Canad. Math. Bull. Vol. **42** (3), 1999 pp. 335–343

Cyclic Subgroup Separability of HNN-Extensions with Cyclic Associated Subgroups

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Abstract. We derive a necessary and sufficient condition for HNN-extensions of cyclic subgroup separable groups with cyclic associated subgroups to be cyclic subgroup separable. Applying this, we explicitly characterize the residual finiteness and the cyclic subgroup separability of HNN-extensions of abelian groups with cyclic associated subgroups. We also consider these residual properties of HNN-extensions of nilpotent groups with cyclic associated subgroups.

1 Introduction

A group G is said to be *cyclic subgroup separable*, or briefly π_c, if for each pair of elements $x, y \in G$ such that $y \notin \langle x\rangle$, there exists a normal subgroup N of finite index in G, briefly $N \triangleleft_f G$, such that $y \notin N\langle x\rangle$. Since every π_c group is residually finite ($\mathcal{RF}$), π_c is a stronger property than $\mathcal{RF}$. On the other hand it is much weaker than subgroup separability. Subgroup separability is such a strong property that only very few classes of groups have this property. Cyclic subgroup separability is possessed by a much larger classes of groups. Fortunately in the study of residual properties and separability properties of generalized free products with cyclic amalgamation and HNN-extensions with cyclic associated subgroups only π_c is needed instead of subgroup separability (see [6], [5], [13]). The concept was also useful in the study of 1–relator groups [1], [2]. In [14], Thurston asked whether Kleinian groups are subgroup separable. In fact he asked whether these groups are separable with respect to any special subgroups. Subgroup separability of Kleinian groups is probably very difficult to prove. Maybe it is more feasible to prove that they are π_c. Separability properties of HNN-extensions were not much known, since one of the simplest type of HNN-extensions, the Baumslag-Solitar group, $\langle b, t : t^{-1}b^2t = b^3\rangle$ is not even residually finite. However, in [11], [7], Stebe and Meskin gave characterizations for 1-relator groups of the form $\langle b, t : t^{-1}b^{\beta}t = b^{\lambda}\rangle$ to be π_c or residually finite, respectively. Hence the residual finiteness and cyclic subgroup separability of HNN-extensions of a cyclic group are known. Andreadakis, Raptis and Varsos [3] characterized the residual finiteness of HNN-extensions of abelian groups. For HNN-extensions with cyclic associated subgroups, Resenberger and Sasse [10] proved a criterion for such HNN-extensions to be $\mathcal{RF}$ or π_c. In [5], Kim and Tang considered the conjugacy separability of HNN-extensions of abelian groups. In this paper

Received by the editors October 11, 1996; revised November 10, 1998.

The first author was partly supported by GARC-KOSEF and KOSEF 961-0101-002-2. The second author gratefully acknowledges the partial support by the Natural Science and Engineering Research Council of Canada, Grant No. A-4064.

AMS subject classification: 20E26, 20E06, 20F10.

Keywords: HNN-extension, nilpotent groups, cyclic subgroup separable (π_c), residually finite.

we derive a simple necessary and sufficient condition for HNN-extensions of π_c groups with cyclic associated subgroups to be π_c (Theorem 2.9). Applying this result we can easily characterize HNN-extensions of abelian group with cyclic associated subgroups to be residually finite or π_c (Theorem 2.16 and Corollary 2.12). We also consider those properties for HNN-extensions of nilpotent groups.

2 Main Results

HNN-extensions of residually finite (or π_c) groups with finite associated subgroups are residually finite (or π_c, respectively). In this paper we shall consider whether HNN-extensions of finitely generated nilpotent groups with infinite cyclic associated subgroups are residually finite or π_c. For HNN-extensions of cyclic groups, we have the followings:

Theorem 2.1 (**[11]**) *The group $\langle b, t : t^{-1}b^{\beta}t = b^{\gamma}\rangle$ is π_c if and only if $\beta = \pm\gamma$.*

Theorem 2.2 (**[7]**) *The group $\langle b, t : t^{-1}b^{\beta}t = b^{\gamma}\rangle$ is residually finite if and only if $|\beta| = 1$ or $|\gamma| = 1$ or $\beta = \pm\gamma$.*

A useful criterion for HNN-extensions to be π_c is the following:

Theorem 2.3 (**[4]**) *Let $H, K < A$ such that $\phi\colon H \to K$ is an isomorphism. Let $\Delta = \{P \lhd_f A : \phi(P \cap H) = P \cap K\}$. Assume that*

(a) $\bigcap_{P\in\Delta} HP = H$ and $\bigcap_{P\in\Delta} KP = K$,
(b) $\bigcap_{P\in\Delta} P\langle x\rangle = \langle x\rangle$ for all $x \in A$.

