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RADICAL PROPERTIES DEFINED LOCALLY BY POLYNOMIAL IDENTITIES I

B. J. GARDNER

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Abstract

Bijective correspondences are established between the radical classes $\mathscr{R}$ (in a variety $\mathscr{W}$ of rings) with the property that a ring A is in $\mathscr{R}$ exactly when its finitely generated subrings are all in $\mathscr{R}$, and certain filters of ideals in a free $\mathscr{W}$-ring. It follows that such classes are determined by the polynomial identities satisfied by the finite subsets of their members. Analogous considerations are applied to radical classes $\mathscr{R}$ which, for a fixed integer n, have the property that a ring is in $\mathscr{R}$ if and only if its subrings generated by at most n elements are in $\mathscr{R}$.

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1. Introduction

Throughout this paper we shall use a variety $\mathscr{W}$ of rings as our universal class, and, on occasion, refer to $\mathscr{W}$ as the 'universal variety'. In such a setting there are several connections between radical classes, semi-simple classes and varieties (that is subvarieties of $\mathscr{W}$). Specifically, it is known when a variety is a radical class and when a variety is a semi-simple class (compare Gardner (1975), Theorems 1.4 and 1.5, respectively) and there is also some information available, at least when $\mathscr{W}$ is the class of associative rings, concerning radical classes which are contained in proper varieties, that is whose members all satisfy some set of polynomial identities (see Gardner (1977)). There is, however, a further way in which polynomial identities can intervene in the description of a radical class. We illustrate this with two examples from the class of associative rings (the first example has much wider validity).

The nil radical class $\mathscr{N}$ consists of all rings such that every one-element subset (or, equivalently, every finite subset) satisfies a polynomial identity $x^n = 0$. The locally nilpotent radical class $\mathscr{L}$ consists of all rings in which every finite subset satisfies a linear monomial identity $x_1 x_2 \dots x_n = 0$.

We shall investigate connections between radical classes like these and some other properties of radical classes which we now define.

DEFINITION 1.1. A radical class $\mathscr{R}$ in $\mathscr{W}$ is called *local* if it satisfies

$$A \in \mathscr{R} \quad \text{if and only if every finitely generated subring of } A \text{ is in } \mathscr{R}.$$

A radical class $\mathscr{R}$ in $\mathscr{W}$ is an n-*radical class* if it satisfies

$$A \in \mathscr{R} \quad \text{if and only if every subring generated by} \leqslant n \text{ elements is in } \mathscr{R}.$$

Note that local and n-radical classes are *strongly hereditary*, that is closed under formation of subrings.

Radical classes with these properties have been studied by Ryabukhin (1965) and Stewart (1972a); $\mathscr{L}$ is an example of the first, $\mathscr{N}$ of the second, for $n = 1$. The properties are reminiscent of a characteristic of torsion classes of modules: a torsion class is completely determined by the one-generator modules it contains. Consideration of annihilators of elements leads to the establishment of a bijection between torsion classes and certain filters of left ideals of the scalar ring. This was first demonstrated by Gabriel (1962).

For 1-radicals, Ryabukhin (1965) established a result which is quite closely analogous to Gabriel's module result; he showed that with each 1-radical class one can associate a certain filter of ideals in a free ring F_1 on one generator (and conversely). For each element a of a ring, the ideal of one-variable polynomials vanishing on a is the kernel of a natural map from F_1 to the subring $\langle a \rangle$ generated by a. The filter associated with a 1-radical class $\mathscr{R}$ consists of all these ideals for $a \in A \in \mathscr{R}$.

An attempt to generalize this result to n-radicals, utilizing filters of ideals of a free ring F_n on n generators comes up against the difficulty that the kernel of the obvious natural map from F_n to a ring $\langle a_1, \dots, a_n \rangle$ generated by $\{a_1, \dots, a_n\}$ is not the same thing as the ideal of polynomials vanishing on $\{a_1, \dots, a_n\}$. It is however, in a sense, the ideal of polynomials vanishing on the ordered n-tuple $(a_1, \dots, a_n)$. We find it profitable to establish a bijective correspondence between n-radical classes and filters of these kernels and then show that, for an n-radical class $\mathscr{R}$, a ring A belongs to $\mathscr{R}$ if and only if for every subset S of cardinality $\leqslant n$, the ideal of n-variable polynomials which vanish on S is in the filter associated with $\mathscr{R}$.

For local radicals, there are further complications; filters of ideals of a free ring on $\aleph_0$ generators are associated with radical classes, but these ideals have a fairly tenuous connection with sets of polynomials vanishing on finite sets. However, the filter-radical correspondences lead to a demonstration that local radical classes are 'locally equationally determined', and it turns out that a local radical class is the same thing as a locally equational class in the sense of Hu (1973) which is also closed under extensions.

Some applications of the results and methods of this paper will be presented in a subsequent one.

We shall use the following notation

$A \lhd B$: A is an ideal of B,
$A \leqslant B$: A is a subring of B,
$\langle a_1, \ldots, a_n \rangle$: subring generated by $\{a_1, \ldots, a_n\}$.

2. Local radical classes

Let F be a free ring (in some universal variety $\mathscr{W}$) on generators $x_1, x_2, x_3, \ldots$. We shall implicitly regard an element of F as a polynomial $p(x_1, x_2, x_3, \ldots)$ in all variables.

DEFINITION 2.1. Let $v_1, v_2, v_3, \ldots, v_m$ be elements (not necessarily distinct) of a ring in $\mathscr{W}$. Let

$$I(v_1, \ldots, v_m) = \{p \in F \mid p(v_1, v_2, \ldots, v_m, 0, 0, 0, \ldots) = 0\}.$$

Note that $I(v_1, \ldots, v_n)$ is associated with the ordered m-tuple $(v_1, \ldots, v_m)$, rather than with the set $\{v_1, \ldots, v_m\}$.

DEFINITION 2.2. A set $\mathscr{F}$ of ideals of F is called a *radical filter* if it satisfies the following conditions.

(i) $J \in \mathscr{F}$ implies that $x_i \in J$ for almost all i;
(ii) $J \in \mathscr{F}$, $J \subseteq K \lhd F$ implies that $K \in \mathscr{F}$;
(iii) $J \in \mathscr{F}$, $\beta_1, \ldots, \beta_n \in F$, $n \in \mathbf{Z}^+$ implies that $I(\beta_1 + J, \ldots, \beta_n + J) \in \mathscr{F}$;
(iv) $K \lhd F$, $x_i \in K$ for almost all i, $J \in \mathscr{F}$, $I(j_1 + K, \ldots, j_m + K) \in \mathscr{F}$ for all $j_1, \ldots, j_m \in J$, and all $m \in \mathbf{Z}^+$ implies that $K \in \mathscr{F}$.

THEOREM 2.3. *There are bijections between the collection of all local radical classes in $\mathscr{W}$ and the collection of all radical filters of ideals in F, given by*

$$\mathscr{R} \mapsto \mathscr{F}_{\mathscr{R}} = \{I(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in A,\ n \in \mathbf{Z}^+,\ A \in \mathscr{R}\};$$

$$\mathscr{F} \mapsto \mathscr{R}_{\mathscr{F}} = \{A \mid I(a_1, \ldots, a_n) \in \mathscr{F} \text{ for all } a_1, \ldots, a_n \in A,\ n \in \mathbf{Z}^+\}.$$

PROOF. Let $\mathscr{R}$ be a local radical class. If $a_1, \ldots, a_n \in A \in \mathscr{R}$, then $x_i \in I(a_1, \ldots, a_n)$ for $i > n$ and so $\mathscr{F}_{\mathscr{R}}$ satisfies (i).

