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CORRIGENDUM

On the quasi-ergodicity of absorbing Markov chains with unbounded transition densities, including random logistic maps with escape – CORRIGENDUM

MATHEUS M. CASTRO†, VINCENT P. H. GOVERSE†,
JEROEN S. W. LAMB†‡§ and MARTIN RASMUSSEN†

† *Department of Mathematics, Imperial College London, London, SW7 2AZ, UK*
(e-mail: m.de-castro20@imperial.ac.uk, vincent.goverse21@imperial.ac.uk, jsw.lamb@imperial.ac.uk, m.rasmussen@imperial.ac.uk)
‡ *International Research Center for Neurointelligence, The University of Tokyo, Tokyo 113-0033, Japan*
§ *Centre for Applied Mathematics and Bioinformatics, Department of Mathematics and Natural Sciences, Gulf University for Science and Technology, Halwally, Kuwait*

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1. *Introduction*

In the paper [1], the technical Lemmas 4.5 and 4.6 are incorrect. This cascaded into the proofs of Proposition 5.2 and Theorems 2.2, 2.3 and 2.4. Although some of the main theorems of the original paper were impacted, the ideas in [1] are robust enough to correct the original proof. In this corrigendum, we provide the necessary modifications to the statements and proofs.

Furthermore, Lemma 3.1(i) has a typo, which propagated to Proposition 4.2(i) and Theorem 2.2(ii). We give the corrected statements below and note that the proof of these results remains correct.

Finally, Theorem 2.4 lacks a condition, which we provide below. Its proof remains essentially the same.

2. *Lemma 3.1(i) and Proposition 4.2(i)*

Lemma 3.1(i) was incorrectly quoted from [2, Theorem 3.3.5] and [3, Corollary V.8.1]. The correct statement is as follows.

LEMMA 3.1.
(i) *For every* $f \in L^1(M, \rho)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} T^i f = \eta \frac{\mathbb{E}_\rho[f \mid \mathcal{I}(T, \rho)]}{\mathbb{E}_\rho[\eta \mid \mathcal{I}(T, \rho)]} \quad \rho\text{-almost surely (a.s.)}.$$

This affected the statements of Proposition 4.2(i), which are corrected as follows.

PROPOSITION 4.2.
(i) *For every* $f \in L^1(M, \mu)$,

$$\frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{\lambda^i} \mathcal{P}^i f \xrightarrow{n\to\infty} \eta \int_M f(y)\mu(dy) \quad \text{in } L^1(M, \mu) \text{ and } \mu\text{-a.s.}$$

3. *Lemmas 4.5 and 4.6*
By labelling $\{g_i\}_{i=0}^{m-1}$ and $\{C_i\}_{i=0}^{m-1}$ in Lemma 4.4, we assume that the permutation σ satisfies $\sigma(i) = i - 1 \pmod m$. We write $g_j = g_{j \pmod m}$ and $C_j = C_{j \pmod m}$ for every $j \in \mathbb{N}$.

With this convention, Lemmas 4.5 and 4.6, should be combined in a single lemma and corrected as follows.

LEMMA 4.5. *Suppose the absorbing Markov chain* X_n *satisfies Hypothesis H1. Then for every bounded and measurable function* $h : M \to \mathbb{R}$ *and* $\ell \in \{0, 1, \dots, m-1\}$,

$$\frac{1}{\lambda^{mn+\ell}} \mathcal{P}^{mn+\ell} h \xrightarrow[L^1(M,\mu)]{n\to\infty} \sum_{s=0}^{m-1} g_s \int_{C_{s+\ell}} h \, d\mu, \tag{3.1}$$

and

$$\frac{1}{nm+\ell} \sum_{i=0}^{mn+\ell-1} \frac{1}{\lambda^i} \mathcal{P}^i \left(h \frac{1}{\lambda^{n-i}} \mathcal{P}^{n-i} \mathbb{1}_M \right) \xrightarrow[L^1(M,\mu)]{n\to\infty} \sum_{s=0}^{m-1} \mu(C_{\ell+s}) g_s \int_M h\eta \, d\mu. \tag{3.2}$$

Proof. Due to Proposition 4.2, there exists $\alpha_0, \dots, \alpha_{m-1} \in \mathbb{C}$ and $v \in E_{\text{aws}}$ such that $h = \sum_{s=0}^{m-1} \alpha_s g_s + v$.

