultrafilter. Let $\mathcal{U}$ be a non-principal ultrafilter on a set I, and let E be a Banach space. We form the Banach space

$$\ell^\infty(E, I) = \left\{ (x_i)_{i\in I} \subseteq E : \|(x_i)\| := \sup_{i\in I} \|x_i\| < \infty \right\},$$

and define the closed subspace

$$\mathcal{N}_\mathcal{U} = \left\{ (x_i)_{i\in I} \in \ell^\infty(E, I) : \lim_{i\to\mathcal{U}} \|x_i\| = 0 \right\}.$$

Thus we can form the quotient space, called the *ultrapower of E with respect to $\mathcal{U}$*,

$$(E)_\mathcal{U} := \ell^\infty(E, I)/\mathcal{N}_\mathcal{U}.$$

In general, this space will depend on $\mathcal{U}$ (and upon, for example, if the continuum hypothesis holds), though many properties of $(E)_\mathcal{U}$ turn out to be independent of $\mathcal{U}$, as long as $\mathcal{U}$ is sufficiently 'large' in some sense.

We can verify that, if $(x_i)_{i\in I}$ represents an equivalence class in $(E)_\mathcal{U}$, then

$$\|(x_i)_{i\in I} + \mathcal{N}_\mathcal{U}\| = \lim_{i\to\mathcal{U}} \|x_i\|.$$

We shall abuse notation and write (x_i) for the equivalence class it represents; of course, it can be checked that any definition we make is independent of the choice of representative of equivalence class. There is a canonical isometry $E \to (E)_\mathcal{U}$ given by sending $x \in E$ to the constant family (x). We again abuse notation and write $x \in (E)_\mathcal{U}$, identifying E with a closed subspace of $(E)_\mathcal{U}$.

Recall the notion of a *countably incomplete* ultrafilter, for which see [**13**]. To avoid set-theoretic complications, we shall henceforth assume that all our ultrafilters are countably incomplete.

There is a canonical map $(E')_\mathcal{U} \to (E)'_\mathcal{U}$ given by

$$\langle (\mu_i), (x_i) \rangle = \lim_{i\to\mathcal{U}} \langle \mu_i, x_i \rangle \qquad ((\mu_i) \in (E')_\mathcal{U}, (x_i) \in (E)_\mathcal{U}).$$

This map is an isometry, and so we identify $(E')_\mathcal{U}$ with a closed subspace of $(E)'_\mathcal{U}$. It is shown in [**13**, Proposition 7.1] that $(E)'_\mathcal{U} = (E')_\mathcal{U}$ if and only if $(E)_\mathcal{U}$ is reflexive.

For Banach spaces E and F, we write $\mathcal{B}(E, F)$ for the space of bounded linear operators from E to F. Then there is a canonical isometric map $(\mathcal{B}(E, F))_\mathcal{U} \hookrightarrow \mathcal{B}((E)_\mathcal{U}, (F)_\mathcal{U})$ given by

$$T(x) = (T_i(x_i)) \qquad (T = (T_i) \in (\mathcal{B}(E, F))_\mathcal{U}, x = (x_i) \in (E)_\mathcal{U}).$$

We shall often identify $(\mathcal{B}(E, F))_\mathcal{U}$ with its image in $\mathcal{B}((E)_\mathcal{U}, (F)_\mathcal{U})$.

When $\mathcal{A}$ is a Banach algebra, $(\mathcal{A})_{\mathcal{U}}$ becomes a Banach algebra under the pointwise product. This follows, as it is easy to show that $\mathcal{N}_{\mathcal{U}}$ is a closed ideal in the Banach algebra $\ell^\infty(\mathcal{A}, I)$. Then $\mathcal{A}$ is commutative if and only if $(\mathcal{A})_{\mathcal{U}}$ is; $(\mathcal{A})_{\mathcal{U}}$ is unital if $\mathcal{A}$ is unital. If $\mathcal{A}$ is a Banach $*$-algebra (see [**5**, Chapter 3]) or a C*-algebra, then $(\mathcal{A})_{\mathcal{U}}$ is a Banach $*$-algebra or a C*-algebra, respectively, with the involution defined pointwise. Thus, as the class of $C(K)$ spaces for compact, Hausdorff spaces K is the class of commutative, unital C*-algebras, we see that the ultrapower of a $C(K)$ space is again a $C(K)$ space.

We now recall the Arens products on $\mathcal{A}''$. Firstly, we turn $\mathcal{A}'$ into a $\mathcal{A}$-bimodule in the usual fashion,

$$\langle a \cdot \mu, b\rangle = \langle \mu, ba\rangle, \quad \langle \mu \cdot a, b\rangle = \langle \mu, ab\rangle \qquad (a, b \in \mathcal{A}, \mu \in \mathcal{A}').$$

In a similar way, $\mathcal{A}''$, $\mathcal{A}'''$, and so forth also become $\mathcal{A}$-bimodules. Then we define bilinear map $\mathcal{A}'' \times \mathcal{A}', \mathcal{A}' \times \mathcal{A}'' \to \mathcal{A}'$ by

$$\langle \Phi \cdot \mu, a\rangle = \langle \Phi, \mu \cdot a\rangle, \quad \langle \mu \cdot \Phi, a\rangle = \langle \Phi, a \cdot \mu\rangle \qquad (\Phi \in \mathcal{A}'', \mu \in \mathcal{A}', a \in \mathcal{A}).$$

Finally, we define bilinear map $\Box, \Diamond : \mathcal{A}'' \times \mathcal{A}'' \to \mathcal{A}''$ by

$$\langle \Phi \Box \Psi, \mu\rangle = \langle \Phi, \Psi \cdot \mu\rangle, \quad \langle \Phi \Diamond \Psi, \mu\rangle = \langle \Psi, \mu \cdot \Phi\rangle \qquad (\Phi, \Psi \in \mathcal{A}'', \mu \in \mathcal{A}').$$

These are associative products which extend the natural action of $\mathcal{A}$ on $\mathcal{A}''$, called the *first* and *second Arens products*. See [**5**, Section 3.3] or [**17**, Section 1.4] for further details. Thus $\Box$ and $\Diamond$ agree with the usual product on $\kappa_{\mathcal{A}}(\mathcal{A})$. When $\Box$ and $\Diamond$ agree on all of $\mathcal{A}''$, we say that $\mathcal{A}$ is *Arens regular* (see also Section 5). As stated above, for suitable $\mathcal{U}$, given $\Phi, \Psi \in \mathcal{A}''$, we can find bounded families (a_i) and (b_i) with (a_i) tending to Φ weak* along $\mathcal{U}$, and (b_i) tending to Ψ. Then

$$\langle \Phi \Box \Psi, \mu\rangle = \lim_{j\to\mathcal{U}} \lim_{i\to\mathcal{U}} \langle \mu, a_i b_j\rangle, \quad \langle \Phi \Diamond \Psi, \mu\rangle = \lim_{i\to\mathcal{U}} \lim_{j\to\mathcal{U}} \langle \mu, a_i b_j\rangle \qquad (\mu \in \mathcal{A}').$$

We show that when $\mathcal{A}$ is Arens regular, we can find a more 'symmetric' version of these formulae.

Recall the map $\star = \underset{K,\mathcal{U}}{\star}$ defined above. We shall henceforth always assume that $K : \mathcal{A}'' \to (\mathcal{A})_{\mathcal{U}}$ is such that $\sigma \circ K$ is the identity on $\mathcal{A}''$ and that $K \circ \kappa_{\mathcal{A}}$ is the canonical map $\mathcal{A} \to (\mathcal{A})_{\mathcal{U}}$. This is enough to ensure

- for $\Phi \in \mathcal{A}''$ and $a \in \mathcal{A}$, we have $\Phi \star \kappa_{\mathcal{A}}(a) = \Phi \cdot a$ and $\kappa_{\mathcal{A}}(a) \star \Phi = a \cdot \Phi$;
- if $\mathcal{A}$ has a unit $e_{\mathcal{A}}$, then $\kappa_{\mathcal{A}}(e_{\mathcal{A}})$ is a unit for $\star$.

In [**14**], the authors make the further assumption that K is always an *isometry*. Under this extra condition, from the proof of [**12**, Corollary II.2], it follows that when $\mathcal{A}$ is a C*-algebra the map $\underset{K,\mathcal{U}}{\star}$ always agrees with $\Box = \Diamond$ (recall that a C*-algebra is always Arens regular, see for example [**12**, Corollary I.2]). We shall show that without this isometric condition on K, we do not always have $\underset{K,\mathcal{U}}{\star} = \Box$, even for some commutative C*-algebras. In [**12**, Example 3, Page 55], it is shown that when $\mathcal{A} = \ell^1$ with pointwise product (which is easily seen to be Arens regular), then for some K, we do not have that $\underset{K,\mathcal{U}}{\star}$ agrees with the Arens products.

In [**14**, Definition 5], the authors say that $\underset{K,\mathcal{U}}{\star}$ is *regular* if it is separately weak*-continuous. They show in [**14**, Proposition 6] that if $\underset{K,\mathcal{U}}{\star}$ is regular for some K and $\mathcal{U}$, then $\mathcal{A}$ is Arens regular, and that $\underset{K,\mathcal{U}}{\star} = \Box = \Diamond$. This follows fairly easily from weak*-continuity. Conversely, in [**14**, Theorem 12], the authors show that if $\mathcal{A}$ is commutative and not Arens regular, then $\underset{K,\mathcal{U}}{\star}$, as it is always a commutative bilinear map, never agrees with either Arens product. Let E be a reflexive Banach space with the *approximation property* (see [**22**, Section 4] or [**10**, Section VIII]). Let $\mathcal{A} = \mathcal{K}(E)$, the algebra of compact operators on E. Then [**14**, Corollary 14] shows that $\underset{K,\mathcal{U}}{\star}$ is associative on $\mathcal{A}$ if and only if it is regular. The second remark after this result in [**14**] asks if there is in general any link between associativity and regularity of $\underset{K,\mathcal{U}}{\star}$, something we do not consider further here.

