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REPRESENTATIONS AND DIVISIBILITY OF OPERATOR POLYNOMIALS

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Introduction. Let $\mathscr{X}$ be a complex Banach space and $\mathbf{L}(\mathscr{X})$ the algebra of bounded linear operators on $\mathscr{X}$. In this paper we study functions from the complex numbers to $\mathbf{L}(\mathscr{X})$ of the form

$$(1) \qquad L(\lambda) = I\lambda^l + A_{l-1}\lambda^{l-1} + \ldots + A_1\lambda + A_0$$

where $A_0, A_1, \ldots, A_{l-1}$ and the identity I are members of $\mathbf{L}(\mathscr{X})$. Such a function is referred to as a *monic operator polynomial* and will be abbreviated to a m.o.p. For the case in which $\mathscr{X}$ is of finite dimension, the subject matter of this paper has been investigated by the authors in two earlier papers, [**3**] and [**4**]. Here, we pay special attention to the case in which $\mathscr{X}$ is of infinite dimension and emphasize those new features introduced by this more general hypothesis. Although this paper can be read independently, we shall rely heavily on the two earlier papers for those proofs in which the dimension of $\mathscr{X}$ plays no part. References to theorems and equations in those papers will be distinguished with subscripts I and II, as appropriate.

In Section 1 of this paper several concepts are introduced leading up to the three standard forms for a m.o.p. described in Theorem 1 and concludes with some applications to differential and difference equations. Inverse theorems and spectral properties associated with eigenvalues are also developed. Section 2 consists entirely of new material which arises very naturally on consideration of spaces $\mathscr{X}$ of infinite dimension, although the questions concerning one-sided invertibility are new and significant in the finite-dimensional context. In Section 3 divisors of a m.o.p. are characterized using the concept of supporting subspace and spectral properties of products and quotients are investigated.

Theorem 27_{II} concerning "reducible" systems in the theory of control can be generalized to admit spaces of infinite dimension in an obvious way. This will not be presented explicitly.

1. Representations and applications.

1.1 *Standard pairs and standard triples.* Let $\mathbf{L}(\mathscr{X}, \mathscr{Y})$ denote the linear space of bounded linear operators having Banach space $\mathscr{X}$ as domain and range in Banach space $\mathscr{Y}$. By $\mathscr{X}^r$, r a positive integer, we mean the direct sum of r copies of $\mathscr{X}$.

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Consider an m.o.p. L as described in Equation (1). In the finite demensional case it has been shown that information about all eigenvalues and generalized eigenvectors of L can be concentrated in two matrices one of which is in Jordan normal form. On assuming $\mathscr{X}$ to be infinite dimensional this device is no longer available but it will be shown that the spectral information about L can be summarized in the properties of just two operators. Such a pair is as follows: Operators $X \in \mathbf{L}(\mathscr{X}^l, \mathscr{X})$ and $T \in B(\mathscr{X}^l)$ form a *standard pair* for the m.o.p. L of (1) if the operator $Q(X, T) \in \mathbf{L}(\mathscr{X}^l)$ defined by

$$(2) \qquad Q(X, T) = \begin{bmatrix} X \\ XT \\ \cdot \\ \cdot \\ \cdot \\ XT^{l-1} \end{bmatrix}$$

is invertible and

$$(3) \qquad A_0X + A_1XT + \ldots + A_{l-1}XT^{l-1} + XT^l = 0.$$

For the existence of a standard pair we have simply to take $T = C_1$, the *first companion operator* for L given by

$$(4) \qquad C_1 = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ 0 & 0 & \cdots & 0 & I \\ -A_0 & -A_1 & \cdots & -A_{l-2} & -A_{l-1} \end{bmatrix},$$

and $X = [I \ 0 \ldots 0]$. In this case $Q(X, C_1) = I$ and $XT^l = [-A_0 \ -A_1 \ldots -A_{l-1}]$ from which (3) follows.

For any $X \in \mathbf{L}(\mathscr{X}^l, \mathscr{X})$ and $T \in \mathbf{L}(\mathscr{X}^l)$ it is easily verified that, with $Q(X, T)$ defined by (2),

$$(5) \qquad Q(X, T)T = C_1Q(X, T).$$

Thus, X, T a standard pair implies that T is similar to C_1.

For brevity, we now drop the explicit dependence of Q on X and T from the notation. There is an asymmetry about a standard pair X, T which is removed by going to the notion of a *standard triple*. Let R be the invertible map in $\mathbf{L}(\mathscr{X}^l)$ defined by the *biorthogonality relation*

$$(6) \qquad RBQ = I$$

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where $B \in \mathbf{L}(\mathscr{X}^l)$ is the invertible operator given by

$$(7) \qquad B = \begin{bmatrix} A_1 & A_2 & \cdots & A_{l-1} & I \\ A_2 & A_3 & \cdots & I & 0 \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ A_{l-1} & I & & & \\ I & 0 & & & 0 \end{bmatrix}$$

Then define the *second companion operator* C_2 for L by

$$(8) \qquad C_2 = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & -A_0 \\ I & 0 & \cdot & \cdot & \cdot & 0 & -A_1 \\ 0 & I & & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & & I & 0 & -A_{l-2} \\ 0 & 0 & \cdots & & 0 & I & -A_{l-1} \end{bmatrix}$$

and it can be verified that $C_2 = BC_1B^{-1}$. Then, using (4) and (5),

$$C_2 = BC_1B^{-1} = B(QTQ^{-1})B^{-1} = R^{-1}TR$$

so that $RC_2 = TR$. The structure of C_2 demands the following representation for R (dual to that for Q):

$$(9) \qquad R = [Y\,TY \ldots T^{l-1}Y]$$

where $Y \in B(\mathscr{X}, \mathscr{X}^l)$, together with the relation

$$(10) \quad YA_0 + TYA_1 + \ldots + T^{l-1}YA_{l-1} + T^lY = 0.$$

Now the representation of B^{-1} as a matrix of operators will also have triangular form (ref. Equation (7)). Exploiting this fact, as was done in [**3**], it follows that X, T, Y satisfy

$$(11) \quad XT^{r-1}Y = \begin{cases} 0 & r = 1, 2, \ldots, l-1 \\ I & r = l \end{cases}$$

and we call $X \in B(\mathscr{X}^l, \mathscr{X})$, $T \in B(\mathscr{X}^l)$ and $Y \in B(\mathscr{X}, \mathscr{X}^l)$ a *standard triple* for L.

Give that X, T are a standard pair, the third member Y of a standard triple is uniquely determined by the above construction. Conversely, if T, Y belong to a standard triple then $R = R(T, Y)$ defined by (9) is invertible and X can be determined. We note that the relations (11) can also be written in the equivalent forms:

$$(12)\quad Q(X, T)Y = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ I \end{bmatrix}, \quad XR(T, Y) = [0 \dots 0\ I].$$

Beginning with the standard pair $[I\ 0 \dots 0]$, C_1 we find from (6) that $R(T, Y) = B^{-1}$ which implies that

$$(13)\quad X_1 = [I\ 0 \dots 0], \quad C_1, \quad \text{and} \quad Y_1 = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ I \end{bmatrix}$$

form a standard triple. Similarly, there is a standard triple

$$X_2 = [0 \dots 0\ I], \quad C_2, \quad \text{and} \quad Y_2 = \begin{bmatrix} I \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}.$$

If (X, T, Y), (X_1, T_1, Y_1) are both standard triples for the m.o.p. L, then it is easily verified that there exists an invertible $M \in B(\mathscr{X}^l)$ such that

$$X_1 = XM, \quad T_1 = M^{-1}TM, \quad Y_1 = M^{-1}Y.$$

Indeed, the map M is given by $M = Q(X, T)^{-1}Q(X_1, T_1)$.

