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SOME SUFFICIENT CONDITIONS FOR MAXIMAL-RESOLVABILITY(1)

BY

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1. **Introduction.** A topological space X is called maximally resolvable if it admits a largest possible family of pairwise disjoint, "maximally dense" subsets. More precisely, if $\Delta(X)$ denotes the least among the cardinal numbers of the nonvoid open subsets of X, then X is *maximally resolvable* if it has isolated points or there exists a family $\{R_\alpha\}_{\alpha<\Delta(X)}$ of subsets of X, called a *maximal resolution* for X, such that $\bigcup\{R_\alpha \mid \alpha<\Delta(X)\}=X$, $R_\gamma \cap R_\delta=\phi$ if $\gamma\neq\delta$, and, for each α and each nonvoid open subset V of X, the cardinality of $R_\alpha \cap V$ is not less than $\Delta(X)$. In [1] J. G. Ceder showed among other things that all locally compact Hausdorff spaces and spaces which possess at each point a local base linearly ordered by inclusion, are maximally resolvable. More generally, if $\chi(X)$ denotes the *weight* of X (i.e. $\chi(X)=\min\{\text{card } \mathscr{B} \mid \mathscr{B}$ is a base for the open sets in $X\}$), a sufficient condition for maximal-resolvability is that $\aleph_0\leq\chi(X)\leq\Delta(X)$ [1, Theorem 3].

In this paper we investigate some further sufficient conditions for maximal-resolvability. We first prove a theorem which relaxes the most general sufficient condition for maximal-resolvability as given in [1]; following this it is shown that under certain conditions separability and generalized separability (in a sense to be defined below) imply maximal-resolvability. Finally, an example is given which answers an unsolved problem posed in [2].

2. **Generalization of a theorem of Ceder.** In the sequel we restrict attention to spaces in which each nonvoid open subset has transfinite cardinality. We consider cardinals and ordinals as defined, for example, by J. L. Kelley in [4], so that an ordinal is equal to the set of its predecessors and a cardinal is an ordinal which is not equipollent with any smaller ordinal. Ordinals will be denoted by lower case Greek letters, cardinals by the aleph notation or bold-face Latin type; in particular, $\mathbf{c}$ will denote the power of the continuum. The cardinality, or power, of a set A will be written $|A|$.

THEOREM 1. *Let $(X, \mathscr{T})$ be a topological space. If there is a base $\mathscr{B}$ for $\mathscr{T}$ which admits a collection $\{\mathscr{B}_\xi\}_{\xi<\mathbf{m}}$ of subfamilies such that $\bigcup\{\mathscr{B}_\xi \mid \xi<\mathbf{m}\}=\mathscr{B}$, $\mathbf{m}\leq\Delta(X)$, and $|\bigcap\{B \mid B\in\mathscr{B}_\xi\}|\geq\Delta(X)$ for each $\xi<\mathbf{m}$, then X is maximally resolvable.*

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Proof. For each $\xi < \mathbf{m}$, let $I_\xi = \bigcap \{B \mid B \in \mathscr{B}_\xi\}$, and let W be the set consisting of all ordered triples of the form $\langle I_\xi, \rho, \sigma \rangle$, where ρ and σ are ordinals less than $\Delta(X)$ and $\xi < \mathbf{m}$. Then $|W| = \mathbf{m} \cdot \Delta(X) \cdot \Delta(X)$, so W can be well-ordered by $\Delta(X)$ as $\{w_\alpha\}_{\alpha < \Delta(X)}$. For each α, denote the triple corresponding to w_α by $\langle I_{\xi(\alpha)}, \rho(\alpha), \sigma(\alpha) \rangle$. We shall construct a maximal resolution for X by transfinite induction on the cardinal $\Delta(X)$. Suppose a point $x_\alpha \in X$ has been chosen for each $\alpha < \beta$, where β is an ordinal less than $\Delta(X)$, and consider $w_\beta = \langle I_{\xi(\beta)}, \sigma(\beta), \rho(\beta) \rangle$. Now $|\{x_\alpha \mid \alpha < \beta\}| = |\beta| < \Delta(X)$, and $|I_{\xi(\beta)}| \geq \Delta(X)$; hence $I_{\xi(\beta)} - \{x_\alpha \mid \alpha < \beta\} \neq \phi$ and we can choose $x_\beta \in I_{\xi(\beta)} - \{x_\alpha \mid \alpha < \beta\}$. Having defined the set $\{x_\alpha \mid \alpha < \Delta(X)\}$ of distinct points, we put