If $J \in \mathscr{F}_{\mathscr{R}}$, then $J = I(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in A \in \mathscr{R}$. The map f from F to $\langle a_1, \ldots, a_n \rangle$, defined by $x_1 \mapsto a_1, \ldots, x_n \mapsto a_n$, $x_{n+1} \mapsto 0, x_{n+2} \mapsto 0, \ldots$, has kernel

J, so that $F/J \cong \langle a_1, \ldots, a_n \rangle \in \mathscr{R}$. If now $J \subseteq K \lhd F$, then

$$F/K \cong (F/J)/(K/J) \cong \langle \bar{a}_1, \ldots, \bar{a}_n \rangle \in \mathscr{R},$$

where $\bar{a}_i = a_i + f(K)$. Since $\langle \bar{a}_1, \ldots, \bar{a}_n \rangle \in \mathscr{R}$, we have $I(\bar{a}_1, \ldots, \bar{a}_n) \in \mathscr{F}_{\mathscr{R}}$. If $\gamma \in K$, then $\gamma(a_1, a_2, \ldots, a_n, 0, 0, 0, \ldots) = f(\gamma) \in f(K)$, so that

$$\gamma(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n, 0, 0, 0, \ldots) = \gamma(a_1, a_2, \ldots, a_n, 0, 0, 0, \ldots) + f(K) = 0,$$

that is $\gamma \in I(\bar{a}_1, \ldots, \bar{a}_n)$. Conversely, for any $\beta \in I(\bar{a}_1, \ldots, \bar{a}_n)$, we have

$$\beta(a_1, \ldots, a_n, 0, 0, 0, \ldots) \in f(K)$$

and so

$$\beta(a_1, \ldots, a_n, 0, 0, 0, \ldots) = f(\delta) = \delta(a_1, \ldots, a_n, 0, 0, 0, \ldots)$$

for some $\delta \in K$, whence $\beta - \delta \in I(a_1, \ldots, a_n) = J \subseteq K$ and so $\beta \in K$. It follows that $K = I(\bar{a}_1, \ldots, \bar{a}_n) \in \mathscr{F}_{\mathscr{R}}$ and so $\mathscr{F}_{\mathscr{R}}$ satisfies (ii).

If $J \in \mathscr{F}_{\mathscr{R}}$ and $\beta_1, \ldots, \beta_n \in F$, let $J = I(a_1, \ldots, a_m)$ for some $a_1, \ldots, a_m \in A \in \mathscr{R}$. Then as before $F/J \cong \langle a_1, \ldots, a_m \rangle \leqslant A \in \mathscr{R}$ and so $F/J \in \mathscr{R}$. But then

$$\beta_1 + J, \ldots, \beta_n + J \in F/J \in \mathscr{R},$$

so that $I(\beta_1 + J, \ldots, \beta_n + J) \in \mathscr{F}_{\mathscr{R}}$. This establishes (iii) for $\mathscr{F}_{\mathscr{R}}$.

Finally we check (iv). Let K be an ideal of F containing almost all x_i and let $J \in \mathscr{F}_{\mathscr{R}}$ be such that $I(j_1 + K, \ldots, j_m + K) \in \mathscr{F}_{\mathscr{R}}$ for all $j_1, \ldots, j_m \in J$. Now $J + K \in \mathscr{F}_{\mathscr{R}}$ by (ii) (already verified) and so $J + K = I(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in A \in \mathscr{R}$. Consider the ring $(J+K)/K$. If $j_1, \ldots, j_m \in J$, then $I(j_1 + K, \ldots, j_m + K) \in \mathscr{F}_{\mathscr{R}}$ and so for some $c_1, \ldots, c_l \in C \in \mathscr{R}$, we have

$$I(j_1 + K, \ldots, j_m + K) = I(c_1 \ldots, c_l).$$

But then

$$\langle j_1 + K, \ldots, j_m + K \rangle \cong F/I(j_1 + K, \ldots, j_m + K) = F/I(c_1, \ldots, c_l) \cong \langle c_1, \ldots, c_l \rangle \leqslant C \in \mathscr{R}.$$

Thus every finitely generated subring of $(J+K)/K$ is in $\mathscr{R}$ and so $(J+K)/K \in \mathscr{R}$. From the exact sequence

$$\begin{array}{c} 0 \to (J+K)/K \to F/K \to F/(J+K) \to 0 \\ \qquad\qquad\qquad\qquad\qquad \Big| \\ \qquad\qquad\qquad\qquad\qquad \langle a_1, \ldots, a_n \rangle \leqslant A \in \mathscr{R}, \end{array}$$

where the vertical line denotes isomorphism, we conclude that $F/K \in \mathscr{R}$. By assumption concerning K, we have $F/K = \langle x_1 + K, \ldots, x_r + K \rangle$ for some $r \in \mathbf{Z}^+$, and $x_i \in K$ for $i > r$. Since $F/K \in \mathscr{R}$, $I(x_1 + K, \ldots, x_r + K) \in \mathscr{F}_{\mathscr{R}}$. Any polynomial $\gamma \in I(x_1 + K, \ldots, x_r + K)$ can be written in the form $\delta + \mu$, where δ involves only $x_1, \ldots, x_r$ and every term of μ involves at least one other x_i. We have

$$0 = \gamma(x_1 + K, \ldots, x_r + K, 0, 0, 0, \ldots) = \gamma(x_1, \ldots, x_r, 0, 0, 0, \ldots) + K$$

and also

$$\mu(x_1+K, \ldots, x_r+K, 0, 0, 0, \ldots) = \mu(x_1, \ldots, x_r, 0, 0, 0, \ldots)+K,$$

since every term of μ involves some x_i with $i>r$. Hence

$$\delta(x_1+K, \ldots, x_r+K, 0, 0, 0, \ldots) = 0, \quad \text{that is } \delta(x_1, \ldots, x_r, 0, 0, 0, \ldots) \in K.$$

But $\delta(x_1, \ldots, x_r) = \delta$ and so $\delta \in K$. Since also μ is in the ideal generated by $x_{r+1}, x_{r+2}, \ldots$, we have $\mu \in K$. Thus $\gamma = \delta+\mu \in K$ and $I(x_1, \ldots, x_r) \subseteq K$. Conversely, if $\lambda \in K$, then $\lambda = \rho+\sigma$, where every term of σ involves an x_i with $i>r$ (and thus $\sigma \in K$) and no term of ρ does. Then $\rho \in K$ and

$$\rho(x_1, \ldots, x_r, 0, 0, 0, \ldots) = \rho \in K; \quad \sigma(x_1, \ldots, x_r, 0, 0, 0, \ldots) = 0,$$

whence $\lambda(x_1, \ldots, x_r, 0, 0, 0, \ldots) \in K$, that is $\lambda \in I(x_1+K, \ldots, x_r+K)$. We conclude that $K = I(x_1+K, \ldots, x_r+K) \in \mathscr{F}_{\mathscr{R}}$. $\mathscr{F}_{\mathscr{R}}$ therefore satisfies (iv) and is a radical filter.

We turn now to the reverse correspondence. Given a radical filter $\mathscr{F}$, we wish to show that $\mathscr{R}_{\mathscr{F}}$ is a local radical class. This involves showing that $\mathscr{R}_{\mathscr{F}}$ is a strongly hereditary radical class containing every ring of which it contains all the finitely generated subrings. We note (for we will need the fact below) that $\mathscr{R}_{\mathscr{F}}$ is clearly strongly hereditary.

If $L \lhd A \in \mathscr{R}_{\mathscr{F}}$ and $a_1, \ldots, a_n \in A$, then

$$I(a_1+L, \ldots, a_n+L) \supseteq I(a_1, \ldots, a_n) \in \mathscr{F},$$

and so $I(a_1+L, \ldots, a_n+L) \in \mathscr{F}$, and thus $A/L \in \mathscr{R}_{\mathscr{F}}$. So $\mathscr{R}_{\mathscr{F}}$ is homomorphically closed.