1.2 *Standard forms.* Parts (i) and (ii) of the next theorem have their origin in a search for an appropriate generalization of the Jordan normal form from $n \times n$ matrices to $n \times n$ matrix polynomials. Part (iii) began as a matrix polynomial version of the spectral resolution of a resolvent operator. These ideas have been developed in [**3**] and [**4**]. First define the resolvent set for L by

$$\text{Res}\ (L(\lambda)) = \{\lambda \in \mathbf{C} : L^{-1}(\lambda) \text{ exists in } \mathbf{L}(\mathscr{X})\}.$$

THEOREM 1. *Let L be a* m.o.p. *of degree l with standard triple X, T, Y. Then*

(i) $L(\lambda) = I\lambda^l - XT^l(V_1 + V_2\lambda + \dots + V_l\lambda^{l-1})$

where $V_i \in \mathbf{L}(\mathscr{X}, \mathscr{X}^l)$, *for* $i = 1, \dots, l$ *and*

$$[V_1\ V_2 \dots V_l] = Q(X, T)^{-1}$$

(ii) $L(\lambda) = I\lambda^l - (Z_1 + Z_2\lambda + \dots + Z_l\lambda^{l-1})T^lY$

where $Z_i \in \mathbf{L}(\mathscr{X}^l, \mathscr{X})$ *for* $i = 1, 2, \ldots, l$ *and*

$$\begin{bmatrix} Z_1 \\ Z_2 \\ \cdot \\ \cdot \\ \cdot \\ Z_l \end{bmatrix} = R(T, Y)^{-1}$$

(iii) *If* $\lambda \in \operatorname{Res} L$, *then*

$$L^{-1}(\lambda) = X(I\lambda - T)^{-1}Y.$$

Part (i) of the theorem presents a *right standard form* for L and the proof is that of Theorem 1_{I}. Part (ii) gives a *left standard form* and is proved as in Theorem 3_{I}. Part (iii) is called a *resolvent form* and the proof is that of Theorem 13_{II}. A result of this kind for operator polynomials but proved under more restrictive hypotheses was presented earlier by Pattabhiraman and Lancaster [**11**]. Extending the result of Corollary 2 to Theorem 13_{II} we have:

COROLLARY. *For* $\lambda \in \operatorname{Res} L$,

$$\lambda^r L^{-1}(\lambda) = \begin{cases} XT^r(I\lambda - T)^{-1}Y & r = 0, 1, \ldots, l-1. \\ XT^l(I\lambda - T)^{-1}Y + I, & r = l. \end{cases}$$

1.3 *Inverse theorems.* Given operators $X \in L(\mathscr{X}^l, \mathscr{X})$, $T \in \mathbf{L}(\mathscr{X}^l)$ which are candidates for a standard pair we show when, and how, a corresponding m.o.p. may be constructed.

THEOREM 2. *Let* $X \in \mathbf{L}(\mathscr{X}^l, \mathscr{X})$, $T \in \mathbf{L}(\mathscr{X}^l)$ *and assume that* $Q(X, T)$ *defined by* (2) *is invertible. Define* $G_1, \ldots, G_l \in \mathbf{L}(\mathscr{X}, \mathscr{X}^l)$ *by*

$$[G_1\, G_2 \ldots G_l] = Q(X, T)^{-1}.$$

Then X, T, $Y = G_l$ *form a standard triple for*

$$L(\lambda) = I\lambda^l - XT^l(G_1 + G_2\lambda + \ldots + G_l\lambda^{l-1})$$

and L *is the unique* m.o.p. *having* X, T *as a standard pair.*

Proof. Let $G_1, \ldots, G_l$ be as in the statement of the theorem and define

$$A_i = -XT^lG_{i+1} \quad \in \mathbf{L}(\mathscr{X}),\ i = 0, 1, \ldots, l-1.$$

Then let $L(\lambda) = \sum_{i=0}^{l} A_i\lambda^i$ with $A_l = I$. We have

$$\begin{aligned} &A_0X + A_1XT + \ldots + A_{l-1}XT^{l-1} \\ &\quad = -XT^l(G_1X + G_2XT + \ldots + G_lXT^{l-1}). = -XT^l\,GQ = -XT^l, \end{aligned}$$

which is condition (3). This, together with the invertibility of Q, ensures that X, T form a standard pair for L.

But then the definition of G_l implies

$$Q(X, T)G_l = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ I \end{bmatrix}$$

and by comparison with the first of equations (12) we deduce that if $Y = G_l$, then X, T, Y is a standard triple for L. The uniqueness follows immediately.

In a similar way we obtain:

THEOREM 3. *Let $Y \in \mathbf{L}(\mathscr{X}, \mathscr{X}^l)$ and $T \in \mathbf{L}(\mathscr{X}^l)$ and assume that $R(T, Y)$ defined by* (9) *is invertible. Define $Z_1, \ldots, Z_l \in \mathbf{L}(\mathscr{X}^l, \mathscr{X})$ by*

$$\begin{bmatrix} Z_1 \\ \cdot \\ \cdot \\ \cdot \\ Z_l \end{bmatrix} = R(T, Y)^{-1}$$

Then $X = Z_1$, T, Y form a standard triple for

$$L(\lambda) = I\lambda^l - (Z_1 + Z_2\lambda + \ldots + Z_l\lambda^{l-1})T^lY$$

and L is the unique m.o.p. *having T, Y as members of a standard pair.*

In the finite dimensional case there is an inverse theorem involving the resolvent form of the following kind: If there exist operators X, T, Y for which (14) is true then they form a standard triple. This result is not generally true in the infinite dimensional case. The most we can expect is the following.

THEOREM 4. *Let L be a* m.o.p. *and assume there is an $X \in \mathbf{L}(\mathscr{X}^l, \mathscr{X})$, $T \in \mathbf{L}(\mathscr{X}^l)$ and $Y \in \mathbf{L}(\mathscr{X}, \mathscr{X}^l)$ such that*

(14) $\quad L^{-1}(\lambda) = X(I\lambda - T)^{-1}Y$

when $\lambda \in$ Res $(L(\lambda))$. *Then $Q(X, T)$ has a right inverse and $R(T, Y)$ has a left inverse.*

Proof. Following the argument of Theorem 14_{II} it is found that conditions (11) are satisfied and these imply that

$$Q(X, T)R(T, Y) = \begin{bmatrix} X \\ XT \\ \cdot \\ \cdot \\ \cdot \\ XT^{l-1} \end{bmatrix} [Y\ TY \ldots TY^{l-1}] = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & 0 & I \\ \cdot & & & & I & \theta \\ \cdot & & & \cdot^{\cdot^{\cdot}} & & \cdot \\ \cdot & & & & & \cdot \\ 0 & I & & & & \cdot \\ I & \theta & \cdot & \cdot & \cdot & \theta \end{bmatrix},$$

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a triangular matrix of operators in which Θ denotes operators of no immediate concern. Since the operator on the right is invertible the conclusion follows.

To see that (14) cannot generally imply the invertibility of $Q(X, T)$ consider $L(\lambda) = I\lambda - I$ and let X, Y be any operators on $\mathscr{X}$ for which $XY = I$, $YX \neq I$. Then $L^{-1}(\lambda) = X(I\lambda - I)^{-1}Y$.

The implications of the one-sided invertibility of Q will be explored in some detail in Section 2.

1.4 *Eigenpairs.* In this section we develop an earlier remark on the significance of a standard pair X, T for spectral theory. In the case of a matrix polynomial acting on $\mathbf{C}^n$, T can take a Jordan normal form and the columns of matrix X display the Jordan chains of L. An appropriate generalization for the operator case will now be introduced. First a formal definition.

Let X, T be a standard pair for m.o.p. L and let $\mathscr{L} \subset \mathscr{X}^l$ be an invariant subspace of T. Then the pair of restrictions $(X|_{\mathscr{L}}, T|_{\mathscr{L}})$ is an *eigenpair* for L. Note that, in the sequel, when $(X|_{\mathscr{L}}, T|_{\mathscr{L}})$ is asserted to be an eigenpair for L it is assumed implicitly that X, T is a standard pair and that $\mathscr{L}$ is an invariant subspace of T.

Two eigenpairs $(X|_{\mathscr{L}}, T|_{\mathscr{L}})$ and $(X_1|_{\mathscr{L}_1}, T_1|_{\mathscr{L}_1})$ for the m.o.p. L are *equivalent* if there is an invertible $D \in \mathbf{L}(\mathscr{L}_1, \mathscr{L})$ which is onto $\mathscr{L}$ and for which

$$\mathscr{L}_1 = D^{-1}\mathscr{L}, \quad X_1 = XD, \quad T_1 = D^{-1}TD.$$

We illustrate this idea with some examples.

Example 1. Consider the standard pair $T = C_1$ and $X = [I\ 0 \ldots 0]$ of Equation (13). If $\mathscr{L}$ is an invariant subspace of C_1 then $X|_{\mathscr{L}}$, $C_1|_{\mathscr{L}}$ form a standard pair and, in this case $X|_{\mathscr{L}}$ is the projection on $\mathscr{L}$ mapping $x \in \mathscr{L} \subset \mathscr{X}^l$ onto its first component. In particular, if $\mathscr{L}$ has dimension one, then $x \in \mathscr{L}$ if and only if $x = (x_1, \lambda x_1, \ldots, \lambda^{l-1}x_1)$ and $L(\lambda)x_1 = 0$ for some $\lambda \in \mathbf{C}$ and $x_1 \neq 0$ in $\mathscr{X}$. Thus, $X|_{\mathscr{L}}$ projects $x \in \mathscr{L}$ onto an eigenvector x_1 of L.

Example 2. Suppose now that X, T is an arbitrary standard pair and that $\mathscr{L}$ is a one-dimensional invariant subspace of T. If λ_0 is the complex number for which $Tx = \lambda_0 x$, $x \in \mathscr{L}$ and $x \neq 0$, then Xx is an eigenvector of L corresponding to the eigenvalue λ_0.