Then the HNN-extension $G = \langle A, t : t^{-1}ht = \phi(h), h \in H\rangle$ is π_c.

In [8], Niblo defined that a group A has *regular quotients* at $\{h, k\}$ if there exists a positive integer r, such that for each positive integer s, there exists $N \lhd_f A$ such that $N \cap \langle h\rangle = \langle h^{rs}\rangle$ and $N \cap \langle k\rangle = \langle k^{rs}\rangle$. He found a condition for a free group to have regular quotients at $\{h, k\}$ [8, Proposition 1.1].

Definition 2.4 Let A be a group and let $h, k \in A$ be of infinite order. Then A is said to be *quasi-regular at* $\{h, k\}$ if, for each given integer $\epsilon > 0$, there exist an integer $\lambda_\epsilon > 0$ and $N_\epsilon \lhd_f A$, depending on ϵ, such that $N_\epsilon \cap \langle h\rangle = \langle h^{\epsilon\lambda_\epsilon}\rangle$ and $N_\epsilon \cap \langle k\rangle = \langle k^{\epsilon\lambda_\epsilon}\rangle$.

Remark The followings are some simple facts about regular quotients and quasi-regularity.

1. If A has regular quotient at $\{h, k\}$ then A is quasi-regular at $\{h, k\}$.
2. A has regular quotients at $\{h, h\}$ if and only if A is weak $\langle h\rangle$-potent [12].
3. If A is weak $\langle h\rangle$-potent and $h \sim_A k$, then A has regular quotients at $\{h, k\}$. Hence A is quasi-regular at $\{h, k\}$.
4. If A is quasi-regular at $\{h, k\}$ and $\langle h\rangle \cap \langle k\rangle \neq 1$, then we have $h^s = k^{\pm s}$ for some $s > 0$. To see this, let $h^s = k^t$, where $s > 0$. By quasi-regularity, there exist an integer λ and $N \lhd_f A$ such that $N \cap \langle h\rangle = \langle h^{\lambda s}\rangle$ and $N \cap \langle k\rangle = \langle k^{\lambda s}\rangle$. Since $h^{\lambda s} = k^{\lambda t} \in N \cap \langle k\rangle = \langle k^{\lambda s}\rangle$, $s \mid t$. Similarly $t | s$. Hence $h^s = k^{\pm s}$ for some $s > 0$.

Lemma 2.5 *Let A be π_c and let $h, k \in A$ be of infinite order. Let A be weak $\langle h\rangle$-potent and $\langle h\rangle \cap \langle k\rangle \neq 1$. Then A is quasi-regular at $\{h, k\}$ if and only if $h^\delta = k^{\pm\delta}$ for some $\delta > 0$.*

Proof As mentioned before, if A is quasi-regular at $\{h, k\}$ and $\langle h\rangle \cap \langle k\rangle \neq 1$ then $h^\delta = k^{\pm\delta}$ for some $\delta > 0$. Conversely, we suppose $h^\delta = k^{\pm\delta}$ for some $\delta > 0$. Let $\epsilon > 0$ be a given integer. Since A is π_c and $|k| = \infty$, there exists $N_1 \lhd_f A$ such that $k^i \notin N_1\langle k^\delta\rangle$ for all $1 \leq i < \delta$. Let $N_1 \cap \langle k\rangle = \langle k^{n\delta}\rangle$. Since A is weak $\langle h\rangle$-potent, we can find $N_2 \lhd_f A$ such that $N_2 \cap \langle h\rangle = \langle h^{n\epsilon\delta\lambda}\rangle$ for some λ. Let $N = N_1 \cap N_2$. Then $N \lhd_f A$ and we have $N \cap \langle h\rangle = N_1 \cap \langle h^{n\epsilon\delta\lambda}\rangle = N_1 \cap \langle k^{n\epsilon\delta\lambda}\rangle = \langle k^{n\epsilon\delta\lambda}\rangle = \langle h^{n\epsilon\delta\lambda}\rangle$ and $N \cap \langle k\rangle = N_2 \cap \langle k^{n\delta}\rangle = N_2 \cap \langle h^{n\delta}\rangle = \langle h^{n\epsilon\delta\lambda}\rangle$. Hence A is quasi-regular at $\{h, k\}$. ■

Let $Z_i(A)$, or simply Z_i, denote the i-th term of the upper central series of A and let $Z(A) = Z_1(A)$ be the center of A.