Consider now a ring A with an ideal L such that L and A/L are in $\mathscr{R}_{\mathscr{F}}$. Let $a_1, \ldots, a_n$ be in A and consider the homomorphism $F \to \langle a_1, \ldots, a_n\rangle$ given by $x_i \mapsto a_i$ if $i \leqslant n$ and $x_i \mapsto 0$ if $i>n$. We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 \to & \langle a_1, \ldots, a_n\rangle \cap L & \to & \langle a_1, \ldots, a_n\rangle & \to & \langle a_1, \ldots, a_n\rangle / \langle a_1, \ldots, a_n\rangle \cap L & \to 0 \\
 & & & \Big| & & \Big| & \\
 & & & & & (\langle a_1, \ldots, a_n\rangle + L)/L & \\
 & & & & & \Big| & \\
 & & & & & \langle a_1+L, \ldots, a_n+L\rangle & \\
 & & & \Big| & & \Big| & \\
0 \to & I(a_1+L, \ldots, a_n+L)/I(a_1, \ldots, a_n) & \to & F/I(a_1, \ldots, a_n) & \to & F/I(a_1+L, \ldots, a_n+L) & \to 0,
\end{array}$$

where the vertical lines denote isomorphism. From this it follows that

$$I(a_1+L, \ldots, a_n+L)/I(a_1, \ldots, a_n) \cong \langle a_1, \ldots, a_n\rangle \cap L \leqslant L \in \mathscr{R}_{\mathscr{F}}.$$

By definition of $\mathscr{R}_{\mathscr{F}}$, and the fact that $\mathscr{R}_{\mathscr{F}}$ is known to be strongly hereditary, if

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$\alpha_1, \ldots, \alpha_m \in I(a_1+L, \ldots, a_n+L)$, then $I(\alpha_1+I(a_1, \ldots, a_n), \ldots, \alpha_m+I(a_1, \ldots, a_n)) \in \mathscr{F}$. Since $I(a_1+L, \ldots, a_n+L) \in \mathscr{F}$, defining condition (iv) for radical filters implies that $I(a_1, \ldots, a_n) \in \mathscr{F}$. This being so for all $a_1, \ldots, a_n \in A$, we conclude that $A \in \mathscr{R}_{\mathscr{F}}$ and $\mathscr{R}_{\mathscr{F}}$is closed under extensions

If a ring A has a chain $\{L_\theta | \theta \in \Theta\}$ of ideals in $\mathscr{R}_{\mathscr{F}}$ and if $b_1, \ldots, b_n \in \bigcup_\theta L_\theta$, then $b_1, \ldots, b_n \in$ some L_θ and so $I(b_1, \ldots, b_n) \in \mathscr{F}$. Hence $\bigcup_\theta L_\theta$ is in $\mathscr{R}_{\mathscr{F}}$. We have now proved that $\mathscr{R}_{\mathscr{F}}$ is a radical class.

As noted, $\mathscr{R}_{\mathscr{F}}$ is strongly hereditary. If $\mathscr{R}_{\mathscr{F}}$ contains every finitely generated subring of a ring A and if $a_1, \ldots, a_n \in A$, then $a_1, \ldots, a_n \in \langle a_1, \ldots, a_n \rangle \in \mathscr{R}_{\mathscr{F}}$ and so $I(a_1, \ldots, a_n) \in \mathscr{F}$ and thus $A \in \mathscr{R}_{\mathscr{F}}$. This means that $\mathscr{R}_{\mathscr{F}}$ is a local radical class, as asserted.

We conclude the proof by showing that the correspondences we have set up are bijections. Let $\mathscr{R}$ be a local radical class. If $a_1, \ldots, a_n \in A \in \mathscr{R}$, then $I(a_1, \ldots, a_n) \in \mathscr{F}_{\mathscr{R}}$, so $A \in \mathscr{R}_{\mathscr{F}_{\mathscr{R}}}$, that is $\mathscr{R} \subseteq \ldots \mathscr{R}_{\mathscr{F}_{\mathscr{R}}}$. Suppose there is a ring $B \in \mathscr{R}_{\mathscr{F}_{\mathscr{R}}} \backslash \mathscr{R}$. Since $\mathscr{R}$ is local, we may assume B is finitely generated. Let $B = \langle b_1, \ldots, b_m \rangle \cong F/I(b_1, \ldots, b_m)$. Since $B \in \mathscr{R}_{\mathscr{F}_{\mathscr{R}}}$, $\mathscr{F}_{\mathscr{R}}$ contains $I(b_1, \ldots, b_m)$. But then $I(b_1, \ldots, b_m) = I(c_1, \ldots, c_k)$ for some $c_1, \ldots, c_k \in C \in \mathscr{R}$, whence $B \cong F/I(b_1, \ldots, b_m) = F/I(c_1, \ldots, c_k) \cong \langle c_1, \ldots, c_k \rangle \in \mathscr{R}$—a contradiction. Hence $\mathscr{R} = \mathscr{R}_{\mathscr{F}_{\mathscr{R}}}$.

Finally, let $\mathscr{F}$ be a radical filter with $J \in \mathscr{F}$. Since J contains almost all x_i, we can write $F/J = \langle x_1+J, \ldots, x_r+J \rangle$. By condition (iii) for radical filters,

$$I(\beta_1+J, \ldots, \beta_n+J) \in \mathscr{F}$$

for all $\beta_1, \ldots, \beta_n \in F$. This means that $F/J \in \mathscr{F}$. Hence $I(x_1+J, \ldots, x_r+J) \in \mathscr{F}_{\mathscr{R}_{\mathscr{F}}}$. Arguing as in an earlier part of the proof, we can show that $I(x_1+J, \ldots, x_r+J) = J$ (using the fact that J contains almost all x_i). Thus $\mathscr{F} \subseteq \mathscr{F}_{\mathscr{R}_{\mathscr{F}}}$. Conversely, if $K \in \mathscr{F}_{\mathscr{R}_{\mathscr{F}}}$, then $K = I(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in A \in \mathscr{R}_{\mathscr{F}}$. But if $A \in \mathscr{R}_{\mathscr{F}}$, then $I(a_1, \ldots, a_n) \in \mathscr{F}$ and so $K \in \mathscr{F}$. Hence $\mathscr{F} = \mathscr{F}_{\mathscr{R}_{\mathscr{F}}}$ and the proof of the theorem is complete.

We have now demonstrated a strong connection between local radical classes and families of ideals of the free ring. These ideals, however, are not the ideals of identities satisfied by finite sets, or even by ordered n-tuples; for instance, for any ordered pair (a, b) of elements of a ring, we have $x_1 x_2 x_3 \in I(a, b)$, though clearly $x_1 x_2 x_3$ does not vanish on all two-element sets of all rings. Our next task is to establish a connection between radical filters and the ideals of identities satisfied by finite sets.

Let F_n be the free ring on $\{x_1, \ldots, x_n\}$ (a subring of F). Thus the elements of F_n are polynomials $p(x_1, \ldots, x_n)$.