To see this we use part (i) of Theorem 1 and write

$$(15)\quad L(\lambda) = I\lambda^l - XT^l \sum_{i=1}^{l} V_i\lambda^{i-1}$$

where $[V_1\ V_2 \ldots V_l] = Q(X, T)^{-1}$. Then, if $x \in \mathscr{L}$, $x \neq 0$

$$(16)\quad x = \left(\sum_{i=1}^{l} V_i X T^{i-1}\right) x = \sum_{i=1}^{l} \lambda_0^{\,i-1} V_i X x$$

and we deduce $Xx \neq 0$. Then use (15) to write

$$L(\lambda_0)Xx = \left[I\lambda_0{}^l - XT^l \sum_{i=1}^{l} \lambda_0{}^{i-1}V_i\right]Xx$$
$$= X(\lambda_0{}^l x) - XT^l \sum_{i=1}^{l} V_i XT^{i-1}x$$
$$= X(\lambda_0{}^l x) - X(T^l x)$$

using (16). Since $T^l x = \lambda_0{}^l x$ it follows that Xx is the required eigenvector of L.

Example 3. More generally, let X, T be a standard pair, and $\mathscr{L}$ a T-invariant subspace with basis $e_1, e_2, \ldots$. Assume further that there are complex numbers $\lambda_1, \lambda_2, \ldots$ for which $Te_i = \lambda_i e_i$, $i = 1, 2, \ldots$. Then vectors $Xe_1, Xe_2, \ldots$ are eigenvectors of L with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots$.

We omit the proof but note that the hypotheses of this theorem correspond to the study of subspaces $\mathscr{L}$ for which the eigenvalues λ_i (not necessarily distinct) of $T|_{\mathscr{L}}$ (and hence of L) have only linear associated elementary divisors.

Example 4. The prototype result for nonlinear divisors is presented in the next result. First we need the definition (now widely used in the literature) of a Jordan chain (chain of generalized eigenvectors) for L. The vectors $x_1, x_2, \ldots, x_k \in \mathscr{X}$ form a *Jordan chain of length k* corresponding to eigenvalue λ_0 of L if $x_1 \neq 0$ and, for $\rho = 0, 1, \ldots, k - 1$,

$$\sum_{i=0}^{\rho} \frac{1}{i!} L^{(i)}(\lambda_0)x_{\rho-i+1} = 0$$

THEOREM 5. *Let $(X|_{\mathscr{L}}, T|_{\mathscr{L}})$ be an eigenpair for* m.o.p. L *and let $\mathscr{L}$ be a finite dimensional subspace of $\mathscr{X}^l$. Let $e_1, e_2, \ldots, e_k$ be a basis for $\mathscr{L}$ and assume that the representation of $T|_{\mathscr{L}}$ in this basis is a Jordan normal form. If $e_i, e_{i+1}, \ldots, e_{i+r}$ is a basis for a Jordan cell of $T|_{\mathscr{L}}$ corresponding to eigenvalue λ_0 then $\{Xe_i, Xe_{i+1}, \ldots, Xe_{i+r}\}$ is a Jordan chain for L corresponding to eigenvalue* λ_0.

Proof. Without loss of generality, assume that $T|_{\mathscr{L}}$ is unicellular, i.e. $e_1, \ldots, e_k$ is a basis of the unique Jordan-cell of $T|_{\mathscr{L}}$ corresponding to eigenvalue λ_0. Defining $e_i = 0$ if $i < 1$ we first prove

$$(17) \quad T^m e_\rho = \sum_{i=0}^{m} \binom{m}{i} \lambda_0{}^{m-i} e_{\rho-i}, \quad \begin{cases} m = 0, 1, \ldots, \\ \rho = 1, 2, \ldots, k. \end{cases}$$

Proceeding by induction on m note the result is trivially true when $m = 0$. Then

$$T^{m+1}e_\rho = \sum_{i=0}^{m} \binom{m}{i} \lambda_0{}^{m-i} Te_{\rho-i} = \sum_{i=0}^{m} \binom{m}{i} \lambda_0{}^{m-i}(\lambda_0 e_{\rho-i} + e_{\rho-i-1})$$
$$= \binom{m}{0} \lambda_0{}^{m+1} e_\rho + \sum_{i=1}^{m} \left[\binom{m}{i} \lambda_0{}^{m+1-i} + \binom{m}{i-1} \lambda_0{}^{m+1-i}\right] e_{\rho-i}$$
$$+ \binom{m}{m} e_{\rho-m-1} = \sum_{i=0}^{m+1} \binom{m+1}{i} \lambda_0{}^{m+1-i} e_{\rho-i},$$

as required.

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The definition of the V_i implies

$$\sum_{i=0}^{l-1} V_{i+1} X T^i = I$$

so that, using (17) we may write

$$(18) \quad e_{\rho+1} = \sum_{i=0}^{l-1} V_{i+1} X (T^i e_{\rho+1}) = \sum_{i=0}^{l-1} V_{i+1} X \sum_{j=0}^{i} \binom{i}{j} \lambda_0{}^{i-j} e_{\rho+1-j}.$$

We can now proceed to verify the theorem:

$$\sum_{i=0}^{\rho} \frac{1}{i!} L^{(i)}(\lambda_0) X e_{\rho-i+1}$$
$$= \sum_{i=0}^{l} \frac{1}{i!} \left[\frac{i!}{(l-i)!} \lambda_0{}^{l-i} I - X T^l \sum_{j=1}^{l-1} \frac{j!}{(j-i)!} \lambda_0{}^{j-i} V_j \right] X e_{\rho-i+1}$$
$$= \sum_{i=0}^{l} \binom{l}{i} \lambda_0{}^{l-i} X e_{\rho-i+1} - X T^l \sum_{i=0}^{l} \sum_{j=i}^{l-1} \binom{j}{i} \lambda_0{}^{j-i} V_j X e_{\rho-i+1}.$$

Now use (18) followed by (17) to obtain

$$\sum_{i=0}^{\rho} \frac{1}{i!} L^{(i)}(\lambda_0) X e_{\rho-i+1} = \sum_{i=0}^{l} \binom{l}{i} \lambda_0{}^{l-i} X e_{\rho-i+1} - X T^l e_{\rho+1}$$
$$= \sum_{i=0}^{l} \binom{l}{i} \lambda_0{}^{l-i} X e_{\rho-i+1} - X \sum_{i=0}^{l} \binom{l}{i} \lambda_0{}^{l-i} e_{\rho-i+1} = 0.$$

1.5 *Applications.* In this section we consider briefly some fundamental applications to constant coefficient differential and difference equations on Banach spaces. The proofs are generally omitted and are natural generalizations of those in [**3**]. First consider a homogeneous constant coefficient differential equation of order l on $\mathscr{X}$. Thus, if $A_0, A_1, \ldots, A_{l-1}$ are the coefficients of an m.o.p. as in (1), we consider the equation

$$(19) \quad u^{(l)}(t) + A_{l-1} u^{(l-1)}(t) + \ldots + A_1 u^{(1)}(t) + A_0 u(t) = 0$$

where t is the real independent variable.

PROPOSITION. *Let X, T be a standard pair for the* m.o.p. *of Equation* (1). *Then every solution of* (19) *has the form*

$$(20) \quad u(t) = X e^{T(t-t_0)} c$$

for some real t_0 and some $c \in \mathscr{X}^l$.

Proof. With C_1 the first companion operator (4) we can write (19) in the form $v^{(1)}(t) = C_1 v(t)$ where

$$v(t) = \begin{bmatrix} x(t) \\ x^{(1)}(t) \\ \cdot \\ \cdot \\ \cdot \\ x^{(l-1)}(t) \end{bmatrix}.$$

It is well-known that every solution of this equation has the form $v(t) = e^{C_1(t-t_0)}c_1$ for some real t_0 and $c_1 \in \mathscr{X}^l$. Examining the image of $v(t)$ under the map $X_1 = [I\ 0 \ldots 0]$, we obtain $x(t) = X_1 e^{C_1(t-t_0)}c_1$. But we have seen that X_1, C_1 is a standard pair for L (Equation (13)) and the conclusion follows from the remark at the end of Section 1.1.

Now let $\mathscr{U}$ be the linear space of piecewise continuous functions (for example) on $[t_0, \infty)$. For the inhomogeneous problem we have:

PROPOSITION. *Let X, T, Y be a standard triple for* m.o.p. L *of Equation* (1) *and let $f \in \mathscr{U}$. Then every solution of*

(21) $u^{(l)}(t) + A_{l-1}u^{(l-1)}(t) + \ldots + A_0u(t) = f(t)$

has the form

(22) $u(t) = Xe^{T(t-t_0)}c + X\int_{t_0}^{t} e^{T(t-\tau)}Yf(\tau)d\tau$

for some real t_0 and some $c \in \mathscr{X}^l$.