$$A_\delta = \{x_\alpha \mid \sigma(\alpha) = \delta\}$$

for $0 < \delta < \Delta(X)$, and

$$A_0 = X - \bigcup \{A_\delta \mid 0 < \delta < \Delta(X)\}.$$

It is easily checked that $\{A_\delta\}_{\delta < \Delta(X)}$ is a maximal resolution for X.

Corollary 1. (*Ceder* [1], *Theorem* 3.) *If* $\chi(X) \leq \Delta(X)$, *then* X *is maximally resolvable.*

3. **Maximal-resolvability and separability.** It is reasonable to conjecture that any uncountable, separable space is maximally resolvable, but an example to be given at the end of this paper shows that this is not the case. On the other hand, every regular, separable space X for which $\Delta(X) \geq \mathbf{c}$ is maximally resolvable; this is an easy consequence of our next theorem. If X is a topological space, let $\delta(X)$ denote the *density character* of X, that is,

$$\delta(X) = \min \{|A| \mid A \text{ is a dense subset of } X\}.$$

We now prove:

Theorem 2. *If* X *is regular and* $2^{\delta(X)} \leq \Delta(X)$, *then* X *is maximally resolvable.*

Proof. Let D be a dense subset of X such that $|D| = \delta(X)$, and put

$$\mathscr{S} = \{\text{Int } \bar{S} \mid S \subset D\},$$

$\bar{S}$ being the closure of S in X. We first show that $\mathscr{S}$ is a base for the open subsets of X. Let $V \subset X$ be open, $x \in V$. By regularity, there exists an open set B such that $x \in B \subset \bar{B} \subset V$. Obviously $\text{Int } \overline{B \cap D} \in \mathscr{S}$, and it is readily verified that $x \in \text{Int } \overline{B \cap D} \subset V$. To complete the proof we note that

$$|\mathscr{S}| \leq 2^{|D|} = 2^{\delta(X)} \leq \Delta(X),$$

so that $\chi(X) \leq \Delta(X)$; hence, by Corollary 1, X is maximally resolvable.

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Since $2^{\aleph_0}=\mathbf{c}$, we have immediately:

COROLLARY 2. *If X is a separable, regular space and $\Delta(X)\geq c$, then X is a maximally resolvable.*

Assuming the Generalized Continuum Hypothesis, we deduce from the theorem:

COROLLARY 3. *If X is regular and $\delta(X)<\Delta(X)$, then X is maximally resolvable.*

We shall say that a topological space X is *ν-separable* if there is a dense subset of X whose cardinality does not exceed $\aleph_\nu$, and that X is *hereditarily ν-separable* if every subspace of X is ν-separable. Thus, the hereditarily 0-separable spaces are just the usual hereditarily separable spaces.