DEFINITION 2.4. Let $a_1, \ldots, a_n$ be elements (not necessarily distinct) of a ring in $\mathscr{W}$. Let

$$I_n(a_1, \ldots, a_n) = \{p = p(x_1, \ldots, x_n) \in F_n | p(a_1, \ldots, a_n) = 0\}.$$

LEMMA 2.5. *For each n, let π_n: $F \to F_n$ be defined by*

$$\pi_n(x_i) = \begin{cases} x_i & \text{if } i \leqslant n, \\ 0 & \text{if } i > n. \end{cases}$$

Then for every ordered n-tuple $(a_1, \ldots, a_n)$ of elements of a ring in $\mathscr{W}$ we have

$$I(a_1, \ldots, a_n) = \pi_n^{-1}(I_n(a_1, \ldots, a_n)).$$

PROOF. Every polynomial $\alpha \in F$ can be written in the form $\beta + \gamma$, where β involves only $x_1, \ldots, x_n$, while every term of γ involves at least one x_i with $i > n$. Then

$$\alpha(a_1, \ldots, a_n, 0, 0, 0, \ldots) = \beta(a_1, \ldots, a_n, 0, 0, 0, \ldots) + \gamma(a_1, \ldots, a_n, 0, 0, 0, \ldots)$$
$$= \beta(a_1, \ldots, a_n, 0, 0, 0).$$

If $\alpha \in I(a_1, \ldots, a_n)$, then $\beta \in I(a_1, \ldots, a_n)$ and so $\pi_n(\alpha) = \pi_n(\beta) \in I_n(a_1, \ldots, a_n)$. Conversely, if $\pi_n(\alpha) \in I_n(a_1, \ldots, a_n)$, then $\pi_n(\beta) \in I_n(a_1, \ldots, a_n)$ and so

$$\beta(a_1, \ldots, a_n, 0, 0, 0, \ldots) = \beta(a_1, \ldots, a_n) = 0.$$

Thus $\beta \in I(a_1, \ldots, a_n)$ and so $\beta \in I(a_1, \ldots, a_n)$.

LEMMA 2.6. *If $J_1, \ldots, J_m$ are ideals in a radical filter $\mathscr{F}$, then $J_1 \cap \ldots \cap J_m \in \mathscr{F}$.*

PROOF. Let $\mathscr{R}$ be the local radical class associated with $\mathscr{F}$. Then $J_1 = I(a_1, \ldots, a_k)$ for some $a_1, \ldots, a_k \in A \in \mathscr{R}$, and so $F/J_1 = \langle a_1, \ldots, a_k \rangle \in \mathscr{R}$. Let $\langle a_1, \ldots, a_n \rangle = B_1$. Similarly, each other F/J_i is isomorphic to a finitely generated ring $B_i \in \mathscr{R}$. It follows that $F/J_1 \cap \ldots \cap J_m$ is isomorphic to a subring of $B_1 \oplus \ldots \oplus B_m$, and so $F/J_1 \cap \ldots \cap J_m \in \mathscr{R}$. Now each J_i contains almost all the free generators of F; so, therefore, does $J_1 \cap \ldots \cap J_m$, and we can write

$$F/J_1 \cap \ldots \cap J_m = \langle x_1 + J_1 \cap \ldots \cap J_m, \ldots, x_u + J_1 \cap \ldots \cap J_m \rangle.$$

Arguing as in the proof of Theorem 2.3, we see that

$$J_1 \cap \ldots \cap J_m = I(x_1 + J_1 \cap \ldots \cap J_m, \ldots, x_u + J_1 \cap \ldots \cap J_m) \in \mathscr{F}.$$

DEFINITION 2.7. For a subset S of a ring in $\mathscr{W}$, $I_n^*(S)$ is the set of polynomials in $x_1, \ldots, x_n$ which vanish on S.

THEOREM 2.8. *Let $\mathscr{R}$ be a local radical class with radical filter $\mathscr{F}$. A ring A belongs to $\mathscr{R}$ if and only if $\pi_n^{-1}(I_n^*(S)) \in \mathscr{F}$ for every finite subset S of A and for every $n \in \mathbf{Z}^+$.*

PROOF. Let A be a ring satisfying the second condition and let $a_1, \ldots, a_n$ be in A. If $\alpha = \alpha(x_1, \ldots, x_n) \in I_n^*(\{a_1, \ldots, a_n\})$, then, in particular, $\alpha(a_1, \ldots, a_n) = 0$, that is $I_n^*(\{a_1, \ldots, a_n\}) \subseteq I_n(a_1, \ldots, a_n)$. By Lemma 2.5, we have

$$I(a_1, \ldots, a_n) = \pi_n^{-1}(I_n(a_1, \ldots, a_n)) \supseteq \pi_n^{-1}(I_n^*(a_1, \ldots, a_n)) \in \mathscr{F}.$$

By condition (ii) for radical filters, $I(a_1, \ldots, a_n) \in \mathscr{F}$. We conclude that $A \in \mathscr{R}$.

Conversely, let A be a ring in $\mathscr{R}$ and consider a finite subset S. It is easy to see that

$$I_n^*(S) = \bigcap \{I_n(a_1, \ldots, a_n) \mid (a_1, \ldots, a_n) \in S \times S \times \ldots \times S (n \text{ factors})\}.$$

Thus

$$\pi_n^{-1}(I_n^*(S)) = \bigcap \{\pi_n^{-1}(I_n(a_1, \ldots, a_n)) \mid (a_1, \ldots, a_n) \in S \times S \times \ldots \times S\}$$
$$= \bigcap \{I(a_1, \ldots, a_n) \mid (a_1, \ldots, a_n) \in S \times S \times \ldots \times S\} \in \mathscr{F}$$

by Lemmas 2.5 and 2.6.

Definition 2.9. For a subset S of a ring R, $I^*(S)$ is the set of polynomials which vanish on S.

Theorem 2.10. *Let $\mathscr{R}$ be a local radical class, A a ring in $\mathscr{W}$. Then $A \in \mathscr{R}$ if and only if $F/I^*(S) \in \mathscr{R}$ for each finite subset S of A.*

Proof. If $F/I^*(S) \in \mathscr{R}$, then

$$F_n/I_n^*(S) = F_n/[F_n \cap I^*(S)] \cong [F_n + I^*(S)]/I^*(S) \in \mathscr{R}.$$

But $F_n/I_n^*(S) \cong \pi_n^{-1}(F_n)/\pi_n^{-1}(I_n^*(S)) = F/\pi_n^{-1}(I_n^*(S))$ and so since $\pi_n^{-1}(I_n^*(S))$ contains almost all the x_i, a by now familiar argument shows that $\pi_n^{-1}(I_n^*(S)) \in \mathscr{F}$, the radical filter associated with $\mathscr{R}$. By Theorem 2.7, A belongs to $\mathscr{R}$.

Conversely, if A is in $\mathscr{R}$, then for every finite subset S of A, we have, by Theorem 2.8, $\pi_n^{-1}(I_n^*(S)) \in \mathscr{F}$, the radical filter associated with $\mathscr{R}$. Hence for some

$$b_1, \ldots, b_m \in B \in \mathscr{R},$$

we have $\pi_n^{-1}(I_n^*(S)) = I(b_1, \ldots, b_m)$, so that

$$(F_n + I^*(S))/I^*(S) \cong F_n/[F_n \cap I^*(S)] = F_n/I_n^*(S)) \cong F/\pi_n^{-1}(I_n^*(S)) \cong \langle b_1, \ldots, b_m \rangle \in \mathscr{R}.$$

For a finite set X of positive integers, let F_X be the free ring on $\{x_i \mid i \in X\}$. Then $F_X \cong F_{|X|}$, $(F_X + I^*(S))/I^*(S) \cong (F_{|X|} + I^*(S))/I^*(S)$ and

$$F/I^*(S) = \bigcup \{(F_X + I^*(S))/I^*(S) \mid X \subseteq \mathbf{Z}^+, |X| < \aleph_0\}.$$

If R is a finitely generated subring of $F/I^*(S)$, then $R \leqslant (F_X + I^*(S))/I^*(S)$ for some X, and so $R \in \mathscr{R}$. Since $\mathscr{R}$ is local, we have $F/I^*(S) \in \mathscr{R}$.

Definition 2.11. (Hu (1973).) Let $\mathscr{A}$ be a class of rings in $\mathscr{W}$ which is closed under finite direct sums. $E_L(\mathscr{A})$ is the class

$\{A \in \mathscr{W} \mid$ for every finite subset S of A, there exists a ring $B \in \mathscr{A}$ and a finite subset T of B such that every polynomial identity of T is a polynomial identity of $S\}$,

and $\mathscr{A}$ is called locally equational if $E_L(\mathscr{A}) = \mathscr{A}$.