The proof of this theorem is by verification and uses part (iii) of Theorem 1 along with the biorthogonality conditions (11). Using this representation of the general solution it is possible to obtain an explicit formula for the solution of the initial value problem with $u^{(r)}(t_0) = u_r$, $r = 0, 1, \ldots, l-1$. Thus, differentiate (22) $l-1$ times and put $t = t_0$ to obtain

$$\begin{bmatrix} X \\ XT \\ \cdot \\ \cdot \\ \cdot \\ XT^{l-1} \end{bmatrix} c = \begin{bmatrix} u_0 \\ u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_{l-1} \end{bmatrix}$$

We recognize the operator on the left as $Q(X, T)$ whose inverse can be written down using (7) and (9). Thus

$$c = R(T, Y)B \begin{bmatrix} u_0 \\ u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_{l-1} \end{bmatrix}$$

An attack can be made on *two-point* boundary value problems by assuming that there is a decomposition $\mathscr{X}^l = \mathscr{Y} \oplus \mathscr{Z}$ where both $\mathscr{Y}$ and $\mathscr{Z}$ are invariant subspaces of T and, as before, (X, T, Y) is a standard triple for L. If P is the projection on $\mathscr{Y}$ along $\mathscr{Z}$, define

$$X_1 = XP, \quad X_2 = X(I-P), \quad Y_1 = PY, \quad Y_2 = (I-P)Y,$$
$$T_1 = PTP, \quad T_2 = (I-P)T(I-P).$$

Then define the operator valued function $G(t, \tau)$ on $[a, b] \times [a, b]$ by

$$G(t, \tau) = \begin{cases} -X_1 e^{T_1(t-\tau)} Y_1 & \text{if } a \leqq t \leqq \tau \\ X_2 e^{T_2(t-\tau)} Y_2 & \text{if } \tau \leqq t \leqq b, \end{cases}$$

and it can be verified using Theorem 1, part (iii) and equations (11) that G has the properties required of a Green's function:

(a) As a function of t, G satisfies (19) on $[a, b] \times [a, b)$ provided $t \neq \tau$.
(b) Differentiating with respect to t,

$$G^{(r)}|_{t=\tau_+} - G^{(r)}|_{t=\tau_-} = \begin{cases} 0 & \text{if } r = 0, 1, \ldots, l-2 \\ I & \text{if } r = l-1. \end{cases}$$

(c) The function

$$u(t) = \int_a^b G(t, \tau) f(\tau) d\tau$$

is a solution of (21).

Finally, we comment on the *difference equation*

(23) $\quad A_0 u_r + A_1 u_{r+1} + \ldots + A_{l-1} u_{r-1} + u_r = f_r, \quad r = 1, 2, \ldots,$

where $\{f_r\}_{r=1}^{\infty} \subset \mathscr{X}$ is given and solution sequences $\{u_r\}_{r=1}^{\infty} \subset \mathscr{X}$ are sought. Again, the operator coefficients $A_0, A_1, \ldots, A_{l-1}$ define the m.o.p. L of (1).

PROPOSITION. *Let X, T, Y be a standard triple for L. Then every solution of* (23) *has the form* $u_1 = Xc$,

(24) $\quad u_r = XT^{r-1}c + \sum_{k=1}^{r-1} XT^{r-k-1} f_k, \quad r = 2, 3, \ldots,$

for some $c \in \mathscr{X}^l$,

As in the case of differential equations, consider the solution determined by initial conditions. In this case, $u_r = v_r$, say, for $r = 1, 2, \ldots, l$. Then the vector c of (24) must have the form:

$$c = R(T, Y)B \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_l \end{bmatrix}$$

where $R(T, Y)$ and B are defined by (9) and (7).

2. Generalizations of the invertibility of Q.

2.1 *One-sided invertibility of the operator* Q. We abbreviate $Q(X, T)$ and $R(T, Y)$ to Q and R where this causes no confusion. In section 1.3 it has been shown that the invertibility of Q, or of R, is sufficient for the solution of the inverse problem. If either of these operators has only a one-sided inverse we are still able to draw some interesting conclusions. Suppose now that both $\mathscr{X}$ and $\mathscr{Y}$ are Banach spaces.

THEOREM 6. *Let* $X \in \mathbf{L}(\mathscr{Y}, \mathscr{X})$ *and* $T \in \mathbf{L}(\mathscr{Y})$ *be such that the operator* $Q(X, T) \in \mathbf{L}(\mathscr{Y}, \mathscr{X}^l)$ *has a left inverse* Q_L^{-1}. *Define operators* $A_i \in \mathbf{L}(\mathscr{X})$ *by* $A_i = -XT^l G_{i+1}$, $i = 0, 1, \ldots, l-1$ *where* $[G_1 \ldots G_l] = Q_L^{-1} \in \mathbf{L}(\mathscr{X}^l, \mathscr{Y})$. *Then*

(25) $A_0X + A_1XT + \ldots + A_{l-1}XT^{l-1} + XT^l = 0.$

Let L be the m.o.p. *on* $\mathscr{X}$ *defined by* $L(\lambda) = \sum_{i=0}^{l} A_i\lambda^i$, $A_l = I$, *and let* C_1 *be the first companion operator of* L. *Then*

(26) $T = Q_L^{-1}|_{\mathscr{M}}\, C_1|_{\mathscr{M}}Q$

where $\mathscr{M} = \operatorname{Im} Q$.

Proof. Using the definition of the A_i we have

$$A_0X + \ldots + A_{l-1}XT^{l-1} = [A_0\, A_1 \ldots A_{l-1}]Q = -XT^l[G_1 \ldots G_l]Q = -XT^l,$$

which gives (25). With L and C_1 as defined it is easily verified that, as a consequence of (25), $QT = C_1Q$. It is apparent from this relation that $\mathscr{M}$ is invariant under C_1. Using the decomposition $\mathscr{X}^l = (\operatorname{Ker} Q_L^{-1}) \oplus \mathscr{M}$, the representation (26) follows from this fact.

Note also that the left inverse is unique if and only if the A_i are uniquely defined, and in this case Q must be invertible.

This theorem admits the *extension* of operator pairs X, T ($T \in \mathbf{L}(\mathscr{Y})$) to standard pairs; the prototype standard pair being $[I\,0 \ldots 0]$, C_1 ($C_1 \in \mathbf{L}(\mathscr{X}^l)$). The need for such extensions has arisen in the work of Gohberg, Kaashoek and Rodman [**2**], for example. A special case of the theorem gives important new information in the finite dimensional case when X, T are, say, $n \times p$ and $p \times p$ complex matrices with T in Jordan normal form. Suppose that we are to construct a monic matrix polynomial L of order l for which the columns of X form corresponding Jordan chains. Suppose, however, that X does not have enough columns in the sense $p < ln$. The theorem then says that as long as Q has full rank, in which case a left inverse exists, then the above construction can be used to determine an L which will have a standard pair which extend X and T appropriately.

Next we have a dual theorem:

THEOREM 7. *Let* $X \in \mathbf{L}(\mathscr{Y}, \mathscr{X})$ *and* $T \in \mathbf{L}(\mathscr{Y})$ *be such that the operator* $Q(X, T) \in \mathbf{L}(\mathscr{Y}, \mathscr{X}^l)$ *has a right inverse* Q_R^{-1}. *Define operators* $Z_i \in \mathbf{L}(\mathscr{X})$ *by* $A_i = -XT^lV_{i+1}$, $i = 0, 1, \ldots, l-1$ *where* $[V_1 \ldots V_l] = Q_R^{-1} \in \mathbf{L}(\mathscr{X}^l, \mathscr{Y})$,

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and hence the monic operator polynomial $L(\lambda) = \sum_{i=0}^{l} A_i\lambda^i$, $A_l = I$. *Then if* $\mathscr{M} = \operatorname{Im} Q_R^{-1}$ *and* P *is the projection in* $\mathbf{L}(\mathscr{Y})$ *on* $\mathscr{M}$ *along* $\operatorname{Ker} Q$, *we have*

(27) $\quad C_1 = Q(PT)|_{\mathscr{M}} Q_R^{-1}.$

Proof. Using the definition of the A_i and the fact that $QQ_R^{-1} = I$ it can be verified that

$$QTQ_R^{-1} = \begin{bmatrix} XT \\ \cdot \\ \cdot \\ \cdot \\ XT^l \end{bmatrix} [V_1 \dots V_l] = C_1.$$

In view of the decomposition $\mathscr{Y} = (\operatorname{Ker} Q) \oplus \mathscr{M}$ the relation (27) follows immediately.

The interpretation of this result is that existence of a *right* inverse for Q admits the construction of a standard pair by *restriction* of the given pair X, T. Heuristically, if X, T contain redundancies or inadmissible duplications then, provided Q_R^{-1} exists, these are removed in the construction of a standard pair via equation (27).