THEOREM 3. *Let X be a topological space. If $\Delta(X)=\aleph_\lambda$ and X is hereditarily ν-separable for some $\nu<\lambda$, then X is maximally resolvable.*

Proof. We first construct a family of pairwise disjoint dense subsets of X by transfinite induction on $\Delta(X)$. Suppose D_α has been defined for each $\alpha<\beta$, where $\beta<\Delta(X)$, such that each D_α is dense in X, $|D_\alpha|\leq\aleph_\nu$, and $D_\gamma\cap D_\delta=\phi$ for $\gamma,\delta<\beta$, $\gamma\neq\delta$. Consider the subspace $Y_\beta=X-\bigcup\{D_\alpha \mid \alpha<\beta\}$; by hypothesis Y_β contains a dense subset D_β whose cardinality is not greater than $\aleph_\nu$. But D_β is also dense in X. To see this, we note first that every nonvoid open subset W of X must meet Y_β; in fact, $W\cap Y_\beta=\phi$ implies $W\subset\bigcup\{D_\alpha \mid \alpha<\beta\}$, and this condition is impossible since

$$\left|\bigcup\{D_\alpha \mid \alpha<\beta\}\right| = \sum_{\alpha<\beta}|D_\alpha| \leq |\beta|\cdot\aleph_\nu < \aleph_\lambda \leq |W|.$$

Hence $W\cap Y_\beta\neq\phi$ and, since D_β is dense in Y_β, $W\cap Y_\beta\cap D_\beta\neq\phi$. Therefore, $W\cap D_\beta\neq\phi$, and D_β is dense in X as claimed. Finally, since $D_\alpha\cap D_\beta=\phi$ for all $\alpha<\beta$, the desired family $\{D_\alpha\}_{\alpha<\Delta(X)}$ is obtained by induction.

Now let $\{K_\eta\}_{\eta<\Delta(X)}$ be a family of mutually disjoint subsets of the cardinal $\Delta(X)$ such that $\bigcup\{K_\eta \mid \eta<\Delta(X)\}=\Delta(X)$ and $|K_\eta|=\Delta(X)$ for each η. Put

$$A_\eta = \bigcup\{D_\alpha \mid \alpha\in K_\eta\}$$

for $0<\eta<\Delta(X)$,

$$A_0 = X-\bigcup\{A_\eta \mid 0<\eta<\Delta(X)\}.$$

Then $\{A_\eta\}_{\eta<\Delta(X)}$ is a maximal resolution for X.

COROLLARY 4. *Every hereditarily separable space X with $\Delta(X)\geq\aleph_1$ is maximally resolvable.*

REMARK. Katětov [3] has constructed a class of spaces of which each representative X is hereditarily separable, has $\Delta(X)=\aleph_0$, and is not maximally resolvable.

4. **Maximal-resolvability of certain topological groups.** By a *topological group* we mean a triple $(X,+,\mathscr{T})$ such that $(X,+)$ is a group, $(X,\mathscr{T})$ is a topological

space, and the function f: $X\times X\to X$ defined by $f(x, y)=x-y$ is continuous relative to the product topology for $X\times X$. No separation axiom is assumed. A *dense subgroup* of X is a subgroup G having the property that $\bar{G}=X$ (with respect to the topology for X).

THEOREM 4. *Every topological group X having a dense subgroup of cardinality less than $|X|$ is maximally resolvable.*

Proof. Let D_0 be a dense subgroup of X such that $|D_0|<|X|$. We shall construct a family $\{D_\alpha\}_{\alpha<|X|}$ of mutually disjoint, dense subsets of X by transfinite induction on $|X|$. Let β be an ordinal less than $|X|$ and assume that a family $\{D_\alpha\}_{\alpha<\beta}$ of subsets of X has been chosen so that

(1) for each α, $D_\alpha=x_\alpha+D_0$ for some $x_\alpha\in X$;
(2) $D_\gamma\cap D_\delta=\phi$ if γ, $\delta<\beta$, $\gamma\neq\delta$; and
(3) each D_α is dense in X.