Step 1. We show that $v \in E_{\text{aws}}$ if and only if $\int_{C_i} v \, d\mu = 0$ for every $i \in \{0, 1, \dots, m-1\}$. Suppose first that $v \in E_{\text{aws}}$. We claim that $\mathbb{1}_{C_i} v \in E_{\text{aws}}$ for all $i \in \{0, 1, \dots, m-1\}$. Indeed, if $\mathbb{1}_{C_i} v \notin E_{\text{aws}}$, then $v = \alpha_i g_i + w + \sum_{j\neq i} \mathbb{1}_{C_j} v$ with $\alpha_i \neq 0$ and $w \in E_{\text{aws}}$. Since $\mu(C_i \cap C_j) = 0$ for all $j \neq i$, we obtain that $v \notin E_{\text{aws}}$. It follows that $|\int_{C_i} v \, d\mu| = |\int_M \mathbb{1}_{C_i} v \, d\mu| = |\int_M (1/\lambda^n) \mathcal{P}^n (\mathbb{1}_{C_i} v) \, d\mu| \xrightarrow{n\to\infty} 0$.

Reciprocally, assume that $\int_{C_i} v \, d\mu = 0$ for every $i \in \{0, 1, \dots, k-1\}$. Write $v = \sum_{i=0}^{k-1} \alpha_i g_i + w$, with $w \in E_{\text{aws}}$. Since $\int g_i \, d\mu = 1$, we have that $\alpha_i = \int_{C_i} \alpha_i g_i \, d\mu = \int_{C_i} (\sum_{j=0}^{k-1} \alpha_j g_j + w) \, d\mu = \int_{C_i} v \, d\mu = 0$. We obtain that $\alpha_i = 0$ for every $i \in \{0, 1, \dots, k-1\}$, which implies $v \in E_{\text{aws}}$.

Step 2. We show that equation (3.1) holds. Integrating $h = \sum_{s=0}^{m-1} \alpha_s g_s + v$ with respect to μ on C_i, from Step 1, we obtain that $h = \sum_{s=0}^{m-1} g_s \int_{C_s} h \, d\mu + v$.

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Therefore,

$$\frac{1}{\lambda^{nm+\ell}}\mathcal{P}^{nm+\ell}h=\sum_{s=0}^{m-1}g_{s-\ell}\int_{C_s}h\,d\mu+\frac{1}{\lambda^{nm+\ell}}\mathcal{P}^{nm+\ell}v\xrightarrow[L^1(M,\mu)]{n\to\infty}\sum_{s=0}^{m-1}g_s\int_{C_{s+\ell}}h\,d\mu.$$

Step 3. We show that equation (3.2) holds and conclude the proof of the lemma. From Step 1, we have that $\mathbb{1}_M=\sum_{s=0}^{m-1}\mu(C_s)g_s+w$ for some $w\in E_{\text{aws}}$. Given $\ell\in\{0,1,\ldots,m-1\}$, define $n_\ell:=mn+\ell$. A direct computation implies that

$$\frac{1}{n_\ell}\sum_{i=0}^{n_\ell-1}\frac{\mathcal{P}^i}{\lambda^i}\bigg(h\frac{\mathcal{P}^{n_\ell-i}}{\lambda^{n_\ell-i}}\mathbb{1}_M\bigg)=\underbrace{\frac{1}{n_\ell}\sum_{s=0}^{m-1}\mu(C_s)\sum_{i=0}^{n_\ell-1}\frac{\mathcal{P}^i}{\lambda^i}(hg_{s-\ell+i})}_{=:I_h^{n_\ell}}+\underbrace{\frac{1}{n_\ell}\sum_{i=0}^{n_\ell-1}\frac{\mathcal{P}^i}{\lambda^i}\bigg(h\frac{\mathcal{P}^{n_\ell-i}}{\lambda^{n_\ell-i}}w\bigg)}_{=:J_h^{n_\ell}}.$$

On the one hand, we have that

$$\|J_h^{n_\ell}\|_{L^1(M,\mu)}\le\frac{1}{n_\ell}\sum_{i=0}^{n_\ell}\bigg\|\frac{\mathcal{P}^i}{\lambda^i}\bigg(\bigg|h\frac{\mathcal{P}^{n_\ell}}{\lambda^{n_\ell-i}}w\bigg|\bigg)\bigg\|_{L^1(M,\mu)}\le\frac{\|h\|_\infty}{n_\ell}\sum_{i=0}^{n_\ell-1}\bigg\|\frac{\mathcal{P}^{n_\ell-i}}{\lambda^{n_\ell-i}}w\bigg\|_{L^1(M,\mu)}.$$

From Step 2, we obtain that $J^{n_\ell}\xrightarrow{n\to\infty}0$ in $L^1(M,\mu)$.