The existence of Q_R^{-1} *and* a set of operators $A_0, \dots, A_{l-1}$ satisfying (25) does not imply that Q is invertible although, in this case, the operators $A_0, \dots, A_{l-1}$ are necessarily unique. For example, take $l = 1$ when $Q = X \in \mathbf{L}(\mathscr{X})$ and assume $XV = I$, $VX \neq I$. Then $\mathscr{X} = (\operatorname{Ker} X) \oplus (\operatorname{Im} V)$. Select any operators $T_1 \in \mathbf{L}(\operatorname{Ker} \mathscr{X})$ and $T_2 \in \mathbf{L}(\operatorname{Im} V)$ and define $T \in \mathbf{L}(\mathscr{X})$ by

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$$

In this decomposition of $\mathscr{X}$, X and V have representations

(28) $\quad X = [0\,X_2], \quad V = \begin{bmatrix} 0 \\ V_2 \end{bmatrix}$

where $X_2 \in \mathbf{L}(\operatorname{Im} V, \mathscr{X})$, $V_2 \in \mathbf{L}(\mathscr{X}, \operatorname{Im} V)$ and $V_2X_2 = I|_{\operatorname{Im} V}$. Thus $VX \neq I$. Furthermore, since $XT = XT\,VX$ the equation $A_0X + XT = 0$ has a solution $A_0 = XTV$.

For the uniqueness, suppose that $\hat{A}_0, \dots, \hat{A}_{l-1}$ also satisfy (25). Then

$$(A_0 - \hat{A}_0)X + \dots + (A_{l-1} - \hat{A}_{l-1})XT^{l-1} = 0$$

and the existence of a right inverse for Q implies $\hat{A}_i = A_i$ for $i = 0, 1, \dots, l - 1$.

To see that the existence of Q_R^{-1} does not imply the existence of operators A_i for which (25) is true, consider again the case $l = 1$ with $X \in \mathbf{L}(\mathscr{X})$, $XV = I$, $VX \neq I$.

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Let $T \in \mathbf{L}(\mathscr{X})$ have the representation

$$T = \begin{bmatrix} T_1 & T_4 \\ T_3 & T_2 \end{bmatrix}$$

with respect to the decomposition $(\text{Ker } X) \oplus (\text{Im } V)$ of $\mathscr{X}$ and assume $T_3 \neq 0$. Recall the representations (28) for X and V. Now the relation $A_0 X + XT = 0$ has the form

$$[0 \;\; A_0 X_2] + [X_2 T_3 \;\; X_2 T_2] = 0$$

and $T_3 \neq 0$, Z_2 invertible mean that there is no solution A_0.

We conclude this section with a result characterizing the case in which Q is right invertible and operators $A_0, \ldots, A_{l-1} \in B(\mathscr{X})$ exist for which (25) is true. First, we need the following concept introduced by Gohberg, Kaashoek and Rodman [**2**]. Given $X \in \mathbf{L}(\mathscr{X}^l, \mathscr{X})$ and $T \in \mathbf{L}(\mathscr{X}^l)$ let $Q_r(X, T)$, (more briefly Q_r) denote the operator in $\mathbf{L}(\mathscr{X}^l, \mathscr{X}^r)$ defined by

$$(29) \quad Q_r(X_1 T) = \begin{bmatrix} X \\ XT \\ \cdot \\ \cdot \\ \cdot \\ XT^{r-1} \end{bmatrix}$$

and note that, in the notation of the earlier remarks, $Q_l(X, T) = Q_l = Q$. Define the *index of stabilization* of the pair (X, T) to be the least positive integer p for which $\text{Ker } Q_p = \text{Ker } Q_r$ for all $r \geqq p$. Write $p = \text{ind } (X, T)$.

THEOREM 8. *Let $X \in \mathbf{L}(\mathscr{X}^l, \mathscr{X})$, $T \in \mathbf{L}(\mathscr{X}^l)$ be such that $Q_l(X, T)$ is right invertible. Then there exist $A_0, A_1, \ldots, A_{l-1} \in \mathbf{L}(\mathscr{X})$ for which* (25) *is satisfied if and only if* $\text{ind } (X, T) \leqq l$.

Proof. (a) We show first that $\text{ind } (X, T) \leqq l$ is equivalent to

$$(30) \quad T(\text{Ker } Q_l) \subset \text{Ker } Q_l.$$

Given this inclusion, if $x \in \text{Ker } Q_l$, then $T^r x \in \text{Ker } Q_l$ for $r = 0, 1, 2, \ldots$. It follows that $x \in \text{Ker } (Q_l T^r)$ and $x \in \text{Ker } Q_r$ for $r = 1, 2, \ldots$. Thus, (30) implies $\text{Ker } Q_l \subset \text{Ker } Q_r$ for $r \geqq l$ and hence $\text{ind } (X, T) \leqq l$.

Conversely, suppose $\text{ind } (X, T) \leqq l$. Then by definition of the index, $r \geqq l$ implies $\text{Ker } Q_l = \text{Ker } Q_r$. But it is easily seen that $\text{Ker } Q_{l+1} \subset \text{Ker } (Q_l T)$. Thus, $x \in \text{Ker } Q_l = \text{Ker } Q_{l+1}$ implies $x \in \text{Ker } (Q_l T)$ and this is equivalent to $Tx \in \text{Ker } Q_l$.

(b) As a second preliminary we need the decomposition

$$(31) \quad \mathscr{X}^l = \text{Ker } Q \oplus (\text{Ker } X \cap \text{Im } Q_R^{-1}) \oplus \text{Im } V_1.$$

To see this note first that $XV_1 = I$ implies $\mathscr{X}^l = \text{Ker } X \oplus \text{Im } V_1$ so we are

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done when we show that

$$\text{Ker } Q \oplus (\text{Ker } X \cap \text{Im } Q_R^{-1}) = \text{Ker } X.$$

Now $QQ_R^{-1} = I$ yields $\mathscr{X}^l = \text{Ker } Q \oplus \text{Im } Q_R^{-1}$ so the sum on the left is certainly direct. Then $\text{Ker } Q \subset \text{Ker } X$ means the subspace on the left is contained in that on the right. For the reverse inclusion let $x \in \text{Ker } X$ and write $x = x_1 + x_2$ where $x_1 \in \text{Ker } Q$, $x_2 \in \text{Im } Q_R^{-1}$. Then $x_2 = x - x_1 \in \text{Ker } X$ so that $x \in \text{Ker } Q \oplus (\text{Ker } X \cap \text{Im } Q_R^{-1})$ as required.

(c) Now to go directly to the proof of the theorem, observe that $A_0, A_1, \ldots, A_{l-1} \in \mathbf{L}(\mathscr{X})$ satisfy (25) if and only if they have the form $A_i = -XT^lV_{i+1}$, $i = 0, 1, \ldots, l-1$ (as in Theorem 7). This means that (25) may be written in the form

$$-(XT^l)VQ_l + XT^l = XT^l(I - VQ_l) = 0.$$

Let $P = I - VQ_l$, the projector on $\text{Ker } Q_l$ along $\text{Im } V = \text{Im } Q_R^{-1}$, so that the existence of $A_0, A_1, \ldots, A_{l-1} \in \mathbf{L}(\mathscr{X})$ is equivalent to the statement $XT^lP = 0$.

Consider the representations of operators X, T^i, P with respect to the decomposition (31):

$$X = [0\ 0\ X_2], \quad T^i = \begin{bmatrix} S_{1i} & S_{2i} & S_{3i} \\ S_{4i} & S_{5i} & S_{6i} \\ S_{7i} & S_{8i} & S_{9i} \end{bmatrix}, \quad P = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and note that $X_2 \in \mathbf{L}(\text{Im } V_1, \mathscr{X})$ is invertible and $XT^i = [S_{7i} S_{8i} S_{9i}]$. The condition $XT^i|_{\text{Ker} Q} = 0$, $i = 0, 1, \ldots, l-1$ then implies that $S_{7i} = 0$ for $i = 1, 2, \ldots, l-1$ and the condition $XT^lP = 0$ may be written $S_{7l} = 0$. Thus,

$$\text{Ker } Q_{l+1} = \text{Ker} \begin{bmatrix} X \\ XT \\ \cdot \\ \cdot \\ \cdot \\ XT^l \end{bmatrix} \supset \text{Ker } Q_l.$$

But we have seen in part (a) that this is equivalent to $T(\text{Ker } Q_l) \subset \text{Ker } Q_l$ and this, in turn, is equivalent to $\text{ind } (X, T) \leqq l$.