Now

$$|\bigcup\{D_\alpha \mid \alpha<\beta\}| = \sum_{\alpha<\beta}|D_\alpha| = |\beta|\cdot|D_0| < |X|,$$

so $X-\bigcup\{D_\alpha \mid \alpha<\beta\}\neq\phi$; choose $x_\beta\in X-\bigcup\{D_\alpha \mid \alpha<\beta\}$ and put $D_\beta=x_\beta+D_0$. We now verify that the family $\{D_\alpha\}_{\alpha\le\beta}$ satisfies conditions (1), (2), and (3). Obviously (1) holds. To check (2), suppose there is some $\alpha<\beta$ such that $D_\alpha\cap D_\beta\neq\phi$, i.e. $(x_\alpha+D_0)\cap(x_\beta+D_0)\neq\phi$. Then there are elements d_0 and d_0' of D_0 such that $x_\alpha+d_0=x_\beta+d_0'$, whence $x_\beta=x_\alpha+d_0-d_0'$. This means that $x_\beta\in x_\alpha+D_0=D_\alpha$, contradicting the fact that $x_\beta\notin\bigcup\{D_\alpha \mid \alpha<\beta\}$. Finally, since D_β is a translate of a dense subset of the topological group X, D_β is itself dense in X and (3) is satisfied.

Having defined the family $\{D_\alpha\}_{\alpha<|X|}$, we let $\{K_\lambda\}_{\lambda<|X|}$ be a family of pairwise disjoint subsets of $|X|$ such that $\bigcup\{K_\lambda \mid \lambda<|X|\}=|X|$ and $|K_\lambda|=|X|$ for each λ, and put

$$R_\lambda = \bigcup\{D_\alpha \mid \alpha\in K_\lambda\}$$

for $0<\lambda<|X|$,

$$R_0 = X-\bigcup\{R_\lambda \mid 0<\lambda<|X|\}.$$

The family $\{R_\lambda\}_{\lambda<|X|}$ is a maximal resolution for X, and the proof is complete.

Retaining the same notation, we note that for each nonvoid open subset V of X,

$$|V| = |\bigcup\{V\cap R_\lambda \mid \lambda<|X|\}| = \sum_{\lambda<|X|}|V\cap R_\lambda| \ge |X|.$$

The theorem therefore has the following rather curious consequence:

COROLLARY 5. *If a topological group X has a dense subgroup G such that $|G|<|X|$, then $|V|=|X|$ for every nonvoid open subset V of X.*

5. **An example.** A subset M of a topological space X is said to be $\mathbf{m}$-*dense* in X if $|M\cap V|\geq \mathbf{m}$ for each nonvoid open subset V of X, and X is $\mathbf{m}$-*resolvable* if there exists a family $\{M_\alpha\}_{\alpha<\mathbf{m}}$ of pairwise disjoint, $\mathbf{m}$-dense subsets of X such that $\bigcup\{M_\alpha \mid \alpha<\mathbf{m}\}=X$. The following two questions were posed, and shown to be equivalent, in [2, Proposition 1]:

Q_1: Does there exist a space X which is $\mathbf{n}$-resolvable for some $\mathbf{n}<\Delta(X)$ $(\mathbf{n}\geq\aleph_0)$, but is not maximally resolvable?

Q_2: Does there exist a maximally resolvable space having an open subspace which is not maximally resolvable?

Let $\mathbf{m}$ and $\mathbf{n}$ be infinite cardinals, $\mathbf{m}>\mathbf{n}$. We next construct a T_1-space Z, with $\Delta(Z)=\mathbf{m}$, which is $\mathbf{n}$-resolvable but not $\mathbf{m}$-resolvable, thus answering Q_1 (and of course Q_2) affirmatively.

EXAMPLE 1. Let X be a set of cardinality $\mathbf{m}$, and denote by $\mathscr{V}_0$ the finite complement topology for X. Let Φ be the family of all topologies $\mathscr{V}$ for X which have the following properties:

(1) $\Delta(X,\mathscr{V})=\mathbf{m}$,
(2) $\mathscr{V}\supset\mathscr{V}_0$, and
(3) no two nonempty elements of $\mathscr{V}$ are disjoint.