2. The principle of local reflexivity. The classical principle of local reflexivity states that for a Banach space E, the local (or finite-dimensional) structure of E'' is the same as that of E, taking account of duality. Formally, we have

DEFINITION 2.1. Let E and F be Banach spaces, and let $T \in \mathcal{B}(E, F)$. For $\epsilon > 0$, we say that T is a $(1+\epsilon)$-*isomorphism onto its range* if $(1-\epsilon)\|x\| \leq \|T(x)\| \leq (1+\epsilon)\|x\|$ for each $x \in E$.

THEOREM 2.2. *Let E be a Banach space, and let $M \subseteq E''$ and $N \subseteq E'$ be finite-dimensional subspaces. For each $\epsilon > 0$ there exists a $(1+\epsilon)$-isomorphism onto its range $T : M \to E$ such that*

(1) $\langle \Phi, \mu \rangle = \langle \mu, T(\Phi) \rangle$ *for* $\mu \in N$ *and* $\Phi \in M$;
(2) $T(\kappa_E(x)) = x$ *for* $x \in E$ *such that* $\kappa_E(x) \in M$.

Proof. See [**22**, Section 5.5] for a readable account. $\Box$

We wish to extend this result, using the results of Behrends [**3**]. Before we can do this, we need a word about tensor products of Banach spaces. Let E and F be Banach spaces, and let $E \otimes F$ be the algebraic tensor product of E with F. We define the *projective tensor norm* on $E \otimes F$ by

$$\|u\|_\pi = \inf\left\{ \sum_{i=1}^n \|x_i\|\|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i \right\} \quad (u \in E \otimes F).$$

Then the completion of $E \otimes F$ with respect to $\|\cdot\|_\pi$ is the *projective tensor product*, $E\widehat{\otimes}F$.

See [**8**], [**10**, Section VIII] or [**22**] for further details. In particular, when E is a Banach space and M is a *finite-dimensional* Banach space, then the dual of $\mathcal{B}(M, E)$ may be identified with $M\widehat{\otimes}E'$ by

$$\langle x \otimes \mu, T \rangle = \langle \mu, T(x) \rangle \quad (x \otimes \mu \in M\widehat{\otimes}E', T \in \mathcal{B}(M, E)).$$

For general Banach spaces F and G, the dual of $F\widehat{\otimes}G$ is $\mathcal{B}(F, G')$, under the identification

$$\langle T, x \otimes y \rangle = \langle T(x), y \rangle \quad (x \otimes y \in F \otimes G, T \in \mathcal{B}(F, G')).$$

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Thus $\mathcal{B}(M,E)'' = (M\widehat{\otimes}E')' = \mathcal{B}(M,E'')$, and we can check that the canonical map $\kappa_{\mathcal{B}(M,E)} : \mathcal{B}(M,E) \to \mathcal{B}(M,E'')$ satisfies $\kappa_{\mathcal{B}(M,E)}(T) = \kappa_E \circ T$ for $T \in \mathcal{B}(M,E)$.

The following definitions are from [**3**]. For a Banach space E, we write $\mathrm{FIN}(E)$ for the collection of finite-dimensional subspaces of E.

DEFINITION 2.3. Let E be a Banach space, and let $M \in \mathrm{FIN}(E'')$ and $N \in \mathrm{FIN}(E')$. A map $T : M \to E$ is an *ϵ-isomorphism along N* if T is a $(1+\epsilon)$-isomorphism onto its range such that $\langle \Phi, \mu\rangle = \langle \mu, T(\Phi)\rangle$ for $\mu \in N$ and $\Phi \in M$.

Let $(F_i)_{i=1}^n$ and $(G_j)_{j=1}^m$ be families of Banach spaces. Let $A_i : \mathcal{B}(M,E) \to F_i$ be an operator, for $1 \le i \le n$, and let $\psi_j : \mathcal{B}(M,E) \to G_j$ be an operator, for $1 \le j \le m$. For $1 \le i \le n$, let $f_i \in F_i$, and for $1 \le j \le m$, let $C_j \subseteq G_j$ be a convex set. Then M satisfies

(1) the *exact conditions* (A_i, f_i), for $1 \le i \le n$, and

(2) the *approximate conditions* (ψ_j, C_j), for $1 \le j \le m$,

if for each $N \in \mathrm{FIN}(E')$ and $\epsilon > 0$, there exists an ϵ-isomorphism $T : M \to E$ along N such that $A_i(T) = f_i$, for $1 \le i \le n$, and $\psi_j(T) \in (C_j)_\epsilon = \{y + z : y \in C_j, z \in G_j, \|z\| \le \epsilon\}$, for $1 \le j \le m$.

Given (A_i) as in the above definition, notice that we have $A_i' : F_i' \to M\widehat{\otimes}E'$ and $A_i'' : \mathcal{B}(M,E'') \to F_i''$. For $M \subseteq E''$, let $\iota_M : M \to E''$ be the inclusion map.

THEOREM 2.4. *Let E be a Banach space, $M \in \mathrm{FIN}(E'')$, and let (F_i), (A_i), (y_i), (G_j), (ψ_j) and (C_j) be as defined in the above definition. Then the following are equivalent:*

1. *M satisfies the exact conditions $(A_i, y_i)_{i=1}^n$ and the approximate conditions $(\psi_j, C_j)_{j=1}^m$;*
2. *ι_M is weak*-continuous on the weak*-closure of $A_1'(F_1') + \cdots + A_n'(F_n')$, $A_i''(\iota_M) = \kappa_{F_i}(y_i)$ for each i, and $\psi_j''(\iota_M)$ lies in the weak*-closure of $\kappa_{G_j}(C_j)$, for each j.*

Suppose that the map $T \mapsto (A_i(T))_{i=1}^n$ from $\mathcal{B}(M,E)$ to $A_1 \oplus \cdots \oplus A_n$ has a closed range. Then we may replace ι_M being weak-continuous on the weak*-closure of $\sum_{i=1}^n A_i'(F_i')$ by there existing $T : M \to E$ which satisfies $A_i(T) = y_i$, for $1 \le i \le n$ (T need not satisfy any other condition).*

Proof. This is in [**3**, Theorem 2.3], and the remark thereafter. □

For example, let $A_M : \mathcal{B}(M,E) \to \mathcal{B}(M \cap \kappa_E(E), E)$ be the restriction operator, and let $B_M \in \mathcal{B}(M \cap \kappa_E(E), E)$ be the map $B_M(\kappa_E(x)) = x$. Then the principle of local reflexivity is just the statement that each $M \in \mathrm{FIN}(E'')$ satisfies the exact condition (A_M, B_M). Notice that $A_M'' : \mathcal{B}(M,E'') \to \mathcal{B}(M \cap \kappa_E(E), E'')$ is also the restriction operator, and that $B_M'' : M \cap \kappa_E(E) \to E'$ is the inclusion map, so that condition (2) given above is easily verified in this case (or one can use the remark).

3. Ultrapowers of modules. We wish to extend the principle of local reflexivity to (bi)modules of Banach algebras. Let $\mathcal{A}$ be a Banach algebra, and let E be a Banach left $\mathcal{A}$-module (or a right $\mathcal{A}$-module, or an $\mathcal{A}$-bimodule), so that we can certainly apply the principle of local reflexivity to E. However, we also want to take account of the $\mathcal{A}$-module structure, that is, ensure that $T : M \to E$ is 'in some sense' an $\mathcal{A}$-module homomorphism (of course, M will in general not be a submodule).

It will be helpful to recall that κ_E is an $\mathcal{A}$-module homomorphism. For the following, note that for $L \subseteq \mathcal{A}$ and $M \subseteq E''$ finite-dimensional, we have

$$L \cdot M = \{a \cdot \Phi : a \in L, \Phi \in M\} \in \mathrm{FIN}(E'').$$

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THEOREM 3.1. *Let $\mathcal{A}$ be a Banach algebra and let E be a Banach left $\mathcal{A}$-module. Let $M \subseteq E''$, $L \subseteq \mathcal{A}$ and $N \subseteq E'$ be finite-dimensional, and let $\epsilon > 0$. Let $M_0 \in \mathrm{FIN}(E'')$ be such that $L \cdot M + M \subseteq M_0$. Then there exists $T : M_0 \to E$, a $(1+\epsilon)$-isomorphism onto its range, such that,*

1. $\langle \Phi, \mu \rangle = \langle \mu, T(\Phi) \rangle$ *for* $\Phi \in M_0$ *and* $\mu \in N$;
2. $T(\kappa_E(x)) = x$ *for* $\kappa_E(x) \in M_0 \cap \kappa_E(E)$;
3. $\|a \cdot T(\Phi) - T(a \cdot \Phi)\| \leq \epsilon \|a\| \|\Phi\|$ *for* $a \in L$ *and* $\Phi \in M$.

A similar result holds for Banach right $\mathcal{A}$-modules and Banach $\mathcal{A}$-bimodules with condition (3) changed in the obvious way.