Lemma 2.6 *Let A be a finitely generated nilpotent group. Let $\bar{h}, \bar{k} \in Z(\bar{A})$ be of infinite order, where $\bar{A} = A/Z_i$. If $\langle \bar{h}\rangle \cap \langle \bar{k}\rangle = 1$ or $\bar{h}^\delta = \bar{k}^{\pm\delta}$ for some $\delta > 0$, then A is quasi-regular at $\{h, k\}$.*

Proof (1) Suppose $\langle \bar{h}\rangle \cap \langle \bar{k}\rangle = 1$, where $\bar{A} = A/Z_i$. Let $\epsilon > 0$ be a given integer. Since $\bar{h}, \bar{k} \in Z(\bar{A})$, $\langle \bar{h}^\epsilon\rangle\langle \bar{k}^\epsilon\rangle \lhd \bar{A}$. Let $\tilde{A} = \bar{A}/\langle \bar{h}^\epsilon\rangle\langle \bar{k}^\epsilon\rangle$. Since $\tilde{A}$ is residually finite, we can find $\tilde{N} \lhd_f \tilde{A}$ such that $\tilde{N} \cap \langle \tilde{h}\rangle\langle \tilde{k}\rangle = 1$. Let N be the preimage of $\tilde{N}$ in A. Then $N \lhd_f A$, $N \cap \langle h\rangle = \langle h^\epsilon\rangle$ and $N \cap \langle k\rangle = \langle k^\epsilon\rangle$. Hence A is quasi-regular at $\{h, k\}$.

(2) Suppose $\bar{h}^\delta = \bar{k}^{\pm\delta}$ for some $\delta > 0$, where $\bar{A} = A/Z_i$. Let $\epsilon > 0$ be a given integer. Since $\bar{h}, \bar{k} \in Z(\bar{A})$, $\langle \bar{h}^{\epsilon\delta}\rangle = \langle \bar{k}^{\epsilon\delta}\rangle \lhd \bar{A}$. Let $\tilde{A} = \bar{A}/\langle \bar{h}^{\epsilon\delta}\rangle$. Then, as before, we can find $N \lhd_f A$ such that $N \cap \langle h\rangle = \langle h^{\epsilon\delta}\rangle$ and $N \cap \langle k\rangle = \langle k^{\epsilon\delta}\rangle$. Hence A is quasi-regular at $\{h, k\}$. ■

Lemma 2.5 and 2.6 imply the following:

Corollary 2.7 *Let A be a finitely generated nilpotent group. Let $h, k \in Z(A)$ be of infinite order. Then A is quasi-regular at $\{h, k\}$ if and only if $\langle h\rangle \cap \langle k\rangle = 1$ or $h^\delta = k^{\pm\delta}$ for some $\delta > 0$.*

Here are some other easy examples of quasi-regularity of nilpotent groups.

Lemma 2.8 *Let A be a finitely generated nilpotent group. Let $h, k \in A$ be of infinite order.*

(1) If $h \sim_A k$ then A is quasi-regular at $\{h, k\}$.
(2) Suppose $\langle h, k\rangle$ is a torsion free normal subgroup of A. Then A is quasi-regular at $\{h, k\}$ if and only if $\langle h\rangle \cap \langle k\rangle = 1$ or $h^\delta = k^{\pm\delta}$ for some $\delta > 0$.

Proof (1) Since A is finitely generated nilpotent, A is weak $\langle h\rangle$-potent. Hence, by Remark (3), A is quasi-regular at $\{h, k\}$ if $h \sim_A k$.

(2) By Lemma 2.5 it suffices to show that A is quasi-regular at $\{h, k\}$ if $\langle h\rangle \cap \langle k\rangle = 1$ and $\langle h, k\rangle$ is a torsion-free normal subgroup of A. Let $B = \langle h, k\rangle$. Then $B^\epsilon \lhd A$ for every $\epsilon > 0$. We shall show $B^\epsilon \cap \langle h\rangle = \langle h^\epsilon\rangle$ by induction on the nilpotency class of B. If B is abelian then it clearly holds. Suppose B is of class $c > 1$. Since $B = \langle h, k\rangle$ is torsion-free, we will prove that if $\langle h\rangle\langle k\rangle \cap Z_{c-1}(B) \neq 1$ then $B = \langle Z_{c-1}(B), y\rangle$ for some $y \in B$. Suppose