(The definition of $E_L(\mathscr{A})$ given by Hu is actually a little more complicated than this since $\mathscr{A}$ is not required to be closed under finite direct sums. However, we always have $E_L(\mathscr{A}) = E_L(E_L(\mathscr{A}))$ and locally equational classes are closed under finite direct sums and so every such class has the form $E_L(\mathscr{A})$ for a class with this closure property.)

THEOREM 2.12. *The following conditions are equivalent for a non-empty subclass* $\mathscr{R}$ *of* $\mathscr{W}$.

(i) $\mathscr{R}$ *is extension-closed and locally equational;*

(ii) $\mathscr{R}$ *is a local radical class.*

PROOF. (i) $\Rightarrow$ (ii): Clearly $\mathscr{R}$ is homomorphically closed and strongly hereditary (so in particular it contains, with any ring A, all finitely generated subrings of A). Let R be a ring which is the union of a directed set $\{R_\lambda \mid \lambda \in \Lambda\}$ of subrings, that is $R = \bigcup_\lambda R_\lambda$, and if $\lambda, \mu \in \Lambda$, there exists $\rho \in \Lambda$ with $R_\lambda \subseteq R_\rho$ and $R_\mu \subseteq R_\rho$. If each $R_\lambda \in \mathscr{R}$, then every finite subset of R is in an $\mathscr{R}$-ring and since $\mathscr{R}$ is locally equational, it follows that $R \in \mathscr{R}$. A special case of this situation occurs when R is the union of an ascending chain of ideals, each in $\mathscr{R}$. From this, and the fact that $\mathscr{R}$ is extension-closed, we conclude that $\mathscr{R}$ is a radical class. Another special case occurs when $\{R_\lambda \mid \lambda \in \Lambda\}$ is the set of all finitely generated subrings of R. From this and the strong heredity of $\mathscr{R}$, we conclude that $\mathscr{R}$ is a local radical class.

(ii) $\Rightarrow$ (i): Since $\mathscr{R}$ is closed under extensions and direct sums, we need only show that $\mathscr{R} \supseteq E_L(\mathscr{R})$. Let A be in $E_L(\mathscr{R})$ and let S be any finite subset of A. Then there is a ring $B \in \mathscr{R}$ with a finite subset T such that every polynomial identity satisfied by T is satisfied by S, that is $I^*(T) \subseteq I^*(S)$. By Theorem 2.10, $F/I^*(T) \in \mathscr{R}$ and so $F/I^*(S) \cong (F/I^*(T))/(I^*(S)/I^*(T)) \in \mathscr{R}$; by Theorem 2.10 again, S being an arbitrary finite subset of S, we have $A \in \mathscr{R}$.

Locally equational classes are generalizations of varieties and so Theorem 2.12 is a generalization of the result (proved by Wiegandt (1974) for associative rings and noted by the author (1975) in general) that a variety is a radical class if and only if it is closed under extensions.

3. n-Radical classes

As before, for each positive integer n, we denote by F_n a free ring (in $\mathscr{W}$) on generators $x_1, \ldots, x_n$.

DEFINITION 3.1. An *n-radical filter* is a set $\mathscr{F}$ of ideals of F_n satisfying the following conditions.

(i) $K \in \mathscr{F}$, $K \subseteq J \lhd F$ implies that $J \in \mathscr{F}$.

(ii) $J \in \mathscr{F}$, $\beta_1, \ldots, \beta_n \in F_n$ implies that $I_n(\beta_1 + J, \ldots, \beta_n + J) \in \mathscr{F}$.

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(iii) $K \triangleleft F_n$, $J \in \mathscr{F}$, $I_n(j_1 + K, \ldots, j_n + K) \in \mathscr{F}$ for all $(j_1, \ldots, j_n) \in J \times \ldots \times J$ implies that $K \in \mathscr{F}$.

(Here I_n has the same meaning as previously.) There is the same sort of connection between n-radical classes and n-radical filters as between local radical classes and radical filters. The relevant proofs closely parallel (and are often simpler than) those of the corresponding results in Section 2. We therefore omit the details and summarize the conclusions as follows.

THEOREM 3.2. *The correspondences*

$$\mathscr{R} \mapsto \{I_n(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in A \in \mathscr{R}\},$$

$$\mathscr{F} \mapsto (A \mid a_1, \ldots, a_n \in A \text{ implies that } I_n(a_1, \ldots, a_n) \in \mathscr{F}\}$$

define bijections between the collections of n-radical classes in $\mathscr{W}$ *and n-radical filters of ideals of* F_n, *for each n.* (Compare with Theorem 2.3.)

THEOREM 3.3. *Let* $\mathscr{R}$ *be an n-radical class with n-radical filter* $\mathscr{F}$. *Then a ring A belongs to* $\mathscr{R}$ *if and only if* $I_n^*(S) \in \mathscr{F}$ *for every subset S of A with* $|S| \leqslant n$. (Compare with Lemma 2.6 and Theorem 2.8.)

Note that for a set S with $|S| \leqslant n$, $I_n^*(S)$ is effectively the set of all polynomials vanishing on S. In the case $n = 1$, of course, there is no distinction between polynomials which annihilate sets and polynomials which annihilate n-tuples. Thus Theorem 3.3 with $n = 1$ is the Theorem of Ryabukhin (1965).

4. Examples and miscellaneous comments

Whenever it is a radical class (and for some choices of $\mathscr{W}$ it is not, see Anderson (1974)) the class of locally nilpotent rings is a local radical class. Some examples of local radical classes of associative rings are given in Stewart ((1972a) and (1972b)), for example, the class of rings in which every finitely generated subring is finitely generated as a group, the class of rings in which every finitely generated subring is artinian and the class of rings in which every finitely generated subring is antisimple. The precise relationship between the last-named and the classes of nil and locally nilpotent rings is not known.

Let $\mathscr{C}$ denote the class of commutative associative rings and for a class $\mathscr{K}$ of associative rings let $\mathscr{K} \circ \mathscr{C} = \{A \mid A$ is associative and has an ideal $I \in \mathscr{K}$ with $A/I \in \mathscr{C}\}$. Freidman (1958) has shown that if $\mathscr{R}$ is a hereditary, supernilpotent radical class of associative rings with the property that every ring which is the sum of a directed system of subrings, each in $\mathscr{R}$, is itself in $\mathscr{R}$, then so is $\mathscr{R} \circ \mathscr{C}$. As a special case, in effect, we have

PROPOSITION 4.1. *If $\mathscr{R}$ is a supernilpotent local radical class of associative rings, so is $\mathscr{R}\circ\mathscr{C}$.*

Examples of 1-radical classes are: the class of nil rings (any $\mathscr{W}$), the class of rings satisfying the identity $x^2 = x$ (in any universal variety of power-associative rings; (see Gardner (1975), Theorem 3.1); semi-simple radical classes of associative rings (see, for example, Gardner and Stewart (1975)).

It is easy to see that every 1-radical class is an n-radical class, and every n-radical class is a local radical class. On the other hand, the class of locally nilpotent associative rings is an example of a local radical class which is not a 1-radical. We have not given any examples so far of n-radical classes ($n > 1$) which are not 1-radical classes. Such things are a bit hard to get hold of, though we give a couple of examples below. Before we do this, however, it is appropriate that we mention another way of generating local radical classes.

PROPOSITION 4.2. (i) *Let Φ be a set of polynomials in the free $\mathscr{W}$-ring on one generator x such that if $p(x), q(x) \in \Phi$, then $p(q(x)) \in \Phi$. Then*

$$\mathscr{R}_\Phi = \{A \in \mathscr{W} \mid a \in A \text{ implies there exists } p \in \Phi \text{ with } p(a) = 0\}$$

is a 1-radical class.