Proof. Let $\delta = \epsilon/5$ or 1, whichever is smaller. Let $(a_i)_{i=1}^n$ be a set in L such that $\|a_i\| = 1$ for each i, and such that

$$\min_{1 \leq i \leq n} \|a - a_i\| < \delta \quad (a \in L, \|a\| = 1).$$

For $1 \leq i \leq n$, define $\psi_i : \mathcal{B}(M_0, E) \to \mathcal{B}(M, E)$ by

$$\psi_i(T)(\Phi) = T(a_i \cdot \Phi) - a_i \cdot T(\Phi) \quad (T \in \mathcal{B}(M_0, E), \Phi \in M).$$

Then $\psi_i' : M \widehat{\otimes} E' \to M_0 \widehat{\otimes} E'$, and for $\Phi \in M$, $\mu \in E'$ and $T \in \mathcal{B}(M_0, E)$, we have

$$\langle \psi_i'(\Phi \otimes \mu), T \rangle = \langle \mu, \psi_i(T)(\Phi) \rangle = \langle \mu, T(a_i \cdot \Phi) - a_i \cdot T(\Phi) \rangle,$$

so that $\psi_i'(\Phi \otimes \mu) = a_i \cdot \Phi \otimes \mu - \Phi \otimes \mu \cdot a_i$. Then, for $\Phi \in M$ and $\mu \in E'$, we have

$$\langle \psi_i''(\iota_{M_0}), \Phi \otimes \mu \rangle = \langle \iota_{M_0}, a_i \cdot \Phi \otimes \mu - \Phi \otimes \mu \cdot a_i \rangle = \langle a_i \cdot \Phi, \mu \rangle - \langle \Phi, \mu \cdot a_i \rangle = 0,$$

so that $\psi_i''(\iota_{M_0}) = 0$.

Consider the exact condition (A_{M_0}, B_{M_0}), as after Theorem 2.4. Then clearly we have verified condition (2) for the approximate conditions $(\psi_i, \{0\})$, for $1 \leq i \leq n$. Applying Theorem 2.4, we find $T \in \mathcal{B}(M_0, E)$, a $(1+\delta)$-isomorphism onto its range, with conditions (1) and (2), and such that $\|\psi_i(T)\| < \delta$ for $1 \leq i \leq n$. Then, for $a \in L$ and $\Phi \in M$ with $\|a\| = \|\Phi\| = 1$, we can find i with $\|a - a_i\| < \delta$. Then we have

$$\begin{aligned}
&\|a \cdot T(\Phi) - T(a \cdot \Phi)\| \\
&\qquad \leq \|(a - a_i) \cdot T(\Phi)\| + \|a_i \cdot T(\Phi) - T(a_i \cdot \Phi)\| + \|T(a_i \cdot \Phi - a.\Phi)\| \\
&\qquad < \delta(1+\delta) + \|\psi_i(T)\| + (1+\delta)\delta < 3\delta + 2\delta^2 < \epsilon.
\end{aligned}$$

Thus we are done, as $\delta < \epsilon$.

Similarly, we can easily adapt the above argument to give the result for right $\mathcal{A}$-modules and $\mathcal{A}$-bimodules. □

It would be nice if we could work with the exact conditions $(\psi_i, \{0\})$ given above, but it is far from clear that we can apply condition (2) of Theorem 2.4 in this case. Fortunately, the above is enough for our application.

We now apply this to prove a result about ultrapowers of modules. Notice that for a Banach algebra $\mathcal{A}$, a left $\mathcal{A}$-module E and an ultrafilter $\mathcal{U}$, we have that $(E)_{\mathcal{U}}$ is a left $\mathcal{A}$-module with pointwise module action. As usual, E' becomes a right $\mathcal{A}$-module for the module action given by

$$\langle \mu \cdot a, x \rangle = \langle \mu, a \cdot x \rangle \qquad (a \in \mathcal{A}, x \in E, \mu \in E').$$

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Similarly, E'' becomes a left $\mathcal{A}$-module. Recall the map $\sigma_{\mathcal{U}} : (E)_{\mathcal{U}} \to E''$; this is easily seen to be a left $\mathcal{A}$-module homomorphism, as

$$\langle a \cdot \sigma_{\mathcal{U}}(x), \mu \rangle = \lim_{i \to \mathcal{U}} \langle \mu \cdot a, x_i \rangle = \lim_{i \to \mathcal{U}} \langle \mu, a \cdot x_i \rangle = \langle \sigma_{\mathcal{U}}(a \cdot x), \mu \rangle,$$

for $a \in \mathcal{A}$, $\mu \in E'$ and $x = (x_i) \in (E)_{\mathcal{U}}$. This all holds, with obvious modifications, for right $\mathcal{A}$-modules and $\mathcal{A}$-bimodules.

THEOREM 3.2. *Let $\mathcal{A}$ be a Banach algebra, and let E be a left $\mathcal{A}$-module. Then there exists an ultrafilter $\mathcal{U}$ and a map $K : E'' \to (E)_{\mathcal{U}}$ such that*

1. *K is an isometry and a left $\mathcal{A}$-module homomorphism;*
2. *$\sigma_{\mathcal{U}} \circ K$ is the identity on E'';*
3. *$K \circ \kappa_E : E \to (E)_{\mathcal{U}}$ is the canonical map.*

Similar results hold for right $\mathcal{A}$-modules and $\mathcal{A}$-bimodules.

Proof. We carefully prove this sort of result once, for completeness. Define

$$I = \big\{(M, N, L, \epsilon) : M \in \mathrm{FIN}(E''), N \in \mathrm{FIN}(E'), L \in \mathrm{FIN}(\mathcal{A}), \epsilon > 0\big\},$$

with partial order given by $(M_1, N_1, L_1, \epsilon_1) \geq (M_2, N_2, L_2, \epsilon_2)$ if and only if $M_1 \supseteq M_2$, $N_1 \supseteq N_2$, $L_1 \supseteq L_2$ and $\epsilon_1 \leq \epsilon_2$. Then $(I, \leq)$ becomes a directed set, and let $\mathcal{U}$ be some ultrafilter refining the order-filter on I.

We now define $K : E'' \to (E)_{\mathcal{U}}$. Fix $\Phi \in E''$, and let $K(\Phi) = (x_i)_{i \in I} \in (E)_{\mathcal{U}}$, where, for $i = (M, N, L, \epsilon) \in I$, we let $x_i = T(\Phi)$ where $T : M \to E$ is given by the above theorem (with, say, $M_0 = L \cdot M + M$). Then, if $\Phi, \Psi \in E''$, let $K(\Phi) = (x_i)$, $K(\Psi) = (y_i)$ and $K(\Phi + \Psi) = (z_i)$, so that for $i = (M, N, L, \epsilon)$ with $\Phi, \Psi \in M$, we have $x_i + y_i = T(\Phi) + T(\Psi) = T(\Phi + \Psi) = z_i$. Thus $(x_i) + (y_i) = (z_i)$ in $(E)_{\mathcal{U}}$, so that K is linear. Similarly, we see that K is an isometry, as $\epsilon \to 0$ along $\mathcal{U}$. It follows from conditions (1) and (2) in the above theorem, that, respectively, $\sigma_{\mathcal{U}} \circ K$ is the identity on E'', and $K \circ \kappa_E$ is the canonical map $E \to (E)_{\mathcal{U}}$.

Finally, we show that K is a left $\mathcal{A}$-module homomorphism. Let $a \in \mathcal{A}$ and $\Phi \in E''$, let $K(\Phi) = (x_i)$ and $K(a \cdot \Phi) = (y_i)$, and let $i = (M, N, L, \epsilon)$ with $\Phi, a \cdot \Phi \in M$ and $a \in L$. Then

$$\|a \cdot x_i - y_i\| = \|a \cdot T(\Phi) - T(a \cdot \Phi)\| \leq \epsilon \|a\| \|\Phi\|,$$

by condition (3) in the above theorem. Thus $a \cdot K(\Phi) = K(a \cdot \Phi)$ in $(E)_{\mathcal{U}}$. □

4. Ultrapowers of algebras. Let $\mathcal{A}$ be a Banach algebra. Using the above, as $\mathcal{A}''$ is an $\mathcal{A}$-bimodule, we can find an ultrafilter $\mathcal{U}$ and an isometry $K : \mathcal{A}'' \to (\mathcal{A})_{\mathcal{U}}$ which is an $\mathcal{A}$-module homomorphism, and with $\sigma_{\mathcal{U}} \circ K$ the identity on $\mathcal{A}''$. This is not quite enough to show that $\underset{K,\mathcal{U}}{\star}$ agrees with either Arens product on $\mathcal{A}''$. We now show that, at least when $\mathcal{A}$ is Arens regular, we can do better.