On the other hand, Step 2 yields that

$$\begin{aligned}I_h^{n_\ell}&=\sum_{s=0}^{m-1}\frac{\mu(C_s)}{n_\ell}\sum_{j=0}^{n-1}\frac{\mathcal{P}^{mj}}{\lambda^{mj}}\bigg(\sum_{i=0}^{m-1}\frac{\mathcal{P}^i}{\lambda^i}(hg_{s-\ell+i})\bigg)+\sum_{s=0}^{m-1}\frac{\mu(C_s)}{n_\ell}\frac{\mathcal{P}^{mn-1}}{\lambda^{mn-1}}\bigg(\sum_{i=0}^{\ell}\frac{\mathcal{P}^i}{\lambda^i}(hg_{s-\ell+i}\bigg)\bigg)\\&\xrightarrow[L^1(M,\mu)]{n\to\infty}\sum_{s=0}^{m-1}\frac{\mu(C_s)}{m}\sum_{k=0}^{m-1}g_k\int_{C_k}\sum_{i=0}^{m-1}\frac{\mathcal{P}^i}{\lambda^i}(hg_{s-\ell+i})\,d\mu=\sum_{k=0}^{m-1}\mu(C_{\ell+k})g_k\int_M h\eta\,d\mu.\end{aligned}$$

Hence, $I_h^{n_\ell}+J_h^{n_\ell}\xrightarrow{n\to\infty}\sum_{k=0}^{m-1}\mu(C_{\ell+k})g_k\int_M h\eta\,d\mu$ in $L^1(M,\mu)$, which concludes the proof of Step 3. □

4. *Proposition 5.2*

As a consequence of the corrected Lemma *4.5*, Proposition *5.2* reads as follows.

PROPOSITION 5.2. *Let X_n be an absorbing Markov chain satisfying Hypothesis H1. Suppose that one of the following items holds:*

(a) *there exists $K>0$ such that $\mu(\{K<\eta\})=1$ almost surely;*

(b) *there exists $g\in L^1(M,\mu)$ such that $(1/\lambda^n)\mathcal{P}^n(x,M)\le g$ for every $n\in\mathbb{N}$;*

(c) *the absorbing Markov chain X_n fulfils Hypothesis H1.*

Then for every $h\in L^\infty(M,\mu)$ and $\ell\in\{0,1,\ldots,m-1\}$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{mn+\ell-1}\frac{\mathcal{P}^i}{\lambda^i}\bigg(h\frac{\mathcal{P}^{mn+\ell-i}(\cdot,M)}{\lambda^{mn+\ell-i}}\bigg)\xrightarrow{n\to\infty}\sum_{s=0}^{m-1}\mu(C_{s+\ell})g_s\int_M h(y)\eta(y)\mu(dy)\ \mu\text{-a.s.}\tag{4.1}$$

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In addition,

$$\frac{1}{\lambda^{mn+\ell}}\mathcal{P}^{nm+\ell}h \xrightarrow{n\to\infty} \sum_{s=0}^{m-1}\mu(C_{s+\ell})g_s\int_M h(x)\mu(dx)\ \mu\text{-a.s.} \tag{4.2}$$

Proof. The proof of the theorem assuming that either item (a) or item (b) holds remains mostly the same. The only correction to be made is on page 16 line 5, where the term $(1/\lambda)^n\mathcal{P}(x, M)$ should be replaced by $(1/\lambda)^n\mathcal{P}^n(x, M)$.

Now, we prove item (c). For every $j \in \mathbb{N}$, define the set $K_j := \{x \in M; k(x,\cdot) \in L^\infty(M,\mu)\}$ and the bounded operator $\mathcal{G}_j : L^1(M,\mu) \to L^\infty(K_j,\mu), \mathcal{G}_j f = \mathbb{1}_{K_j}(1/\lambda)\mathcal{P}f$. By composing $\mathcal{G}_j$ to equations (3.1) and (3.2) considering $\ell - 1$ instead of ℓ, from Lemma 4.5 and the fact that $\mathcal{G}_j$ is a bounded operator, we obtain that equations (4.1) and (4.2) converge for μ-almost every $x \in K_j$. Finally, since Hypothesis H1 implies that $\mu(\bigcup_{j\geq 1} K_j) = 1$, we obtain the result. □

5. *Theorem 2.2*

The corrections of Lemma 4.5 also affect Theorem 2.2.