(ii) *Let Ψ be a set of polynomials in the free $\mathscr{W}$-ring on generators $x_1, x_2, \ldots$ satisfying either*

$$p_1(x_{i(1,1)}, \ldots, x_{i(1,n_1)}), \ldots, p_m(x_{i(m,1)}, \ldots, x_{i(m,n_m)}), q(x_{i_1}, \ldots, x_{i_m}) \in \Psi$$
$$\Rightarrow q(p_1(x_{i(1,1)}, \ldots, x_{i(1,n_1)}), p_2(x_{i(1,n_1)+1}, \ldots), \ldots,$$
$$p_m(\ldots, x_{i(1,n_1)} + n_2 + \ldots + n_m) \in \Psi$$

or

$$p_1(x_{i(1,1)}, \ldots, x_{i(1,n_1)}), \ldots, p_m(x_{i(m,1)}, \ldots, x_{i(m,n_m)}), q(x_{i_1}, \ldots, x_{i_m}) \in \Psi$$
$$\Rightarrow q(p_1(x_{i(1,1)}, \ldots, x_{i(1,n_1)}), \ldots, p_m(x_{i(m,1)}, \ldots, x_{i(m,n_m)})) \in \Psi.$$

Then

$$\mathscr{R}_\Psi = \{A \in \mathscr{W} \mid \text{for every finite subset } S \text{ of } A \text{ there is a polynomial in } \Psi \text{ which vanishes on } S\}$$

is a local radical class.

PROOF. Everything is straightforward except possibly for closure under extensions which is ensured by the conditions stated. We illustrate for the first case in (ii). Let A be a ring with an ideal I such that $I, A/I \in \mathscr{R}_\Psi$, and let S be a finite subset of A. Then there is a polynomial $p = p(x_{i_1}, \ldots, x_{i_l}) \in \Psi$ such that every evaluation of p in S produces an element of I. Thus there is a polynomial $q = q(x_{j_1}, \ldots, x_{j_l}) \in \Psi$

which vanishes on $\{p(s_1, \ldots, s_n) | s_1, \ldots, s_n \in S\}$. Then

$$q(p(x_{i_1}, \ldots, x_{i_n}), p(x_{i_n+1}, \ldots, x_{i_n+n}), \ldots, p(x_{i_n+(m-2)n+1}, \ldots, x_{i_n+(m-1)n})),$$

which is in Ψ, vanishes on S and so A is in $\mathscr{R}_\Psi$.

For example, when $\Phi = \{x^n | n = 1, 2, 3, \ldots\}$, $\mathscr{R}_\Phi$ is the class of nil rings and when Ψ is the class of linear monomials, $\mathscr{R}_\Psi$, for associative rings, is the class of locally nilpotent rings. (The set of linear monomials satisfies the first condition mentioned under (ii); the set of all monomials satisfies the second, and defines the nil radical class.)

It is difficult to see exactly how, for $n > 1$, n-radical classes might be generated by families of polynomials *à la* Proposition 4.2. The obvious way is *via* families Λ satisfying the condition

$$p_1, \ldots, p_n, q \in \Lambda \quad \text{implies that} \quad q(p_1, \ldots, p_n) \in \Lambda. \tag{*}$$

Consider the class $\mathscr{X}$ of rings in which every subset of cardinality $\leqslant n$ is annihilated by a polynomial in Λ, and let A be a ring with an ideal I such that $I, A/I \in \mathscr{X}$. If $a_1, \ldots, a_n \in A$ then $p(b_1, \ldots, b_n) \in I$ for each $(b_1, \ldots, b_n) \in \{a_1, \ldots, a_n\}^{(n)}$. It is to be expected that $|\{p(b_1, \ldots, b_n) | (b_1, \ldots, b_n) \in \{a_1, \ldots, a_n\}^{(n)}\}| > n$ in general and so there is no guarantee that the set will satisfy a polynomial identity given by Λ. At the same time, one can see how families of polynomials satisfying (*), for a finite number (>1) of variables, could easily contain the zero polynomial (consider, for example, the family of two-variable polynomials generated by (*) from $x_1 x_2 - x_2 x_1$) and thus correspond to a trivial radical class.

Local radical classes (including n-radical classes with $n > 1$) can be determined, in the manner of Proposition 4.2, by sets of polynomials without any sort of substitution-closure; we shall see some examples shortly. We first revert to consideration of filters, however, to show that for n-radicals a kind of polynomial substitution-closure property is always involved.

PROPOSITION 4.3. *Let $\mathscr{F}$ be an n-radical filter, $J, K \in \mathscr{F}$. Let L denote the ideal of F_n generated by*

$$\{p(q_1, \ldots, q_n) | p = p(x_1, \ldots, x_n) \in J, q_1, \ldots, q_n \in K\}.$$

Then $L \in \mathscr{F}$.

PROOF. For any $q_1, \ldots, q_n \in K$, $p \in J$, we have $p(q_1, \ldots, q_n) \in L$, so that

$$I_n(q_1 + L, \ldots, q_n + L) \supseteq J \in \mathscr{F}, \quad \text{whence } I_n(q_1 + L, \ldots, q_n + L) \in \mathscr{F}.$$

It follows that $L \in \mathscr{F}$.

Now to the other examples. The 1-radical class $\mathscr{B}_1$ of associative rings discussed by Stewart (1970) consists of all rings for which every element satisfies an equation $x^n = x$. The family $\{x^n - x | n = 1, 2, 3, \ldots\}$ is clearly not closed under composition.

Another example: let $\mathscr{W}$ be the class of all Lie rings. Parfenov (1971) has defined a local radical class in $\mathscr{W}$ as follows. Let $p_1, p_2, \ldots$ be the polynomials defined by

$$p_0(x_1) = x_1, p_1(x_1, x_2) = x_1 x_2, \ldots, p_{n+1}(x_1, \ldots, x_{2^{n+1}})$$
$$= p_n(x_1, \ldots, x_{2^n-1}) p_n(x_{2^n}, \ldots, x_{2^{n+1}})$$

and let $\mathscr{S}$, called the class of weakly solvable rings, be the class of rings in which every finite subset satisfies a p_n; $\mathscr{S}$ is then a local radical class.

We now present some examples of n-radicals, with $n > 1$. The first of these is a 2-radical, based on an idea of Morse and Hedlund (1938).

EXAMPLE 4.4. Let $*$ be a binary operation defined on the free $\mathscr{W}$-ring F_2 on two generators x_1, x_2. Let $a_0, b_0, a_1, b_1, \ldots$ be the following polynomials:

$$\begin{aligned} a_0 &= x_1, & b_0 &= x_2, \\ a_1 &= x_1 * x_2, & b_1 &= x_2 * x_1, \\ a_2 &= (x_1 * x_2) * (x_2 * x_1), & b_2 &= (x_2 * x_1) * (x_1 * x_2), \\ a_{n+1} &= a_n(x_1, x_2)\, b_n(x_1, x_2), & b_{n+1} &= b_n(x_1, x_2)\, a_n(x_1, x_2). \end{aligned}$$

A straightforward induction argument shows that

$$a_{n+m} = a_m(a_n, b_n) \quad \text{and} \quad b_{n+m} = b_m(a_n, b_n)$$

for each m, n. Now let

$$\mathscr{M}_* = \{A \in \mathscr{W} \mid u, v \in A \text{ implies there exists } n \text{ with } a_n(u, v) = 0 = b_n(u, v)\}.$$

If a ring $A \in \mathscr{W}$ has an ideal $I \in \mathscr{M}_*$ such that $A/I \in \mathscr{M}_*$, let u, v be in A. Then for some n we have $a_n(u, v), b_n(u, v) \in I$. But then for some m we have

$$0 = a_m(a_n(u, v), b_n(u, v)) = a_{n+m}(u, v)$$

and

$$0 = b_m(a_n(u, v), b_n(u, v)) = b_{n+m}(u, v),$$

whence it follows that $A \in \mathscr{M}_*$. Thus $\mathscr{M}_*$ is closed under extensions. It is straightforward now to complete the proof that $\mathscr{M}_*$ is a 2-radical class.