THEOREM 4.1. *Let $\mathcal{A}$ be an Arens regular Banach algebra, $M \in \mathrm{FIN}(\mathcal{A}'')$, $N \in \mathrm{FIN}(\mathcal{A}')$ and $\epsilon > 0$. Let $M_0 = M + M \Box M$ and $N_0 = N + M \cdot N$. Then there exists a $(1 + \epsilon)$-isomorphism onto its range $T : M_0 \to \mathcal{A}$ such that*

1. *$\langle \Phi, \mu \rangle = \langle \mu, T(\Phi) \rangle$ for $\Phi \in M_0$ and $\mu \in N_0$;*
2. *$T(\kappa_{\mathcal{A}}(a)) = a$ for $\kappa_{\mathcal{A}}(a) \in M_0 \cap \kappa_{\mathcal{A}}(\mathcal{A})$;*
3. *$|\langle \mu, T(\Phi \Box \Psi) - T(\Phi)T(\Psi) \rangle| \leq \epsilon \|\mu\| \|\Phi\| \|\Psi\|$ for $\mu \in N$ and $\Phi, \Psi \in M$.*

Proof. Let $\delta > 0$ be such that $\delta < \epsilon$ and $\delta(1+\delta)(3+\delta) < \epsilon$. Let $(\mu_i)_{i=1}^n \subseteq N$ be such that $\|\mu_i\| = 1$ for each i, and such that we have

$$\min_{1\leq i\leq n} \|\mu_i - \mu\| < \delta \quad (\mu \in N, \|\mu\| = 1).$$

For $1 \leq i \leq n$, define $\psi_i : \mathcal{B}(M_0, \mathcal{A}) \to \mathcal{B}(M_0, \mathcal{A}')$ by

$$\psi_i(T)(\Phi) = T(\Phi) \cdot \mu_i \quad (T \in \mathcal{B}(M_0, \mathcal{A}), \Phi \in M_0),$$

and define $T_i \in \mathcal{B}(M_0, \mathcal{A}')$ by $T_i(\Phi) = \Phi \cdot \mu_i$ for $\Phi \in M_0$. Then we have $\psi_i' : M_0\widehat{\otimes}\mathcal{A}'' \to M_0\widehat{\otimes}\mathcal{A}'$, and, for $T \in \mathcal{B}(M_0, \mathcal{A})$, $\Phi \in M_0$ and $\Lambda \in \mathcal{A}''$, we have

$$\langle \psi_i'(\Phi \otimes \Lambda), T\rangle = \langle \Lambda, \psi_i(T)(\Phi)\rangle = \langle \Lambda, T(\Phi) \cdot \mu_i\rangle = \langle \mu_i \cdot \Lambda, T(\Phi)\rangle.$$

Thus we have $\psi_i'(\Phi \otimes \Lambda) = \Phi \otimes \mu_i \cdot \Lambda$, and so

$$\begin{aligned}\langle \psi_i''(\iota_{M_0}), \Phi \otimes \Lambda\rangle &= \langle \Phi, \mu_i \cdot \Lambda\rangle = \langle \Lambda\Diamond\Phi, \mu_i\rangle \\ &= \langle \Lambda\Box\Phi, \mu_i\rangle = \langle \Lambda, \Phi \cdot \mu_i\rangle = \langle \kappa_{\mathcal{A}'}(T_i(\Phi)), \Lambda\rangle, \end{aligned} \tag{4.1}$$

as $\mathcal{A}$ is Arens regular. Thus $\psi_i''(\iota_{M_0}) = \kappa_{\mathcal{B}(M_0,\mathcal{A}')}(T_i)$. Again, we can then find $T \in \mathcal{B}(M_0, \mathcal{A})$ satisfying (1) and (2), and such that $\|\psi_i(T) - T_i\| < \delta$ for $1 \leq i \leq n$.

For $\mu \in N$ and $\Phi, \Psi \in M$ with $\|\mu\| = \|\Phi\| = \|\Psi\| = 1$, let i be such that $\|\mu - \mu_i\| < \delta$. Then $\Phi\Box\Psi \in M_0$ and $\Psi \cdot \mu \in N_0$ so that we have

$$\langle \mu, T(\Phi\Box\Psi)\rangle = \langle \Phi\Box\Psi, \mu\rangle = \langle \Phi, \Psi \cdot \mu\rangle = \langle \Psi \cdot \mu, T(\Phi)\rangle.$$

As $\|\psi_i(T) - T_i\| < \delta$, we have $\|T(\Psi) \cdot \mu_i - \Psi \cdot \mu_i\| < \delta$, and so

$$\begin{aligned}&\|T(\Psi) \cdot \mu - \Psi \cdot \mu\| \\ &\quad\leq \|T(\Psi) \cdot \mu - T(\Psi) \cdot \mu_i\| + \|T(\Psi) \cdot \mu_i - \Psi \cdot \mu_i\| + \|\Psi \cdot \mu_i - \Psi \cdot \mu\| \\ &\quad< \delta\|T(\Psi)\| + \delta + \delta\|\Psi\| \leq \delta(1+\delta) + 2\delta.\end{aligned}$$

Putting these together, we then get

$$\begin{aligned}|\langle \mu, T(\Phi\Box\Psi) - T(\Phi)T(\Psi)\rangle| &= |\langle \Psi \cdot \mu, T(\Phi)\rangle - \langle T(\Psi) \cdot \mu, T(\Phi)\rangle| \\ &< \|T(\Phi)\|\big(\delta(1+\delta) + 2\delta\big) \leq (1+\delta)(\delta(1+\delta) + 2\delta) \\ &= \delta(1+\delta)(3+\delta) < \epsilon,\end{aligned}$$

as required, completing the proof. □

THEOREM 4.2. *Let $\mathcal{A}$ be an Arens regular Banach algebra. There exists an ultrafilter $\mathcal{U}$ and an isometry $K : \mathcal{A}'' \to (\mathcal{A})_{\mathcal{U}}$ such that*

1. *$\sigma_{\mathcal{U}} \circ K$ is the identity on A'';*
2. *$K \circ \kappa_{\mathcal{A}}$ is the canonical map $\mathcal{A} \to (\mathcal{A})_{\mathcal{U}}$;*
3. *$\underset{K,\mathcal{U}}{\star}$, defined using K, agrees with the Arens products on $\mathcal{A}''$.*

Proof. This follows exactly as for the proof of Theorem 3.2. □

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Let $\mathcal{A}$ be an Arens regular Banach algebra, and form $K : \mathcal{A}'' \to (\mathcal{A})_{\mathcal{U}}$ as in the theorem. For $\Phi, \Psi \in \mathcal{A}''$, let $(a_i) = K(\Phi)$ and $(b_i) = K(\Psi)$, so that for $\mu \in \mathcal{A}'$,

$$\langle \Phi, \mu \rangle = \lim_{i \to \mathcal{U}} \langle \mu, a_i \rangle, \quad \langle \Psi, \mu \rangle = \lim_{i \to \mathcal{U}} \langle \mu, b_i \rangle,$$
$$\langle \Phi \Box \Psi, \mu \rangle = \left\langle \Phi \underset{K,\mathcal{U}}{\star} \Psi \mu \right\rangle = \lim_{i \to \mathcal{U}} \langle \mu, a_i b_i \rangle.$$

Compare this symmetric definition of $\Phi \Box \Psi$ to the formulae in Section 1.1. Of course, here we have to be careful in our choice of K.

As in the introduction, we note that [**14**, Theorem 12] shows that it is too much to expect the above to be true for a non-Arens regular Banach algebra $\mathcal{A}$, with (3) replaced by asking for $\underset{K,\mathcal{U}}{\star}$ to agree with $\Box$ or $\Diamond$. At least, this is true if $\mathcal{A}$ is commutative. It would be interesting to know if we could ever have, say, $\underset{K,\mathcal{U}}{\star} = \Box$ for a non-commutative, non-Arens regular Banach algebra.

4.1. Asking for an algebra homomorphism. A much stronger result than the above would be to find $\mathcal{U}$ and $K : \mathcal{A}'' \to (\mathcal{A})_{\mathcal{U}}$ with K being an algebra homomorphism (presumably assuming that $\mathcal{A}$ is Arens regular). We now present a case when this is possible.

PROPOSITION 4.3. *Let $\mathcal{A}$ be a commutative, Arens regular Banach algebra such that $\mathcal{A}$ is an essential ideal in $\mathcal{A}''$. Suppose that $\mathcal{A}$ has an approximate identity (e_α) consisting of idempotents, which is bounded in the multiplier norm. Then there exists an ultrafilter $\mathcal{U}$ and $K : \mathcal{A}'' \to (\mathcal{A})_{\mathcal{U}}$ with K being an algebra homomorphism, and such that $\sigma \circ K$ is the identity on $\mathcal{A}''$, and $K \circ \kappa_{\mathcal{A}}$ is the canonical inclusion $\mathcal{A} \to (\mathcal{A})_{\mathcal{U}}$.*

Proof. Let (e_α) be indexed by the directed set I, and let $\mathcal{U}$ be an ultrafilter on I refining the order filter. As $\mathcal{A}$ is an ideal in $\mathcal{A}''$, we see that for $\Phi \in \mathcal{A}''$, we have that $\Phi e_\alpha \in \mathcal{A}$ for each $\alpha \in I$. As (e_α) is bounded in the multiplier norm, there exists $M > 0$ such that

$$\|a e_\alpha\| \leq M \|a\| \quad (a \in \mathcal{A}, \alpha \in I).$$

Thus $\|e_\alpha \cdot \mu\| \leq M \|\mu\|$ for $\mu \in \mathcal{A}'$ and each α, and so $\|\Phi e_\alpha\| \leq M \|\Phi\|$ for $\Phi \in \mathcal{A}''$ and each α. Hence we may define K by

$$K(\Phi) = (\Phi e_\alpha) \in (\mathcal{A})_{\mathcal{U}} \quad (\Phi \in \mathcal{A}'').$$

Hence K is linear and bounded, and as (e_α) is an approximate identity, we see that $K \circ \kappa_{\mathcal{A}}$ is the canonical map $\mathcal{A} \to (\mathcal{A})_{\mathcal{U}}$. For each α, we know that $e_\alpha^2 = e_\alpha$, and so

$$K(\Phi)K(\Psi) = (\Phi e_\alpha \Psi e_\alpha) = (\Phi \Psi e_\alpha e_\alpha) = K(\Phi \Psi) \quad (\Phi, \Psi \in \mathcal{A}'').$$

For $\mu \in \mathcal{A}'$ and $a \in \mathcal{A}$, we see that

$$\langle \sigma K(\Phi), a \cdot \mu \rangle = \lim_{\alpha \to \mathcal{U}} \langle \Phi, e_\alpha a \cdot \mu \rangle = \langle \Phi, a \cdot \mu \rangle \quad (\Phi \in \mathcal{A}'').$$

Hence $\sigma K(\Phi) = \Phi$ when restricted to $\mathcal{A} \cdot \mathcal{A}'$. If $\Phi \in (\mathcal{A} \cdot \mathcal{A}')^\perp$, then $\Phi a = 0$ for each $a \in \mathcal{A}$. As $\mathcal{A}$ is an essential ideal in $\mathcal{A}''$, by definition, $\Phi = 0$, and so we conclude that $\sigma \circ K$ is the identity on $\mathcal{A}''$. $\Box$

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The above applies in particular to $\mathcal{A} = c_0$. It would be interesting to know if the same conclusions hold in the *non-commutative* case, namely the compact operators on ℓ^2.