A sequence of parallel remarks can be made, and conclusions drawn, concerning operator pairs $T \in \mathbf{L}(\mathscr{X}^l)$, $Y \in \mathbf{L}(\mathscr{X}, \mathscr{X}^l)$ via the study of operators

$$R_r(T, Y) = [Y\ TY \ldots T^{r-1}Y]$$

from $\mathscr{X}^r$ to $\mathscr{X}^l$. For example, the dual of Theorem 6 is:

THEOREM 10. *Let $T \in \mathbf{L}(\mathscr{Y})$ and $Y \in \mathbf{L}(\mathscr{X}, \mathscr{Y})$ be such that the operator $R_l(T, Y) \in \mathbf{L}(\mathscr{X}^l, \mathscr{Y})$ has a right inverse R_R^{-1}. Define operators $A_i \in \mathbf{L}(\mathscr{X})$*

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by $A_i = -W_{i+1}T^l Y$, $i = 0, 1, \ldots, l-1$ *where*

$$\begin{bmatrix} W_1 \\ \cdot \\ \cdot \\ \cdot \\ W_l \end{bmatrix} = R_R^{-1},$$

and hence the m.o.p. $L(\lambda) = \sum_{i=0}^{l} A_i\lambda^i$, $A_l = I$. *Then*

$$YA_0 + TYA_1 + \ldots + T^{l-1}YA_{l-1} + T^l Y = 0$$

and, if C_2 *is the second companion operator of* L, *then*

$$T = R(PC_2|_{\mathcal{M}})R_R^{-1}$$

where $\mathcal{M} = \operatorname{Im} R_R^{-1}$ *and* $P \in \mathbf{L}(\mathcal{X}^l)$ *is the projection on* $\mathcal{M}$ *along* Ker R.

Once more, a result is obtained admitting the extension of the operator pair to a standard pair.

2.2 *The case of* $Q(X, T)^{-1}$ *unbounded.* As in the preceding section we consider pairs of bounded operators $X \in \mathbf{L}(\mathcal{X}^l, \mathcal{X})$ and $T \in \mathbf{L}(\mathcal{X}^l)$ and the derived operator $Q(X, T) \in \mathbf{L}(\mathcal{X}^l)$. Suppose Ker $Q = 0$ and $\mathcal{M} = \operatorname{Im} Q$ is a proper, dense, subspace of $\mathcal{X}^l$. Then Q has an unbounded inverse Q^{-1} defined on $\mathcal{M}$. Assume that it is possible to write Q^{-1} in the form $[V_1 \ldots V_l]$ where each $V_i : \mathcal{X} \to \mathcal{X}^l$ may be unbounded. For example, this possibility exists if we can write

$$\mathcal{M} = \sum_{i=1}^{l} \oplus \mathcal{M} \cap \mathcal{X}_i$$

and $\mathcal{X}_i$ is the ith component space of $\mathcal{X}^l$.

Then we can define an associated m.o.p. L by

$$L(\lambda) = I\lambda^l - XT^l(V_1 + V_2\lambda + \ldots + V_l\lambda^{l-1})$$

where, in general, the coefficients may be unbounded. However, it is possible for them to be bounded as the following example shows.

Let $l = 2$ and $B \in \mathbf{L}(\mathcal{X})$ with Im B a proper, dense subspace of $\mathcal{X}$ and Ker $B = 0$. Consider the pair X, T defined by

$$X = [I \quad I], \quad T = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}$$

Then

$$Q(X, T) = \begin{bmatrix} X \\ XT \end{bmatrix} = \begin{bmatrix} I & I \\ 0 & B \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} I & -B^{-1} \\ 0 & B^{-1} \end{bmatrix}$$

and

$$L(\lambda) = I\lambda^2 - [0 \quad B^2]\left[\begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} -B^{-1} \\ B^{-1} \end{bmatrix}\right] = I\lambda^2 - B\lambda,$$

as required.

When L has bounded coefficients then, of course, the associated first companion operator C_1 is bounded and we have $QT = C_1Q$. Thus C_1 and T may be said to be similar even though the transforming operator Q has an unbounded inverse.

In general, even though T and C_1 may both be bounded, one cannot assert that they are related by a "bounded" similarity transformation. To see this, consider the above example again and let B be compact. We have

$$T = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & I \\ 0 & B \end{bmatrix}$$

and assume there is an $S \in B(\mathscr{X}^2)$ with bounded inverse for which $ST = C_1S$. Write

$$S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}$$

and it is found that $ST = C_1S$ implies $S_3 = 0$, $S_2B = S_4$ and S_4 has a bounded inverse. Thus, $S_2BS_4^{-1} = I$ which is not consistent with the assumed compactness of B.

In this example it is easily seen that T is *not* a linearization of the m.o.p. $L(\lambda) = I\lambda^2 - B\lambda$ we have associated with (X, T).

3. Multiplication and division theorems.

3.1 *The basic theorems.* Results are presented in this section which develop relationships between the spectra (via standard triples) of products of m.o.p.s and of divisors of m.o.p.s when such exist.

THEOREM 11. *Let L_1, L_2 be* m.o.p. *on $\mathscr{X}$ with standard triples X_1, T_1, Y_1 and X_2, T_2, Y_2, respectively and let $L(\lambda) = L_2(\lambda)L_1(\lambda)$. Then*

(a) $L^{-1}(\lambda) = X(I\lambda - T)^{-1}Y$

where

$$X = [X_1 \quad 0], \quad T = \begin{bmatrix} T_1 & Y_1X_2 \\ 0 & T_2 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 \\ Y_2 \end{bmatrix},$$

(b) *X, T, Y is a standard triple for L.*

Proof. The proof of part (a) is just that of Theorem 17_{II} provided Corollary 1 of Theorem 13_{II} is replaced by its operator version, Theorem 1, part (iii) of this paper.

For part (b) the proof used in II relies in an essential way on the finite dimension of $\mathscr{X}$. More generally, let L_1, L_2 have degrees k_1, k_2 and write $l = k_1 + k_2$. It will be proved that the operator $Q(X, T)$ on $\mathscr{X}^l$ is invertible and this, combined with (a) implies that X, T, Y form a standard triple.

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Using the definitions of X, T we have

$$Q(X, T) = \begin{bmatrix} X_1 & 0 \\ X_1T_1 & X_1Y_1X_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ X_1T_1^{l-1} & \sum_{i=0}^{l-2} (X_1T_1^iY_1)X_2T_2^{l-2-i} \end{bmatrix}$$

and then the biorthogonality relations for X_1, T_1, Y_1 yield

$$Q(X, T) = \begin{bmatrix} X_1 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ X_1T_1^{k_1-1} & 0 \\ X_1T_1^{k_1} & X_2 \\ X_1T_1^{k_1+1} & \sum_{i=0}^{1} (X_1T_1^{i+k_1-1}Y_1)X_2T_2^{1-i} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ X_1T_1^{l-} & \sum_{k=0}^{k_2-1} (X_1T_1^{i+k_1-1}Y_1)(X_2T_2^{k_2-1-i}) \end{bmatrix}$$

In a top left block of this matrix representation we have $Q(X_1, T_1)$ which is invertible and then the complementary bottom right block can be factored in the form

$$\begin{bmatrix} I & 0 & \cdots & & 0 \\ X_1T_1^{k_1}Y_1 & I & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & I & 0 \\ X_1T_1^{l-2}Y_1 & X_1T_1^{l-3}Y_1 & \ldots & X_1T_1^{k_1}Y_1 & I \end{bmatrix} \begin{bmatrix} X_2 \\ X_2T_2 \\ \cdot \\ \cdot \\ \cdot \\ X_2T_2^{k_2-1} \end{bmatrix}$$

which is clearly invertible. It follows that $Q(X, T)$ is invertible, as required.

It turns out that, by representing m.o.p. in right (or left) standard forms, an explicit representation of quotients and remainders in a division process can be obtained. This is described in:

THEOREM 12. *Let L, L_1 be* m.o.p. *on $\mathscr{X}$ of degrees l, k and having standard triples X, T, Y and X_1, T_1, Y_1 respectively. Assume $1 \leqq k < l$ and write L, L_1 in right standard form:*

(32) $L(\lambda) = I\lambda^l - XT^l(F_1 + F_2\lambda + \ldots + F_l\lambda^{l-1})$.

(33) $L_1(\lambda) = I\lambda^k - X_1T_1^k(G_1 + G_2\lambda + \ldots + G_k\lambda^{k-1})$.