As a specific instance, take $\mathscr{W}$ to be the class of all associative rings and let $f * g = fg^2 - g^2 f$ for all $f, g \in F_2$. A fairly routine induction argument establishes that none of the polynomials a_n, b_n, $n = 0, 1, 2, \ldots$ is zero and so $F_2 \notin \mathscr{M}_*$. On the other hand, every one-generator ring is in $\mathscr{M}_*$, so $\mathscr{M}_*$ is not a 1-radical class.

It would be of interest to determine what $\mathscr{M}_*$ looks like when $f * g = fg$ (for associative and other rings): for associative rings it is properly larger than the class of locally nilpotent rings and contained in the class of nil rings.

EXAMPLE 4.5. For the class of associative rings, we define, for $n = 1, 2, \ldots$, the polynomials a_n, b_n, c_n, d_n as follows

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}^n = \begin{bmatrix} a_n(x_1, x_2, x_3, x_4) & b_n(x_1, x_2, x_3, x_4) \\ c_n(x_1, x_2, x_3, x_4) & d_n(x_1, x_2, x_3, x_4) \end{bmatrix}.$$

Let

$$\mathscr{P} = \{A \mid t, u, v, w \in A \text{ implies there exists } n \text{ with } a_n(t, u, v, w) = 0,$$
$$b_n(t, u, v, w) = 0, c_n(t, u, v, w) = 0, d_n(t, u, v, w) = 0\}.$$

Although there is no obvious composition closure for the set of a_n's b_n's c_n's, d_n's, $\mathscr{P}$ is in fact a radical class, and thus a 4-radical class, specifically,

$$\mathscr{P} = \left\{ A \;\middle|\; \begin{bmatrix} A & A \\ A & A \end{bmatrix} \text{ is nil} \right\}.$$

It has been shown by Krempa (1972) that $\mathscr{P}$ is the class of nil rings if and only if the answer to Köthe's problem—do nil one-sided ideals always generate nil two-sided ideals?—is affirmative. Clearly also $\mathscr{P}$ is the class of nil rings if and only if $\mathscr{P}$ is a 1-radical class. Thus we conclude that if Köthe's problem has a negative answer then $\mathscr{P}$ is a 4-radical class which is not a 1-radical class.

EXAMPLE 4.6. By Freidman's result quoted earlier in this section $\mathscr{N} \circ \mathscr{C}$ is a local radical class of associative rings; in fact it is a 2-radical class. If $A \in \mathscr{N} \circ \mathscr{C}$, then $A/\mathscr{N}(A)$ is commutative, so its nilpotent elements form an ideal, I. But then $I = 0$ and so $\mathscr{N}(A)$ is the set of all nilpotent elements of A, and all commutators in A are nilpotent. Conversely, any ring satisfying these two conditions is in $\mathscr{N} \circ \mathscr{C}$. Let A be a ring such that $\langle a, b \rangle \in \mathscr{N} \circ \mathscr{C}$ for all $a, b \in A$. If c, d are nilpotent elements of A, then $c, d \in \langle c, d \rangle \in \mathscr{N} \circ \mathscr{C}$ and so $c - d$ is nilpotent. Also, for any $r \in A$, we have $rc, cr \in \langle c, r \rangle \in \mathscr{N} \circ \mathscr{C}$ and so rc and cr are nilpotent. Thus the set of nilpotent elements of A is an ideal. Finally, for each $a, b \in A$, we have $ab - ba \in \langle a, b \rangle \in \mathscr{N} \circ \mathscr{C}$ and so $ab - ba$ is nilpotent. It follows that $A \in \mathscr{N} \circ \mathscr{C}$, and thus that $\mathscr{N} \circ \mathscr{C}$ is a 2-radical class. It is clearly not a 1-radical class.

It would be interesting to know which 1-radical classes (and local radical classes in general) are representable by families of polynomials as in Proposition 4.2 (or as in the examples just considered) and if there is any sort of uniqueness about the representing families. We close this section with some examples of Proposition 4.2-type representations which possibly are a little unexpected.

For each n, let $T_n = T_n(x)$ denote the Chebyshev polynomial of the first kind of degree n. Then $T_m(T_n(x)) = T_{mn}(x)$ for all m, n (see, for example, Rivlin (1974), pp. 1–5). The odd-degree T_n's have zero constant terms so we can consider, *inter*

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alia, the classes

$$\mathscr{Y} = \{A \mid a \in A \text{ implies that } T_n(a) = 0 \text{ for some odd } n\},$$

$$\mathscr{Y}_p = \{A \mid a \in A \text{ implies that } T_{p^n}(a) = 0 \text{ for some } n\}$$

of associative rings where p is an odd prime. It can be shown that $\mathscr{Y}$ is the class of 2-torsion-free, torsion nil rings, $\mathscr{Y}_p$ the class of nil rings with p-primary additive groups. We do not know of any families of polynomials which determine either the class of all torsion nil rings or the class of all nil 2-rings in this way.

5. The lower local radical construction

Every class (in every $\mathscr{W}$) is contained in a smallest local radical class. We now describe that local radical class. It is constructed in a manner which is reminiscent of the construction given by Tangeman and Kreiling (1972) for the lower radical class.

Let $\mathscr{X}$ be a class of rings which, in view of our aim, can be assumed to be strongly hereditary and homomorphically closed. We define a class $\mathscr{X}_\alpha$ for each ordinal α as follows

$$\mathscr{X}_1 = \mathscr{X}; \quad \mathscr{X}_{\alpha+1} = \mathscr{X}_\alpha \circ \mathscr{X}_\alpha = \{A \mid \text{there exists } I \triangleleft A \text{ with } I, A/I \in \mathscr{X}_\alpha\};$$

$$\mathscr{X}_\beta = \{A \mid A \text{ is the union of a directed system of subrings from } \textstyle\bigcup_{\alpha<\beta} \mathscr{X}_\alpha\}$$

if β is a limit ordinal.

THEOREM 5.1. *The smallest local radical class containing* $\mathscr{X}$ *is* $\bigcup \mathscr{X}_\alpha$.

PROOF. We first show that each $\mathscr{X}_\alpha$ is strongly hereditary and homomorphically closed. This is certainly true for $\mathscr{X}_1$. If some $\mathscr{X}_\alpha$ has these properties, consider a ring $A \in \mathscr{X}_{\alpha+1}$; for some $I \triangleleft A$, we have $I, A/I \in \mathscr{X}_\alpha$. If $B \leqslant A$, then $B \cap I \leqslant I$ and so $B \cap I \in \mathscr{X}_\alpha$, while $B/(B \cap I) \cong (B+I)/I \leqslant A/I \in \mathscr{X}_\alpha$ and so $B \in \mathscr{X}_\alpha \circ \mathscr{X}_\alpha = \mathscr{X}_{\alpha+1}$. If $J \triangleleft A$, then $(I+J)/J \cong I/(I \cap J) \in \mathscr{X}_\alpha$, while $A/(I+J) \cong (A/I)/(I+J/I) \in \mathscr{X}_\alpha$. From the exact sequence

$$0 \to (I+J)/J \to A/J \to A/(I+J) \to 0,$$

we see that $A/J \in \mathscr{X}_{\alpha+1}$. Thus $\mathscr{X}_{\alpha+1}$ is strongly hereditary and homomorphically closed.