Following an example from [**11**], let $\mathcal{A} = C([0,1])$, a commutative C*-algebra. Then $\mathcal{A}''$ is a von Neumann algebra, and so has many (self-adjoint) projections. Let $\mathcal{U}$ be an ultrafilter, and suppose that $\mathcal{A}''$ is isomorphic, as a Banach algebra, to a subalgebra of $(\mathcal{A})_{\mathcal{U}}$. Thus $(\mathcal{A})_{\mathcal{U}}$ contains non-trivial, not necessarily self-adjoint, projections. However, $(\mathcal{A})_{\mathcal{U}}$ is isomorphic to $C(K)$ for some compact Hausdorff K, and any projection in $C(K)$ is automatically self-adjoint. It follows from [**11**, Proposition 2.1] that $(\mathcal{A})_{\mathcal{U}}$ does not contain non-trivial projections, as the only projections in $\mathcal{A}$ are 0 and 1, and so we have a contradiction. It would be interesting to, say, characterise C*-algebras $\mathcal{A}$ which are such that $\mathcal{A}''$ is isomorphic, or $*$-isomorphic, to a subalgebra of an ultrapower of $\mathcal{A}$. This of course has links to the notorious Connes-embedding problem for von Neumann algebras.

Following [**12**, Example III.1], let $\mathcal{A} = \ell^1$ with the pointwise product. Then $c_0 \subseteq (\ell^1)' = \ell^\infty$, and we can decompose $(\ell^1)''$ as $c_0^\perp \oplus \ell^1$, where

$$c_0^\perp = \{\Phi \in (\ell^1)'' : \langle \Phi, x\rangle = 0 \ (x \in c_0)\}.$$

Furthermore, $\|\Phi + a\| = \|\Phi\| + \|a\|$ for $\Phi \in c_0^\perp$ and $a \in \ell^1$. Then the product on $(\ell^1)''$ is simply

$$(\Phi, a)(\Psi, b) = ab \qquad (a, b \in \ell^1,\ \Phi, \Psi \in c_0^\perp).$$

Thus $\mathcal{A}$ is an ideal in $\mathcal{A}''$, and clearly $\mathcal{A}$ has an approximate identity consisting of idempotents, and bounded in the multiplier norm. Notice that $\mathcal{A}$ is certainly not an essential ideal in $\mathcal{A}''$.

Proposition 4.4. *With notation as above, there exists an ultrafilter $\mathcal{U}$ and an isometry $K : (\ell^1)'' \to (\ell^1)_{\mathcal{U}}$ satisfying the conclusions of Theorem 4.2, with K being a homomorphism.*

Proof. Let $M \in \mathrm{FIN}((\ell^1)'')$, $N \in \mathrm{FIN}(\ell^\infty)$ and $\epsilon > 0$. As $(\ell^1)''$ is an $\mathcal{L}_1$ space (see [**22**, Chapter 2], for example), by enlarging M as necessary, we can find a basis $(m_i)_{i=1}^k$ for M such that $\|m_i\| = 1$ for each i, and

$$(1-\epsilon)\sum_{i=1}^k |\alpha_i| \le \left\| \sum_{i=1}^k \alpha_i m_i \right\| \le (1+\epsilon)\sum_{i=1}^k |\alpha_i| \qquad ((\alpha_i)_{i=1}^k \subseteq \mathbb{C}).$$

As $(\ell^1)'' = c_0^\perp \oplus \ell^1$, we may suppose that $m_i \in \ell^1$ for $1 \le i < \hat{k}$ and $m_i \in c_0^\perp$ for $\hat{k} \le i \le k$. Similarly, it is no loss of generality to suppose that N is the linear span of indicator functions $\chi_{A_1}, \ldots, \chi_{A_l}$ with $(A_i)_{i=1}^l \subseteq \mathbb{N}$ pairwise disjoint. Furthermore, we may suppose that

$$\sup\left\{ |\langle m, x\rangle| : x \in N, \|x\| \le 1 \right\} \ge (1-\epsilon)\|m\| \quad (m \in M).$$

Order (A_i) so that A_i is finite for $1 \le i < \hat{l}$, and A_i is infinite otherwise.

Let $a_i = m_i \in \ell^1$ for $1 \le i < \hat{k}$, let $a_i = (a_t^{(i)})_{t\in\mathbb{N}} \in \ell^1$. Suppose we have chosen $a_1, \ldots, a_r$, and let N be such that $\sum_{|t|>N} |a_t^{(i)}| < \epsilon$ for $1 \le i \le r$, and such that $A_j \subseteq$

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$\{1, \cdots, N\}$ for $1 \leq j < \hat{l}$. As $m_{r+1} \in c_0^{\perp}$, for all finite sets $A \subseteq \mathbb{N}$, we have that

$$\langle m_{r+1}, \chi_{A_j} \rangle = 0, \quad (1 \leq j < \hat{l}), \qquad \langle m_{r+1}, \chi_{A_j \setminus A} \rangle = \langle m_{r+1}, \chi_{A_j} \rangle, \quad (\hat{l} \leq j \leq l).$$

Consequently, by a simple argument, we can find $a_{r+1} = (a_t^{(r+1)}) \in \ell^1$ with $\|a_{r+1}\| = \|m_{r+1}\| = 1$, $\langle \chi_{A_j}, a_{r+1} \rangle = \langle m_{r+1}, \chi_{A_j} \rangle$ for $\hat{l} \leq j \leq l$, with $a_t^{(r+1)} = 0$ for $t \leq N$, and with $|a_t^{(r+1)}| < \epsilon$ for all t. Thus $\langle \chi_{A_j}, a_{r+1} \rangle = 0 = \langle m_{r+1}, \chi_{A_j} \rangle$ for $1 \leq j < \hat{l}$.

Hence we find $(a_i)_{i=1}^k \in \ell^1$ such that $\langle x, a_i \rangle = \langle m_i, x \rangle$ for $x \in N$ and $1 \leq i \leq k$. Furthermore, $(a_i)_{i=\hat{k}}^k$ have disjoint support. Consequently, for $(\alpha_i)_{i=1}^k \subseteq \mathbb{C}$,

$$\left\| \sum_{i=1}^k \alpha_i a_i \right\| = \left\| \sum_{i=1}^{\hat{k}-1} \alpha_i m_i \right\| + \sum_{i=\hat{k}}^k |\alpha_i| \leq (1+\epsilon) \sum_{i=1}^k |\alpha_i| \leq \frac{1+\epsilon}{1-\epsilon} \left\| \sum_{i=1}^k \alpha_i m_i \right\|.$$

Similarly,

$$\left\| \sum_{i=1}^k \alpha_i a_i \right\| \geq \sup \left\{ \sum_{i=1}^k \alpha_i \langle x, a_i \rangle : x \in N, \|x\| = 1 \right\} \geq (1-\epsilon) \left\| \sum_{i=1}^k \alpha_i m_i \right\|.$$

Hence the map $T : M \to \ell^1$ defined by $T(m_i) = a_i$ and linearity is a $(1+\hat{\epsilon})$-isomorphism onto its range, for $\hat{\epsilon} = 2\epsilon(1-\epsilon)^{-1}$. Furthermore, $\langle m, x \rangle = \langle x, T(m) \rangle$ for $m \in M$ and $x \in N$, and $T\kappa_{\ell^1}(a) = a$ for $a \in \ell^1$ with $\kappa_{\ell^1}(a) \in M$.

By a now standard argument, we find an isometry $K : (\ell^1)'' \to (\ell^1)_{\mathcal{U}}$ such that $\sigma_{\mathcal{U}} K$ is the identity on $(\ell^1)''$, $K\kappa_{\ell^1}$ is the canonical map $\ell^1 \to (\ell^1)_{\mathcal{U}}$, and for $\Phi, \Psi \in c_0^{\perp}$, we have $K(\Phi)K(\Psi) = 0$. This last fact follows as we chose $(a_i)_{i=\hat{k}}^k$ above with disjoint support, and with $\|a_i\|_\infty$ small for $i \geq \hat{k}$ (this deals with the case that Ψ is a scalar multiple of Φ). By the discussion above hence K is an algebra homomorphism, as required. □

5. Dual Banach algebras and weakly almost periodic functionals. Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{A} = E'$ for some Banach space E. We say that $\mathcal{A}$ is a *dual Banach algebra* if the product on $\mathcal{A}$ is separately weak*-continuous. It is shown in [**21**, Section 1] (see also [**6**, Section 2]) that $\mathcal{A}$ is a dual Banach algebra if and only if $\kappa_E(E) \subseteq E'' = \mathcal{A}'$ is an $\mathcal{A}$-submodule.