THEOREM 2.2. *Let X_n be an absorbing Markov chain fulfilling Hypothesis H1. Then the following assertions hold:*

(i) *there exist a natural number $m \in \mathbb{N}$ and sets $C_0, C_1, \ldots, C_{m-1} =: C_{-1} \in \mathscr{B}(M)$ such that $\{\mathcal{P}\mathbb{1}_{C_i} > 0\} \subset C_{i-1}$ for every $i \in \{0, 1, \ldots, m-1\}$;*

(ii) *for every $f \in L^1(M,\mu)$, $(1/n)\sum_{i=0}^{n-1}(1/\lambda^i)\mathcal{P}^i f \xrightarrow{n\to\infty} \eta\int_M f(y)\mu(dy)$ in $L^1(M,\mu)$ and μ-a.s.;*

(iii) *there exist non-negative functions $g_0, g_1, \ldots, g_{m-1} =: g_{-1} \in L^1(M,\mu)$, satisfying*

$$\mathcal{P}g_j = \lambda g_{j-1} \quad \textit{and} \quad \|g_j\|_{L^1(M,\mu)} = 1$$

for every $j \in \{0, 1, \ldots, n-1\}$, such that given $\ell \in \{0, 1, \ldots, m-1\}$ and $h \in L^\infty(M,\mu)$, the following limit holds:

$$\frac{1}{\lambda^{nm+\ell}}\mathcal{P}^{nm+\ell}h \xrightarrow[L^1(M,\mu)]{n\to\infty} \sum_{s=0}^{m-1} g_s\int_M h(x)\mu(dx);$$

(iv) *if in addition, we assume that M is a Polish space, then for every $h \in L^\infty(M,\mu)$,*

$$\left(x \mapsto \mathbb{E}_x\left[\frac{1}{n}\sum_{i=0}^{n-1} h\circ X_i \mid \tau > n\right]\right) \xrightarrow{n\to\infty} \int_M h(y)\eta(y)\mu(dy) \tag{5.1}$$

in the $L^\infty(M,\mu)$-weak topology. In particular, we obtain that equation (5.1) also converges weakly in $L^1(M,\mu)$.*

Proof. The proof of assertions (i), (ii) and (iii) remains unchanged. To prove assertion (iv), fix $\ell \in \{0, 1, \ldots, m-1\}$, repeating the proof of [**1**, Lemma 2.2] but changing g_n by $g_{mn+\ell}$, we obtain that $g_{nm+\ell}$ converges to the right-hand side of equation (5.1) in $L^\infty(M,\mu)$-weak*. Since $\ell \in \{0, 1, \ldots, m-1\}$ is arbitrary, we obtain that assertion (iv) follows. □

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6. *Theorem 2.3*
The same proof as before holds using the corrected Lemma 4.5 and Proposition 5.2.

7. *Theorem 2.4*
Theorem 2.4 requires an extra assumption.

THEOREM 2.4. *Let X_n be an absorbing Markov chain fulfilling Hypothesis H2, and suppose that $\mathcal{P}f|_{K_i} \in \mathcal{C}^0(K_i)$ for every $f \in L^1(M, \mu)$ and $i \in \mathbb{N}$, where $\{K_i\}_{i\in\mathbb{N}}$ is the nested sequence of compact sets given by the second part of Hypothesis H2. Then, given $h \in L^\infty(M, \mu)$, equation (2.3) holds for every $x \in (\bigcup_{i\in\mathbb{N}} K_i) \cap \{\eta > 0\}$.*

In the case where $m = 1$ in Theorem 2.2(i), equation (2.4) holds for every $x \in (\bigcup_{i\in\mathbb{N}} K_i) \cap \{\eta > 0\}$.

Proof. Observe that $\mathcal{G}_j : L^1(M, \mu) \to \mathcal{C}^0(K_j), \mathcal{G}_j f := \mathbb{1}_{K_j}(1/\lambda)\mathcal{P}f$ is a bounded linear operator since it is a positive operator between two Banach lattices [**3**, Theorem 5.3]. Then the proof follows from the same arguments as given in the new proof of Proposition *5.2*(c) and equation (5.3). □

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