If $\mathscr{X}_\alpha$ is strongly hereditary and homomorphically closed for each $\alpha <$ a limit ordinal β, consider a ring $A \in \mathscr{X}_\beta$. Let A be the union of a directed set $\{B_\rho \mid \rho \in P\}$ of subrings from $\bigcup_{\alpha<\beta} \mathscr{X}_\alpha$. If C is a subring of A, then

$$C = C \cap A = C \cap \textstyle\bigcup_\rho B_\rho = \bigcup_\rho C \cap B_\rho$$

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and each $C \cap B_\rho$ is in some $\mathscr{X}_\alpha$ and so C is in $\mathscr{X}_\beta$. If $K \lhd A$, then

$$A/K = [\bigcup B_\rho]/K = \bigcup [B_\rho + K/K]$$

and so $A/K \in \mathscr{X}_\beta$. Thus each $\mathscr{X}_\alpha$ is strongly hereditary and homomorphically closed and so $\bigcup_\alpha \mathscr{X}_\alpha$ has these properties also.

If a ring $A \in \mathscr{W}$ has an ideal I such that I and A/I are in $\bigcup \mathscr{X}_\alpha$, there are ordinals γ, δ with $I \in \mathscr{X}_\gamma$, $A/I \in \mathscr{X}_\delta$. But then $A \in \mathscr{X}_{max\{\gamma,\delta\}} \circ \mathscr{X}_{max\{\gamma,\delta\}} = \mathscr{X}_{max\{\gamma,\delta\}+1}$ and so $\bigcup \mathscr{X}_\alpha$ is closed under extensions.

If a ring $A \in \mathscr{W}$ is the union of a directed system $\{B_\rho | \rho \in P\}$ of subrings from $\bigcup \mathscr{X}_\alpha$, let B_ρ be in $\mathscr{X}_{\alpha_\rho}$ for each ρ. Then for any limit β which is greater than every α_ρ, we have $A \in \mathscr{X}_\beta$. Thus $\bigcup \mathscr{X}_\alpha$ is closed under directed unions. Arguing as in the proof of Theorem 2.11, we conclude that $\bigcup \mathscr{X}_\alpha$ is a local radical class.

Let $\mathscr{R}$ be a local radical class containing $\mathscr{X} = \mathscr{X}_1$. If, for some limit ordinal β, $\mathscr{X}_\alpha \subseteq \mathscr{R}$ for all $\alpha < \beta$, let A be in $\mathscr{X}_\beta$. Then $A = \bigcup_{\rho \in P} B_\rho$ for some directed set of subrings B_ρ from earlier $\mathscr{X}_\alpha$'s and thus from $\mathscr{R}$. If S is any finitely generated subring of A then $S \leqslant B_\rho$ for some $\rho \in P$, so that $S \in \mathscr{R}$. Since $\mathscr{R}$ is a local radical class, it follows that $A \in \mathscr{R}$. Hence $\mathscr{X}_\beta \subseteq \mathscr{R}$. The induction at non-limit ordinals being straightforward, we conclude that $\bigcup \mathscr{X}_\alpha \subseteq \mathscr{R}$. This completes the proof.

EXAMPLE 5.2. For associative rings, the smallest local radical class containing all commutative rings is $\mathscr{L} \circ \mathscr{C}$, where $\mathscr{L}$ is the class of locally nilpotent rings and $\mathscr{C}$ that of all commutative rings. As noted at the beginning of Section 4, $\mathscr{L} \circ \mathscr{C}$ is indeed a local radical class. Let A be a nilpotent ring. Then there is a series

$$0 = I_0 \lhd I_1 \lhd \ldots \lhd I_n \lhd A$$

with zerorings as factors. Since zerorings are commutative, it follows that all nilpotent rings belong to $\bigcup_{n<\omega} \mathscr{C}_n$, so that $\mathscr{L} \subseteq \mathscr{C}_\omega$ and then $\mathscr{L} \circ \mathscr{C} \subseteq \mathscr{C}_\omega \circ \mathscr{C}_\omega = \mathscr{C}_{\omega+1}$. Since $\mathscr{C} \subseteq \mathscr{L} \circ \mathscr{C}$, we conclude that $\mathscr{L} \circ \mathscr{C}$ is the smallest local radical class containing $\mathscr{C}$.

EXAMPLE 5.3. Let $\mathscr{Z}$ denote the class of zerorings (associative). Then $\mathscr{L} = \mathscr{Z}_\omega$ is the smallest local radical class containing $\mathscr{Z}$.

It is worth noting that the lower radical class defined by a variety need not be local—consider, for example, the variety of associative zerorings. Also, if $\mathscr{X}$ is homomorphically closed and strongly hereditary, the smallest local radical class containing $\mathscr{X}$ also contains all rings A for which there exists a transfinite series

$$0 = I_0 \lhd I_1 \lhd \ldots \lhd I_\alpha \lhd I_{\alpha+1} \lhd \ldots \lhd A,$$

where $I_{\alpha+1}/I_\alpha \in \mathscr{X}$ for each α and $I_\beta = \bigcup_{\alpha<\beta} I_\alpha$ when β is a limit. However, the radical class need not coincide with the class of rings having such series: Ryabukhin (1968) has shown that the class of rings having series like this when $\mathscr{X}$ is the class of zero-rings (associative) is properly contained in $\mathscr{L}$.

References

T. Anderson (1974), 'On the Levitzki radical', *Canad. Math. Bull.* **17**, 5–10.

P. A. Freidman (1958), 'On the theory of the radical of an associative ring', *Isv. Vyshch. Ucheb. Zaved. Mat.* 3(4), 225–232 (in Russian).

P. Gabriel (1962), 'Des catégories abéliennes', *Bull. Soc. Math. France* **90**, 323–448.

B. J. Gardner (1975), 'Semi-simple radical classes of algebras and attainability of identities', *Pacific J. Math.* **61**, 401–416.

B. J. Gardner (1977), 'Polynomial identities and radicals', *Compositio Math.* **35**, 269–279.

B. J. Gardner and P. N. Stewart (1975), 'On semi-simple radical classes', *Bull. Austral. Math. Soc.* **13**, 349–353.

T. K. Hu (1973), 'Locally equational classes of universal algebras', *Chinese J. Math.* **1**, 143–165.

J. Krempa (1972), 'Logical connections between some open problems concerning nil-rings', *Fund. Math.* **76**, 121–130.

M. Morse and G. A. Hedlund (1938), 'Symbolic dynamics', *Amer. J. Math.* **60**, 815–864.

V. A. Parfenov (1971), 'On a weakly solvable radical of Lie algebras', *Siberian Math. J.* **12**, 123–127.

T. J. Rivlin (1974), *The Chebyshev polynomials* (Wiley–Interscience, New York).

Yu. M. Ryabukhin (1965), 'Semi-strictly hereditary radicals in primitive classes of rings', *Issled. po. Obshchei Algebre, Kishinev*, 112–122 (in Russian).

Yu. M. Ryabukhin (1968), 'On a class of locally nilpotent rings', *Algebra i Logika* **7**, 100–108 (in Russian).

P. N. Stewart (1970), 'Semi-simple radical classes', *Pacific J. Math.* **32**, 249–254.

P. N. Stewart (1972a), 'Strongly hereditary radical classes', *J. London Math. Soc.* (2) **4**, 499–509.

P. N. Stewart (1972b), 'On the locally antisimple radical', *Glasgow Math. J.* **13**, 42–46.

R. Tangeman and D. Kreiling (1972), 'Lower radicals in nonassociative rings', *J. Austral. Math. Soc.* **14**, 419–423.

R. Wiegandt (1974), *Radical and semisimple classes of rings* (Queens' Papers in Pure and Applied Mathematics No. 37, Kingston, Ontario).

Dalhousie University
Halifax, N.S.
Canada

Current address:
University of Tasmania
Hobart
Australia