Notice that $\mathcal{A}''$ is always a dual Banach space. It is not hard to show that $\mathcal{A}''$, with either Arens product, is a dual Banach algebra if and only if $\mathcal{A}$ is Arens regular. Similarly, by the definition given in [**14**], the product $\underset{K,\mathcal{U}}{\star}$ is regular if and only if it makes $\mathcal{A}''$ a dual Banach algebra (of course, taking account of the fact that $\underset{K,\mathcal{U}}{\star}$ may not be associative). Hence [**14**, Proposition 6] shows that $(\mathcal{A}'', \underset{K,\mathcal{U}}{\star})$ is a dual Banach algebra (in this not necessarily associative sense) if and only if $\underset{K,\mathcal{U}}{\star} = \Box = \Diamond$ (in which case, $\underset{K,\mathcal{U}}{\star}$ is automatically associative). In particular, $\underset{K,\mathcal{U}}{\star}$ cannot turn $\mathcal{A}''$ into a genuine dual Banach algebra unless $\mathcal{A}$ is already Arens regular.

Let $\mathcal{A}$ be a Banach algebra. We say that $\mu \in \mathcal{A}'$ is *weakly almost periodic* (WAP) if the map $\mathcal{A} \to \mathcal{A}'; a \mapsto a \cdot \mu$ is weakly compact. We write $\mu \in \mathrm{WAP}(\mathcal{A}')$ in this case (some authors write $\mathrm{WAP}(\mathcal{A})$ for this). The space $\mathrm{WAP}(\mathcal{A}')$ has been widely studied, especially in the context of group algebras. We now collect some useful results.

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Proposition 5.1. *Let $\mathcal{A}$ be a Banach algebra. Then* $\mathrm{WAP}(\mathcal{A}')$ *is a closed submodule of $\mathcal{A}'$. For $\mu \in \mathcal{A}'$, we have $\mu \in \mathrm{WAP}(\mathcal{A}')$ if and only if $\langle \Phi \Box \Psi, \mu \rangle = \langle \Phi \Diamond \Psi, \mu \rangle$ for $\Phi, \Psi \in \mathcal{A}''$. In particular, $\mathcal{A}$ is Arens regular if and only if* $\mathrm{WAP}(\mathcal{A}') = \mathcal{A}'$.

Let $X \subseteq \mathcal{A}'$ be a closed submodule, so we identify X' with the quotient $\mathcal{A}''/X^{\perp}$. The following are equivalent:

1. $X \subseteq \mathrm{WAP}(\mathcal{A}')$;
2. *the Arens products drop to a well-defined product on X' turning X' into a dual Banach algebra.*

Proof. These facts are collected in [**6**, Section 2]. The first result is due to John Pym. The second result can be found in many places in the literature; see [**16**, Lemma 1.4] for example, which shows this for commutative Banach algebras. □

Similarly, we say that $\mu \in \mathcal{A}'$ is *almost periodic*, written $\mu \in \mathrm{AP}(\mathcal{A}')$, if the map $\mathcal{A} \to \mathcal{A}'; a \mapsto a \cdot \mu$ is (norm) compact.

Proposition 5.2. *Let $\mathcal{A}$ be a Banach algebra. Then* $\mathrm{AP}(\mathcal{A}')$ *is a closed submodule of $\mathcal{A}'$.*

Let $X \subseteq \mathcal{A}'$ be a closed submodule, so we identify X' with the quotient $\mathcal{A}''/X^{\perp}$. The following are equivalent:

1. $X \subseteq \mathrm{AP}(\mathcal{A}')$;
2. *the Arens products drop to a well-defined product on X' which is jointly continuous on bounded spheres.*

Proof. Lau shows this for a certain class of commutative Banach algebras in [**15**, Theorem 5.8], although the proof is very easy to adapt to the general case. Compare also [**14**, Proposition 7]. □

Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{U}$ be an ultrafilter. Define $\sigma_{\mathcal{U}}^{\mathrm{WAP}} : (\mathcal{A})_{\mathcal{U}} \to \mathrm{WAP}(\mathcal{A}')'$ by

$$\langle \sigma_{\mathcal{U}}^{\mathrm{WAP}}((a_i))\mu \rangle = \lim_{i \to \mathcal{U}} \langle \mu, a_i \rangle \quad ((a_i) \in (\mathcal{A})_{\mathcal{U}}, \mu \in \mathrm{WAP}(\mathcal{A}')).$$

That is, $\sigma_{\mathcal{U}}^{\mathrm{WAP}}$ is simply the map $\sigma_{\mathcal{U}}$ composed with the quotient map $\mathcal{A}'' \to \mathrm{WAP}(\mathcal{A}')'$.

Theorem 5.3. *Let $\mathcal{A}$ be a Banach algebra, let $M \in \mathrm{FIN}(\mathcal{A}'')$, $N \in \mathrm{FIN}(\mathrm{WAP}(\mathcal{A}'))$ and $\hat{N} \in \mathrm{FIN}(\mathcal{A}')$, and let $\epsilon > 0$. Let $M_0 = M + M \Box M$ and $N_0 = N + M \cdot N$. Then there exists a $(1+\epsilon)$-isomorphism onto its range $T : M_0 \to \mathcal{A}$ such that*

1. $\langle \Phi, \mu \rangle = \langle \mu, T(\Phi) \rangle$ *for $\Phi \in M_0$ and $\mu \in \hat{N}$;*
2. $T(\kappa_{\mathcal{A}}(a)) = a$ *for $\kappa_{\mathcal{A}}(a) \in M_0 \cap \kappa_{\mathcal{A}}(\mathcal{A})$;*
3. $|\langle \mu, T(\Phi \Box \Psi) - T(\Phi)T(\Psi) \rangle| \leq \epsilon \|\mu\| \|\Phi\| \|\Psi\|$ *for $\mu \in N$ and $\Phi, \Psi \in M$.*

Proof. If we examine the proof of Theorem 4.1, we see that it will hold for non-Arens regular Banach algebras, so long as $N \subseteq \mathrm{WAP}(\mathcal{A}')$. It is an easy exercise to take account of $\hat{N}$. □

Corollary 5.4. *Let $\mathcal{A}$ be a Banach algebra. There exists an ultrafilter $\mathcal{U}$ and an isometry $K : \mathcal{A}'' \to (\mathcal{A})_{\mathcal{U}}$ such that*

1. $\sigma_{\mathcal{U}} \circ K$ *is the identity on A'';*
2. $K \circ \kappa_{\mathcal{A}}$ *is the canonical map $\mathcal{A} \to (\mathcal{A})_{\mathcal{U}}$;*

3. *Let* $\iota = \iota_{\mathrm{WAP}} : \mathcal{A}'' \to \mathrm{WAP}(\mathcal{A}')'$ *be the quotient map. For* $\Phi, \Psi \in \mathcal{A}''$, *we have*

$$\langle \iota(\Phi)\iota(\Psi), \mu \rangle = \left\langle \sigma_{\mathcal{U}}^{\mathrm{WAP}}(K(\Phi)K(\Psi))\mu \right\rangle \quad (\mu \in \mathrm{WAP}(\mathcal{A}')).$$

It would seem to be natural to ask if we could define K as a map $\mathrm{WAP}(\mathcal{A}')' \to (\mathcal{A})_{\mathcal{U}}$ with $\sigma_{\mathcal{U}}^{\mathrm{WAP}} \circ K$, the identity on $\mathrm{WAP}(\mathcal{A}')'$.

PROPOSITION 5.5. *Let* $\mathcal{A}$ *be a Banach algebra. The following are equivalent:*
1. *There exists a map* $K_{\mathrm{WAP}} : \mathrm{WAP}(\mathcal{A}')' \to (\mathcal{A})_{\mathcal{U}}$ *with* $\sigma_{\mathcal{U}}^{\mathrm{WAP}} \circ K_{\mathrm{WAP}}$, *the identity on* $\mathrm{WAP}(\mathcal{A}')'$;
2. $\mathrm{WAP}(\mathcal{A}')^{\perp}$ *is complemented in* $\mathcal{A}''$ *(that is,* $\mathrm{WAP}(\mathcal{A}')$ *is* weakly complemented*)*;

Proof. If (1) holds, then $L = \sigma_{\mathcal{U}} \circ K$ is a map $\mathrm{WAP}(\mathcal{A}')' \to \mathcal{A}''$. Then $\iota_{\mathrm{WAP}} \circ L$ is the identity on $\mathrm{WAP}(\mathcal{A}')'$, and so $L \circ \iota_{\mathrm{WAP}}$ is a projection of $\mathcal{A}''$ onto the image of L, with complementary space $\mathrm{WAP}(\mathcal{A}')^{\perp}$, and so (2) holds.