Define

(34) $G_{\alpha\beta} = X_1 T_1^{\alpha} G_{\beta}, \quad 1 \leqq \beta \leqq k, \quad \alpha = 0, 1, 2, \ldots$

(35) $F_{\alpha\beta} = X_1 T^{\alpha} F_{\beta}, \quad 1 \leqq \beta \leqq l, \quad \alpha = 0, 1, 2, \ldots.$

Then

$$L(\lambda) = \mathscr{D}_1(\lambda) L_1(\lambda) + \mathscr{D}_2(\lambda)$$

where

$$\mathscr{D}_1(\lambda) = I\lambda^{l-k} + \sum_{p=0}^{l-k-1} \left(G_{l-p-1,k} - \sum_{i=p+2}^{l} F_{li} G_{i-p-2,k} \right) \lambda^p$$

and

$$\mathscr{D}_2(\lambda) = \sum_{j=1}^{k} \left(G_{lj} - \sum_{i=1}^{l} F_{lj} G_{i-1,j} \right) \lambda^{j-1}.$$

The proof of this theorem uses only algebraic properties of the noncommutative algebra $\mathbf{L}(\mathscr{X})$ and makes no reference to the dimensionality of $\mathscr{X}$. Thus, the argument of Theorem 6_{I} applies almost verbatim. There is a dual theorem, the generalization of Theorem 7_{I} cast in terms of *left* standard forms.

3.2 *Characterization of divisors.* Let L be a m.o.p. with standard triple X, T, Y. The next theorem characterizes divisors of L by means of certain invariant subspaces of T. Recall the notation introduced in Equation (29).

THEOREM 13. *Let L be a* m.o.p. *on $\mathscr{X}$ of degree l with standard triple X, T, Y. Let $\mathscr{L}$ be an invariant subspace of T on which $Q_k(X, T)$, viewed as an operator from $\mathscr{L}$ to $\mathscr{X}^k$, is invertible. Then there exists a right divisor L_1 of L with the representation*

(36) $L_1(\lambda) = I\lambda^k - XT^k(W_1 + W_2\lambda + \ldots + W_k\lambda^{k-1})$

where $W_1, \ldots, W_k \in \mathbf{L}(\mathscr{X}, \mathscr{L})$ are defined by

(37) $[W_1\, W_2 \ldots . W_k] = Q_k(X, T)|_{\mathscr{L}}^{-1}.$

Conversely, for every monic right divisor L_1 of L of degree k there is a unique invariant subspace $\mathscr{L}$ of T such that $Q_k(X, T)|_{\mathscr{L}}$ is invertible and (36) *holds.*

The proof is that of Theorem 8_{I}.

Remark 1. The invariant subspace $\mathscr{L}$ of T associated uniquely with the right divisor L_1 of L is called a *supporting subspace* for L with respect to T.

Remark 2. As pointed out in paper I, it follows from the details of proof of Theorem 13 that $Q_k(X, T)|_{\mathscr{L}}$ determines the similarity transforming $T|_{\mathscr{L}}$ to the first companion operator for L_1. Thus:

COROLLARY. *If we write* $L_1(\lambda) = \sum_{i=0}^{k} B_i\lambda^i$, $B_k = I$, *then*

$$Q_k(X, T)|_{\mathscr{L}} T|_{\mathscr{L}} (Q_k(X, T)|_{\mathscr{L}})^{-1} = \begin{bmatrix} 0 & I & 0 & & 0 \\ 0 & 0 & I & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & & & & I \\ -B_0 & -B_1 & & \dots & -B_{k-1} \end{bmatrix}$$

The following theorem is due to Langer [**8**]. It can be proved directly from the more primitive Theorem 12 but we present it here as a consequence of Theorem 13.

THEOREM 14. *Let L be an* m o.p. *on* $\mathscr{X}$ *of degree l and let* $C_1 \in \mathbf{L}(\mathscr{X}^l)$ *be the first companion operator for L. There exists an* m.o.p. L_1 *of degree* $k < l$ *which divides L on the right if and only if there exists an invariant subspace* $\mathscr{L}$ *of* C_1 *of the form*

$$\mathscr{L} = \operatorname{Im}\begin{bmatrix} I_k \\ G \end{bmatrix}$$

for some $G \in \mathbf{L}(\mathscr{X}^k, \mathscr{X}^{l-k})$, (*and* $I_k \in \mathbf{L}(\mathscr{X}^k)$).

Furthermore, if the right divisor L_1 *exists,* $G = [G_{k-1+i,j}]$, $i = 1, \dots, l-k$, *and* $j = 1, \dots, k$, *then*

(38) $L_1(\lambda) = I\lambda^k - (G_{k1} + G_{k2}\lambda + \dots + G_{kk}\lambda^{k-1})$.

Proof. Let C_1 have an invariant subspace $\mathscr{L}$ as described and let $W = \begin{bmatrix} I \\ G \end{bmatrix}$. Now the map $X = [I\ 0 \dots 0]$ from $\mathscr{X}^l$ to $\mathscr{X}$ together with C_1 form a standard pair for L. Thus,

$$Q_k(X, C_1) = [I_k\ 0] \in \mathbf{L}(\mathscr{X}^l, \mathscr{X}^k)$$

and for any $x \in \mathscr{X}^k$, $Q_k(X, C_1)|_{\mathscr{L}}$ maps $\begin{bmatrix} x \\ Gx \end{bmatrix}$ onto x. This operator is obviously invertible. Furthermore, it is apparent that

$$Q_k(X, C_1)|_{\mathscr{L}}^{-1} = \begin{bmatrix} I \\ G \end{bmatrix}.$$

Thus, in (37) we have $[W_1 \dots W_k] = W$ and, since

$$XC_1^k = [0 \dots 0\ I\ 0 \dots 0]$$

with I in the $k+1$ position, $XC_1^kW = [G_{k1} \dots G_{kk}]$. It follows from Theorem 13 that L_1 of (38) is a right divisor of L. In fact, if $[G_1 \dots G_k] = Q_k(X_1, T_1)^{-1}$ for any standard pair X_1, T_1 of L_1 we can go further and identify

(39) $G_{ij} = X_1T_1^iG_j$, $i = 0, 1, 2, \dots,$ $j = 1, 2, \dots, k$.

For the converse, the existence of the right divisor L_1 ensures (by Theorem 13) the existence of an invariant subspace $\mathscr{L}$ of C_1 on which $Q_k(X, C_1)$ is invertible. Since $Q_k(X, C_1) = [I_k 0]$ it follows easily (Part (a) of the lemma for Theorem 9_{I}) that $\mathscr{L} = \text{Im}\begin{bmatrix} I \\ G \end{bmatrix}$ for some $G \in \mathbf{L}(\mathscr{X}^k, \mathscr{X}^{l-k})$. This completes the proof.

The case of divisors of degree one is of particular interest (see Langer [**7**], for example) partly because of the connection with the solution of operator equations provided by the remainder theorem:

There exists a right divisor $L_1(\lambda) = I\lambda - D$ *for* m.o.p. L *if and only if*

(40) $L(D) \equiv A_0 + A_1 D + \ldots + A_{l-1}D^{l-1} + D^l = 0.$

Putting $k = 1$ in Theorem 14 we obtain $G_{i1} = X_1 T_1{}^i X_1{}^{-1}$ from (39) and writing $L_1(\lambda) = I\lambda - D$ we deduce:

COROLLARY. *The* m.o.p. L *has a right divisor* $I\lambda - D$, $D \in \mathbf{L}(\mathscr{X})$ *if and only if there exists an invariant subspace* $\mathbf{L}$ *of* C_1 *of the form*

$$\mathscr{L} = \text{Im}\begin{bmatrix} I \\ D \\ \cdot \\ \cdot \\ \cdot \\ D^{l-1} \end{bmatrix}$$

This result can be found in the work of Langer, and for the case of a spectral divisor, in a paper by Mereutsa [**10**].

3.3 *Characterization of the quotient.* We proceed now to a description of the quotient polynomial resulting when Theorems 13 and 14 apply.

THEOREM 15 *Let* L *be an* m.o.p. *and* X, T, Y *be a standard triple for* L. *Define* $Q_k(X, T)$ *by* (29) *and* $R_{l-k}(T, Y) \in \mathbf{L}(\mathscr{X}^{l-k}, \mathscr{X}^l)$ *by*

$$R_{l-k}(T, Y) = [Y\, TY \ldots T^{l-k-1}Y].$$

Let $\mathscr{L}$ *be an invariant subspace for* T *on which* $Q_k(X, T)|_{\mathscr{L}}$ *is invertible. Then*

$$\mathscr{X}^l = \mathscr{L} \oplus \text{Im}\, R_{l-k}$$

and the map $\mathscr{R} : \mathscr{X}^{l-k} \to \text{Im}\, R_{l-k}$ *generated by* R_{l-k} *is invertible.*

For $1 \leqq i \leqq k$ *and* $1 \leqq j \leqq l - k$ *define* $W_i: \mathscr{X} \to \mathscr{L}$, *and* Z_j: $\text{Im}\, R_{l-k} \to \mathscr{X}$ *by*

$$Q_k(X, T)|_{\mathscr{L}}{}^{-1} = [W_1 \ldots W_k], \quad \mathscr{R}^{-1} = \begin{bmatrix} Z_1 \\ Z_2 \\ \cdot \\ \cdot \\ \cdot \\ Z_{l-k} \end{bmatrix}$$

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Then if $P \in \mathbf{L}(\mathscr{X}^l)$ *is the projection on* $\operatorname{Im} R_{l-k}$ *along* $\mathscr{L}$,

$$(41) \quad L(\lambda) = [I\lambda^{l-k} - (Z_1 + \ldots + Z_{l-k}\lambda^{l-k-1})PT^{l-k}PY] \times [I\lambda^k - XT^k(W_1 + \ldots + W_k\lambda^{k-1})].$$

As indicated in Paper I, the factors in (41) require little modification to produce standard forms. The description of the left factor as a *right* standard form can also be found in that paper.