Clearly, Φ is partially ordered by $\subset$. If Γ is an arbitrary chain in Φ, the topology $\mathscr{V}'$ having as a base the family $\mathscr{S}=\{V \mid V\in\mathscr{V}$ for some $\mathscr{V}\in\Gamma\}$ is an upper bound for Γ, and it is readily verified that $\mathscr{V}'$ has properties (1), (2), and (3). By Zorn's lemma, Φ must have a maximal element $\mathscr{V}^*$.

Now let D_1 be a subset of X which is $\mathbf{m}$-dense relative to $\mathscr{V}^*$. The family of all sets of the form $V_1\cup(V_2\cap D_1)$, where $V_1, V_2\in\mathscr{V}^*$, is a base for a topology $\mathscr{V}''$ for X which satisfies (1), (2), and (3) and contains $\mathscr{V}^*$. By the maximality of $\mathscr{V}^*$, $\mathscr{V}''=\mathscr{V}^*$, so D_1 must be $\mathscr{V}^*$-open. Therefore, if D_2 is any other subset of X which is $\mathbf{m}$-dense relative to $\mathscr{V}^*$, $D_1\cap D_2$ is nonempty, and it follows that $(X,\mathscr{V}^*)$ is not maximally resolvable.

Next, let Y be a set of cardinality $\mathbf{n}$, and let Y have the finite complement topology $\mathscr{W}$. Put $Z=X\cup Y$. The family

$$\mathscr{B}=\{V\cup W \mid V\in\mathscr{V}^*,\ W\in\mathscr{W},\ V\neq\phi,\ W\neq\phi\}\cup\{\phi\}$$

forms a base for a topology $\mathscr{U}$ for Z, and it is evident that $\Delta(Z,\mathscr{U})=\mathbf{m}$. We shall show that $(Z,\mathscr{U})$ is $\mathbf{n}$-resolvable but not $\mathbf{m}$-resolvable.

Let $\{Y_\mu\}_{\mu<\mathbf{n}}$ be a collection of pairwise disjoint subsets of Y such that $|Y_\mu|=\mathbf{n}$ for each $\mu<\mathbf{n}$ and $\bigcup\{Y_\mu \mid \mu<\mathbf{n}\}=Y$. Put

$$T_\mu = Y_\mu$$

for $0<\mu<\mathbf{n}$,

$$T_0 = Y_0\cup X.$$

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Then $\bigcup\{T_\mu \mid \mu<\mathbf{n}\}=Z$, $T_\gamma \cap T_\delta=\phi$ for $\gamma\neq\delta$, and $|T_\mu \cap U|\geq\mathbf{n}$ for each $\mu<\mathbf{n}$ and each $\mathscr{U}$-open set U, that is, $(Z, \mathscr{U})$ is $\mathbf{n}$-resolvable. On the other hand, if $Z=\bigcup\{R_\alpha \mid \alpha<\mathbf{m}\}$, $R_\gamma \cap R_\delta=\phi$ for $\gamma\neq\delta$, and $|R_\alpha \cap U|=\mathbf{m}$ for each $\mathscr{U}$-open subset U of Z, then the family $\{R_\alpha \cap X\}_{\alpha<\mathbf{m}}$ is a maximal resolution for $(X, \mathscr{V}^*)$, contradicting the fact that $(X, \mathscr{V}^*)$ is not maximally resolvable. Hence, $(Z, \mathscr{U})$ is not maximally resolvable.

Taking $\mathbf{n}=\aleph_0$ and $\mathbf{m}=\mathbf{c}$ in Example 1, we obtain a space Z which is separable (Y being a countable dense subset), has $\Delta(Z)=\mathbf{c}$, and is not maximally resolvable; hence, as mentioned earlier, the assumption of regularity cannot be dropped from the hypothesis of Corollary 2. Also in this connection, we mention that Katětov [3] has shown that there exist separable, regular spaces X with $\Delta(X)<\mathbf{c}$ which do not admit two disjoint dense subsets. Such spaces are of course not maximally resolvable.

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