If (2) holds then let $P : \mathcal{A}'' \to \mathrm{WAP}(\mathcal{A}')^{\perp}$ be a projection. Let $K : \mathcal{A}'' \to (\mathcal{A})_{\mathcal{U}}$ be such that $\sigma_{\mathcal{U}} \circ K$ is the identity on $\mathcal{A}''$. We identify $\mathrm{WAP}(\mathcal{A}')'$ with the quotient $\mathcal{A}''/\mathrm{WAP}(\mathcal{A}')^{\perp}$, and let ι_{WAP} be the quotient map. Define $K_{\mathrm{WAP}} : \mathrm{WAP}(\mathcal{A}')' \to (\mathcal{A})_{\mathcal{U}}$ by

$$K_{\mathrm{WAP}}(\iota_{\mathrm{WAP}}(\Phi)) = K(\Phi - P(\Phi)) \quad (\Phi \in \mathcal{A}'').$$

Then, if $\iota_{\mathrm{WAP}}(\Phi) = \iota_{\mathrm{WAP}}(\Psi)$, we have $\Phi - \Psi \in \mathrm{WAP}(\mathcal{A}')^{\perp}$, so that $P(\Phi - \Psi) = \Phi - \Psi$, and so $K(\Phi - \Psi - P(\Phi - \Psi)) = 0$, showing that K_{WAP} is well defined. Then $\sigma_{\mathcal{U}}^{\mathrm{WAP}} K_{\mathrm{WAP}} \iota_{\mathrm{WAP}} = \iota_{\mathrm{WAP}} \sigma_{\mathcal{U}} K(I - P) = \iota_{\mathrm{WAP}}(I - P) = \iota_{\mathrm{WAP}}$, and so (1) holds. □

We currently have no examples showing that $\mathrm{WAP}(\mathcal{A}')^{\perp}$ can be complemented in $\mathcal{A}''$. An obvious place to look is at group algebras $L^1(G)$, for which $\mathrm{WAP}(L^{\infty}(G))$ is (reasonably) well understood (see [**6**, Section 7] and references therein).

We can similarly define $\sigma_{\mathcal{U}}^{\mathrm{AP}}$, and as $\mathrm{AP}(\mathcal{A}') \subseteq \mathrm{WAP}(\mathcal{A}')$, all of the above holds for AP, with suitable modifications. By Proposition 5.2, we see that $\sigma_{\mathcal{U}}^{\mathrm{AP}}$ is actually an *algebra homomorphism*.

6. Automatic regularity for C*-algebras. As stated in the introduction, [**12**, Corollary II.2] shows that when $\mathcal{A}$ is a C*-algebra the map $\underset{K,\mathcal{U}}{\star}$ always agrees with $\Box = \Diamond$, so long as $K : \mathcal{A}'' \to (\mathcal{A})_{\mathcal{U}}$ is an *isometry* (and satisfies our two standing assumptions, namely, $\sigma \circ K$ is the identity on $\mathcal{A}''$, and $K \circ \kappa_{\mathcal{A}}$ is the canonical embedding).

The proof of [**14**, Proposition 7] is trivially adapted to our situation (that is, when K is not assumed to be an isometry) to show that when $\mathrm{AP}(\mathcal{A}') = \mathcal{A}'$, we have that $\underset{K,\mathcal{U}}{\star} = \Box$ (as $\mathrm{WAP}(\mathcal{A}') = \mathcal{A}'$, such algebras are always Arens regular).

Recall, [**23**], that a positive functional of a C*-algebra is *pure* if the associated GNS representation is irreducible. Recall that a C*-algebra is said to be *scattered* if every positive functional is a sum of pure positive functionals. For a commutative C*-algebra, $C(X)$ is scattered if and only if X is scattered, in the topological sense that every subset of X contains an isolated point. Then Quigg [**19**, Theorem 3.2] showed the following:

THEOREM 6.1. *For a C*-algebra* $\mathcal{A}$, $\mathrm{AP}(\mathcal{A}') = \mathcal{A}'$ *if and only if* $\mathcal{A}$ *is scattered and each irreducible representation of* $\mathcal{A}$ *is finite-dimensional.*

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It seems possible that for well-behaved Banach algebras (say, certainly for C*-algebras), that when $\mathcal{A}' \neq \mathrm{AP}(\mathcal{A}')$, we can construct a map $K : \mathcal{A}'' \to (\mathcal{A})_{\mathcal{U}}$ satisfying our two assumptions, but with $\underset{K,\mathcal{U}}{\star} \neq \Box$. We can currently only show this under an extra assumption.

Before we proceed, we provide a general way of finding such maps K.

PROPOSITION 6.2. *Let $\mathcal{A}$ be an Arens regular Banach algebra. Suppose that there exist weakly null nets (a_α) and (b_α), on the same index set, such that $(a_\alpha b_\alpha)$ is not weakly null. Then there exists an ultrafilter $\mathcal{U}$ and a bounded map $K : \mathcal{A}'' \to (\mathcal{A})_{\mathcal{U}}$ which satisfies our standing assumptions, but with $\underset{K,\mathcal{U}}{\star}$ not equal to the Arens products.*

Proof. Let our nets be indexed by the directed set J, and let $\mathcal{V}$ be an ultrafilter refining the order filter on J. Let $\mathcal{W}$ be some ultrafilter such that there exists a bounded map $L : \mathcal{A}'' \to (\mathcal{A})_{\mathcal{W}}$ satisfying the conclusions of Theorem 4.2. Let $\Phi \in \mathcal{A}''$ be the weak*-limit of $(a_\alpha b_\alpha)$ along $\mathcal{V}$, so that $\Phi \neq 0$.

Let $\Psi_1, \Psi_2 \in \mathcal{A}'' \setminus \mathcal{A}$ and $M_1, M_2 \in \mathcal{A}'''$ be such that $\langle M_1, \kappa_{\mathcal{A}}(a)\rangle = \langle M_2, \kappa_{\mathcal{A}}(a)\rangle = 0$ for $a \in \mathcal{A}$, and

$$\langle M_1, \Psi_1\rangle = \langle M_2, \Psi_2\rangle = 1, \quad \langle M_2, \Psi_1\rangle = \langle M_1, \Psi_2\rangle = 0.$$

Recall that we define the ultrafilter $\mathcal{V} \times \mathcal{W}$ on $J \times I$ by, for $K \subseteq J \times I$, setting $K \in \mathcal{V} \times \mathcal{W}$ if and only if

$$\{i \in I : \{\alpha \in J : (\alpha, i) \in K\} \in \mathcal{V}\} \in \mathcal{W}.$$

Then, for a family $(x_{\alpha,i})_{\alpha\in J, i\in I}$ in a Hausdorff space X, we have

$$\lim_{i\to\mathcal{W}} \lim_{\alpha\to\mathcal{V}} x_{\alpha,i} = \lim_{(\alpha,i)\to\mathcal{V}\times\mathcal{W}} x_{\alpha,i},$$

whenever the limits exist. Let $\mathcal{U} = \mathcal{V} \times \mathcal{W}$, and define $K : \mathcal{A}'' \to (\mathcal{A})_{\mathcal{U}}$ as follows. For $\Lambda \in \mathcal{A}''$, let $L(\Lambda) = (c_i)_{i\in I}$, and define

$$K(\Lambda) = (c_i + \langle M_1, \Lambda\rangle a_\alpha + \langle M_2, \Lambda\rangle b_\alpha)_{(\alpha,i)\in J\times I}.$$

Obviously K is linear and bounded. For $a \in \mathcal{A}$, by the choice of M_1 and M_2, we have $K(\kappa_{\mathcal{A}}(a)) = (a)$, so that $K \circ \kappa_{\mathcal{A}}$ is the canonical map $\mathcal{A} \to (\mathcal{A})_{\mathcal{U}}$.

For $\mu \in \mathcal{A}'$ and $\Lambda \in \mathcal{A}''$, let $L(\Lambda) = (c_i)$, so we have

$$\langle \sigma(K(\Lambda)), \mu\rangle = \lim_{(\alpha,i)\to\mathcal{U}} \langle \mu, c_i + \langle M_1, \Lambda\rangle a_\alpha + \langle M_2, \Lambda\rangle b_\alpha\rangle = \lim_{i\to\mathcal{W}} \langle \mu, c_i\rangle = \langle \Lambda, \mu\rangle,$$

as (a_α) and (b_α) are weakly null.

Finally, let $L(\Psi_1) = (c_i)$ and $L(\Psi_2) = (d_i)$, so that

$$K(\Psi_1) = (c_i + a_\alpha), \quad K(\Psi_2) = (d_i + b_\alpha).$$

Thus we see that for $\mu \in \mathcal{A}'$,

$$\begin{aligned}
\langle \Psi_1 \underset{K,\mathcal{U}}{\star} \Psi_2, \mu\rangle &= \lim_{(\alpha,i)\to\mathcal{U}} \langle \mu, (c_i + a_\alpha)(d_i + b_\alpha)\rangle = \lim_{(\alpha,i)\to\mathcal{U}} \langle \mu, c_i d_i + a_\alpha d_i + c_i b_\alpha + a_\alpha b_\alpha\rangle \\
&= \langle \Psi_1 \Box \Psi_2, \mu\rangle + \langle \Phi, \mu\rangle + \lim_{i\to\mathcal{W}} \Big(\lim_{\alpha\to\mathcal{V}} \langle d_i \cdot \mu, a_\alpha\rangle + \langle \mu \cdot c_i, b_\alpha\rangle \Big) \\
&= \langle \Psi_1 \Box \Psi_2 + \Phi, \mu\rangle,
\end{aligned}$$

as (a_α) and (b_α) are weakly null. Hence, $\Psi_1 \underset{K,\mathcal{U}}{\star} \Psi_2 \neq \Psi_1 \Box \Psi_2$ as $\Phi \neq 0$. $\Box$

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Let $\mathcal{A} = \mathcal{K}(\ell^p)$, the compact operators of ℓ^p, for $1 < p < \infty$. Then, as detailed in [**17**, Section 1.4] for example, the dual of $\mathcal{A}$ is the nuclear operators on ℓ^p, and $\mathcal{A}'' = \mathcal{B}(\ell^p)$, with $\Box = \Diamond$ agreeing with the usual product. Let (δ_n) be the standard unit vector basis of ℓ^p, and for each n, let a_n be the rank-one operator which sends δ_n to δ_1, and kills δ_k otherwise, and let b_n be the rank-one operator which sends δ_1 to δ_n, and kills δ_k otherwise. Then (a_n) and (b_n) are both weakly null sequences, but $a_n b_n$ is the projection onto the first coordinate, for each n. Hence the above proposition applies in this case, and in particular it applies to the C*-algebra $\mathcal{K}(\ell^2)$.