If $\mathscr{X}$ is a Hilbert space—inducing the usual Hilbert-space structure on $\mathscr{X}^r$, $r = 1, 2, \ldots$ and $\mathscr{M}$ is the orthogonal complement of $\mathscr{L}$ in $\mathscr{X}^l$, then

$$(R_{l-k}|_{\mathscr{M}})^{-1} T(R_k|_{\mathscr{M}})$$

is the second companion operator associated with the left factor.

It is also pointed out in Paper I that the subspace $\operatorname{Im} R_{l-k}$ plays no essential role; the results can be formulated in terms of *any* complementary subspace for $\mathscr{L}$ in $\mathscr{X}^l$. Using the standard triple (13) the geometry of subspaces is clarified as indicated in the following result. The proof is a straight-forward verification and is therefore omitted. Let $P_k \in \mathbf{L}(\mathscr{X}^l)$ be the projection onto the first k components of $\mathscr{X}^l$ (so that P_1, C_1 form a standard pair).

THEOREM 16. *Let* L *be an* m.o.p. *of degree* l *with associated first companion operator* C_1. *A subspace* $\mathscr{L} \subset \mathscr{X}^l$ *is a supporting subspace of* L *with respect to* C_1 *associated with right divisor* L_1 *of degree* k *if and only if the following conditions hold:*

(i) $\mathscr{L}$ *is an invariant subspace of* C_1.

(ii) *The subspace* $(I - P_k)\mathscr{X}^l$ *of* $\mathscr{X}^l$ *is a direct complement of* $\mathscr{L}$.

When these conditions hold, then

$$L_1(\lambda) = I\lambda^k = I\lambda^k - P_1C_1^k(V_1 + V_2\lambda + \ldots + V_k\lambda^{k-1})$$

where $[V_1 \ldots V_k] = (P_k|_{\mathscr{L}})^{-1}$.

If $L(\lambda) = L_2(\lambda)L_1(\lambda)$ *and we define*

$$Y = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ I \end{bmatrix}, \quad R_{l-k} = [Y\, C_1Y \ldots C_1^{l-k-1}Y],$$

and $\mathscr{R}$ *as the invertible map from* $\mathscr{X}^{l-k}$ *to* $\operatorname{Im} R_{l-k}$ *generated by* R_{l-k}, *then*

$$L_2(\lambda) = I\lambda^{l-k} - (Z_1 + Z_2\lambda + \ldots + Z_{l-k}\lambda^{l-k-1})PC_1^{l-k}PY$$

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where P is the projector on $\operatorname{Im} R_{l-k}$ *along* $\mathscr{L}$ *and*

$$\begin{bmatrix} Z_1 \\ \cdot \\ \cdot \\ \cdot \\ Z_{l-k} \end{bmatrix} = \mathscr{R}^{-1}$$

We turn now to the possibility of extracting successive right divisors.

THEOREM 17. *Let L be an* m.o.p. *of degree l. Let $\mathscr{L}_1, \mathscr{L}_2$ be supporting subspaces for L (with respect to T) associated with monic right divisors L_1, L_2 of L degrees k_1, k_2, respectively. If $k_2 < k_1$ and $\mathscr{L}_2 \subset \mathscr{L}_1$ then L_2 is a right divisor of L_1.*

The proof of this theorem is that of Theorem 10_{I} and the natural extension to a chain of nested supporting subspaces also follows, as in Theorem 11_{I}.

Theorem 15 can be combined with the resolvent form (iii) of Theorem 1 to obtain the next result. The proof is just that of Theorem 18_{II}.

THEOREM 18. *If* m.o.p. *L has a standard triple X, T, Y, $L = L_2L_1$ where L_1, L_2 are* m.o.p. *and $\mathscr{L}$ is the supporting subspace of L_1 with respect to T then, in the notation of Theorem* 15,

(42) $L_1^{-1}(\lambda) = X|_{\mathscr{L}}(I\lambda - T|_{\mathscr{L}})^{-1}W_k,$

(43) $L_2^{-1}(\lambda) = Z_{l-k}(I\lambda - PTP|_{\operatorname{Im} P})^{-1}Y.$

3.4 *Divisors and the spectrum.* It is apparent that, if L_1 is a monic right divisor of L, then the point spectrum of L_1 is contained in that of L. On the other hand, it is not necessarily the case that $\sigma(L_1) \subset \sigma(L)$ as the following example of Marcus and Mereutsa [**9**] shows.

Example. Let V be the unilateral shift operator on l_2 and V^* its Hilbert adjoint. It is well known that $V^*V = I \neq VV^*$ and that $\sigma(I\lambda - V) = \{\lambda \in C\colon |\lambda| \leqq 1\} = \sigma(I\lambda - V^*)$. Define m.o.p. L of degree two by

(44) $L(\lambda) = I\lambda^2 - (V^* + V)\lambda + I = (I\lambda - V^*)(I\lambda - V).$

Then it can be verified that $\sigma(L) = \{\lambda \in C : |\lambda| = 1\}$.

As an illustration of the Corollary to Theorem 14 observe that

$$C_1 = \begin{bmatrix} 0 & I \\ -I & V^* + V \end{bmatrix}$$

and the right divisor displayed in (44) has the associated supporting subspace

$$\mathscr{L} = \operatorname{Im}\begin{bmatrix} I \\ V \end{bmatrix}$$

which is invariant under C_1.

Note also that, in the representation of the quotient given in (41)

$$Z_1 = [0 \quad I], \quad P = \begin{bmatrix} 0 & 0 \\ -V & I \end{bmatrix} \quad Y = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

from which it is easily verified that the quotient is, indeed, $I\lambda - V^*$.

In contrast to the above example, the next theorem characterizes some divisors whose spectra are proper subsets of the spectrum of the "parent" m.o.p. First some definitions: A point $\lambda \in C$ is a *regular point* of m.o.p. L if $L(\lambda)$ has an inverse in $\mathbf{L}(\mathscr{X})$. Let Γ be a contour in C consisting of regular points of L. A monic right divisor L_1 of L is a Γ *spectral right divisor* of $L = L_2L_1$ if $\sigma(L_1)$, $\sigma(L_2)$ are inside and outside Γ, respectively.

THEOREM 15. *Let L be an* m.o.p. *with standard triple X, T, Y and let L_1 be a Γ-spectral right divisor of L having associated supporting subspace $\mathscr{L}$ with respect to T. Then $\mathscr{L}$* = Im R_Γ *where R_Γ is the Riesz projector corresponding to T and Γ*:

$$R_\Gamma = \frac{1}{2\pi i}\oint_\Gamma (I\lambda - T)^{-1}d\lambda$$

Conversely, if Γ is a contour of regular points and $\mathscr{L}$ = Im R_Γ *is the supporting subspace for the monic right divisor L_1 of L with respect to T, then L_1 is a Γ-spectral right divisor.*

Proof. The first statement is proved as in Theorem 20_{II}. For the converse we have that $\mathscr{L}$ = Im R_Γ is the supporting subspace for divisor L_1. As described in Theorem 15 we have the decomposition $\mathscr{X}^l = \mathscr{L} \oplus$ Im R_{l-k} and then $\mathscr{L}$ an invariant subspace of T implies that the representation of T with respect to this decomposition has the form

$$T = \begin{bmatrix} T|_{\mathscr{L}} & T_{12} \\ 0 & PTP \end{bmatrix}$$

where P is the projection on Im R_{l-k} along $\mathscr{L}$. Now, from the classical theory (Dunford and Schwartz [**1**, Theorem VII.3.20]), $\sigma(I\lambda - T|_{\mathscr{L}})$ is precisely the subset of $\sigma(I\lambda - T) = \sigma(L(\lambda))$ inside Γ. Furthermore, the above representation of T implies that

$$\sigma(I\lambda - T) = \sigma(I\lambda - T|_{\mathscr{L}}) \cup \sigma(I\lambda - PTP).$$

Hence, $\sigma(I\lambda - PTP)$ is outside Γ. It follows from (42) and (43) that $\sigma(L_1(\lambda)) = \sigma(I\lambda - T|_{\mathscr{L}})$ and $\sigma(L_2(\lambda)) = \sigma(I\lambda - PTP)$ and so L_1 is a Γ-spectral divisor of L.

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