Definition 6.3. Let E and F be Banach spaces, and let $T : E \to F$ be a bounded linear map. Then T is *totally completely continuous* if, whenever (x_α) is a weakly null net in E, then $\lim_\alpha \|T(x_\alpha)\| = 0$.

Recall that if we only use sequences in the above definition, we get the usual notion of T being *completely continuous* [**1**]. Our work below will show that being totally completely continuous is strictly stronger than being completely continuous. We view being totally completely continuous as a property close to being compact.

Proposition 6.4. *Let $\mathcal{A}$ be a Banach algebra, and let $\mu \in \mathrm{WAP}(\mathcal{A}')$ be such that the map $\mathcal{A} \to \mathcal{A}'; a \mapsto a \cdot \mu$ is not totally completely continuous. Then there exist weakly null nets (a_α) and (b_α), on the same index set, such that $(b_\alpha a_\alpha)$ is not weakly null.*

Proof. Let (a_i) be some weakly null net in $\mathcal{A}$ such that for some $\delta > 0$, we have $\|a_i \cdot \mu\| \geq \delta$ for all i. Let Λ be the collection of finite-dimensional subspaces of $\mathcal{A}'$, partially ordered by reverse inclusion. For each $M \in \Lambda$, if we can find some i and some $b \in \mathcal{A}$ with $\|b\| = 1$, and with

$$|\langle a_i \cdot \mu, b\rangle| \geq \delta/3, \qquad \langle \lambda, b\rangle = 0, \quad |\langle \lambda, a_i\rangle| \leq (\dim M)^{-1}\|\lambda\| \qquad (\lambda \in M),$$

then we are done.

As (a_i) is weakly null, we can ensure the final condition by simply ensuring that $i \geq i_0$, for some i_0 depending upon M. For $c \in \mathcal{A}$, we have

$$\lim_i \langle a_i \cdot \mu, c\rangle = \lim_i \langle \mu \cdot c, a_i\rangle = 0,$$

so we see that $(a_i \cdot \mu)$ is weak*-null.

For fixed i, notice that the Hahn–Banach theorem shows that

$$\begin{aligned}\sup\{|\langle a_i \cdot \mu, b\rangle| \ : \ b \in \mathcal{A}, \|b\| = 1, \langle \lambda, b\rangle = 0\ (\lambda \in M)\}\\ = d(a_i \cdot \mu, M) := \inf\{\|a_i \cdot \mu - \lambda\| : \lambda \in M\}.\end{aligned}$$

For each i, let $\lambda_i \in M$ be such that $\|a_i \cdot \mu - \lambda_i\| = d(a_i \cdot \mu, M)$. Suppose that $d(a_i \cdot \mu, M) \leq \epsilon$ for each $i \geq i_0$. Then (λ_i) is a bounded net, and so, by passing to a subnet if necessary, we may suppose that $\lambda_i \to \lambda \in M$, in norm. For $c \in \mathcal{A}$, we see that

$$|\langle \lambda, c\rangle| = \lim_i |\langle a_i \cdot \mu - \lambda, c\rangle| = \lim_i |\langle a_i \cdot \mu - \lambda_i, c\rangle| \leq \|c\|\epsilon,$$

so that $\|\lambda\| \leq \epsilon$. Hence

$$\delta \leq \liminf_i \|a_i \cdot \mu\| \leq \liminf_i \|a_i \cdot \mu - \lambda\| + \|\lambda\| \leq \epsilon + \liminf_i \|a_i \cdot \mu - \lambda_i\| \leq 2\epsilon,$$

showing that $\epsilon \geq \delta/2$, as required. □

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For more about the following definition, see [**4**, Section 5].

DEFINITION 6.5. A Banach space E has the π-property if there exists a bounded net (T_α) of finite-rank projections such that $\|T_\alpha(x) - x\| \to 0$ for $x \in E$.

Notice that if E has a basis, then it has the π-property. It is easy to show that L^1 spaces (and hence $M(X)$ spaces) have the π-property, so the duals of commutative C*-algebras have the π-property. However, the π-property is stronger than the more usual (bounded) approximation property, and it is known (see [**20**, Remark 6.1.9]) that a von Neumann algebra $\mathcal{M}$ has the approximation property if and only if it is nuclear, which is if and only if $\mathcal{M}$ is a finite sum of algebras of the form $C(X) \otimes \mathbb{M}_n$. As the approximation property passes to preduals, there are plenty of von Neumann algebras $\mathcal{M}$ such that $\mathcal{M}'$ cannot have the π-property.

PROPOSITION 6.6. *Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{A}'$ has the π-property. Let $\mu \in \mathrm{WAP}(\mathcal{A}') \setminus \mathrm{AP}(\mathcal{A}')$. Then the map $\mathcal{A} \to \mathcal{A}'; a \mapsto a \cdot \mu$ is not totally completely continuous.*

Proof. Let (T_α) be a net of finite-rank projections on $\mathcal{A}'$ such that $\|T_\alpha(\lambda) - \lambda\| \to 0$ for each $\lambda \in \mathcal{A}'$. Let $M = \sup_\alpha \|T_\alpha\|$. Suppose, towards a contradiction, that for each $\delta > 0$ there exists some α such that for each $\Phi \in \mathcal{A}''$ with $T'_\alpha(\Phi) = 0$, we have $\|\Phi \cdot \mu\| \leq \delta\|\Phi\|$.

For each δ, choose such an α. For $\Phi \in \mathcal{A}''$ let $\Psi = \Phi - T'_\alpha(\Phi)$, so that $T'_\alpha(\Psi) = 0$, and so $\|\Phi \cdot \mu - T'_\alpha(\Phi) \cdot \mu\| = \|\Psi \cdot \mu\| \leq \delta\|\Psi\| \leq \delta(1 + M)\|\Phi\|$. As $\delta > 0$ was arbitrary, we see that the map $\mathcal{A}'' \to \mathcal{A}'; \Phi \mapsto \Phi \cdot \mu$ can be uniformly approximated by finite-rank operators, and hence is compact, contradicting the fact that $\mu \notin \mathrm{AP}(\mathcal{A}')$.

Thus, for some $\delta > 0$, for each α there exists $\Phi_\alpha \in \mathcal{A}''$ with $\|\Phi_\alpha\| = 1$, $T'_\alpha(\Phi_\alpha) = 0$ and $\|\Phi_\alpha \cdot \mu\| \geq \delta$. For each α, let $a_\alpha \in \mathcal{A}$ be such that $\|a_\alpha\| \leq 2$, a_α agrees with Φ_α on the image of T_α, a finite-dimensional subspace of $\mathcal{A}'$, and $\|a_\alpha \cdot \mu\| \geq \delta/2$. We can ensure the final condition as $a_\alpha \cdot \mu \to \Phi_\alpha \cdot \mu$ weakly, as $\mu \in \mathrm{WAP}(\mathcal{A}')$. Then, for $\lambda \in \mathcal{A}'$, as $T_\alpha(\lambda) \to \lambda$ in norm, we see that

$$\lim_\alpha \langle \lambda, a_\alpha \rangle = \lim_\alpha \langle T_\alpha(\lambda), a_\alpha \rangle = \lim_\alpha \langle \Phi_\alpha, T_\alpha(\lambda) \rangle = 0,$$

so (a_α) is weakly null. $\square$

We can immediately draw conclusions about commutative C*-algebras.

THEOREM 6.7. *Let $\mathcal{A} = C_0(X)$ be a commutative C*-algebra. The following are equivalent:*

1. *X is scattered;*
2. *for any $K : \mathcal{A}'' \to (\mathcal{A})_{\mathcal{U}}$ satisfying our standing assumptions, we have $\underset{K,\mathcal{U}}{\star} = \Box$.*

Asking for K to be an isometry is a strong condition. There is by now a large selection of results in the theory of C*-algebras which weaken the requirement of ‘isometry’ to ‘completely bounded’ (see [**18**], for example). Given a C*-algebra $\mathcal{A}$, there is a canonical way to turn the matrix algebra $\mathbb{M}_n(\mathcal{A})$ into a C*-algebra. Given a linear map $T : \mathcal{A} \to \mathcal{B}$ between two C*-algebras, let T acts pointwise as a map $(T)_n : \mathbb{M}_n(\mathcal{A}) \to \mathbb{M}_n(\mathcal{B})$. Then T is *completely bounded* if and only if $\sup_n \|(T)_n\| < \infty$. It is simple to show the map K which we constructed in Proposition 6.2 is completely bounded: the only non-trivial thing to check is the well-known fact that bounded linear functionals are automatically completely bounded. It would be interesting to know if

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there is any reasonable weakening of the 'isometry' condition on K which still ensures that $\underset{K,\mathcal{U}}{\star} = \Box$ for C*-algebras.

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