$h^n k^m \in Z_{c-1}(B)$ for $n \neq 0 \neq m$ then we may assume that $(n, m) = 1$. This is because B is torsion-free and $B/Z_{c-1}(B)$ is abelian. Let $1 = ns + mt$ for some integers s, t. Thus $hZ_{c-1}(B) = h^{ns+mt} Z_{c-1}(B) = (k^{-s} h^t)^m Z_{c-1}(B)$. Similarly, $kZ_{c-1}(B) = (k^{-s}h^t)^{-n} Z_{c-1}(B)$. Hence $B = \langle h, k\rangle = \langle Z_{c-1}(B), k^{-s}h^t\rangle$ has nilpotency class $\leq c-1$ [9, p. 135]. Also if $h^n \in Z_{c-1}(B)$ (or $k^n \in Z_{c-1}(B)$) for $n \neq 0$ then $h \in Z_{c-1}(B)$. Hence $B = \langle Z_{c-1}(B), k\rangle$ has nilpotency class $\leq c-1$. This implies $\langle h\rangle\langle k\rangle \cap Z_{c-1}(B) = 1$. Hence $\langle \bar{h}\rangle \cap \langle \bar{k}\rangle = 1$, where $\bar{B} = B/Z_{c-1}(B)$. Therefore, by induction, $\bar{B}^\epsilon \cap \langle \bar{h}\rangle = \langle \bar{h}^\epsilon\rangle$. This implies $B^\epsilon \cap \langle h\rangle = \langle h^\epsilon\rangle$. Similarly $B^\epsilon \cap \langle k\rangle = \langle k^\epsilon\rangle$.

Let $\tilde{A} = A/B^\epsilon$. Then $|\tilde{h}| = |\tilde{k}| = \epsilon$. Since $\tilde{A}$ is residually finite, there exists $\tilde{N} \lhd_f \tilde{A}$ such that $\tilde{N} \cap \langle \tilde{h}\rangle\langle \tilde{k}\rangle = 1$. Let N be the preimage of $\tilde{N}$ in A. Then $N \cap \langle h\rangle = \langle h^\epsilon\rangle$ and $N \cap \langle k\rangle = \langle k^\epsilon\rangle$, hence A is quasi-regular at $\{h, k\}$. ∎

We are now ready to prove our main theorem.

Theorem 2.9 *Let A be π_c and let $h, k \in A$ be of infinite order. Then $G = \langle A, t : t^{-1}ht = k\rangle$ is π_c if and only if A is quasi-regular at $\{h, k\}$.*

Proof Suppose G is π_c. Let $\epsilon > 0$ be a given integer. Since G is π_c, there exists $M \lhd_f G$ such that $h^i \notin M\langle h^\epsilon\rangle$ and $k^i \notin M\langle k^\epsilon\rangle$ for all $1 \leq i < \epsilon$. Let $M \cap \langle h\rangle = \langle h^{\epsilon\lambda}\rangle$ for some λ. Since $h \sim_G k$, $M \cap \langle k\rangle = \langle k^{\epsilon\lambda}\rangle$. Let $M \cap A = N$. Then $N \lhd_f A$, $N \cap \langle h\rangle = \langle h^{\epsilon\lambda}\rangle$ and $N \cap \langle k\rangle = \langle k^{\epsilon\lambda}\rangle$. Hence A is quasi-regular at $\{h, k\}$.

Conversely, suppose A is quasi-regular at $\{h, k\}$. We need to show that A satisfies (a) and (b) of Theorem 2.3. Let $a, x \in A$ such that $a \notin \langle x\rangle$. Since A is π_c, there exists $N_1 \lhd_f A$ such that $a \notin N_1\langle x\rangle$. Let $N_1 \cap \langle h\rangle = \langle h^{n_1}\rangle$, $N_1 \cap \langle k\rangle = \langle k^{n_2}\rangle$ and $\epsilon = n_1 n_2$. By quasi-regularity, there exist an integer λ and $N_2 \lhd_f A$ such that $N_2 \cap \langle h\rangle = \langle h^{\lambda\epsilon}\rangle$ and $N_2 \cap \langle k\rangle = \langle k^{\lambda\epsilon}\rangle$. Let $N_a = N_1 \cap N_2$. Then $a \notin N_a\langle x\rangle$, $N_a \cap \langle h\rangle = \langle h^{\lambda\epsilon}\rangle$ and $N_a \cap \langle k\rangle = \langle k^{\lambda\epsilon}\rangle$. Thus (a) and (b) of Theorem 2.3 hold. Hence G is π_c. ∎

The above result together with Lemmas 2.5 and 2.6 and Corollary 2.7 implies:

Corollary 2.10 *Let A be a finitely generated nilpotent group. Let $h, k \in A$ be of infinite order such that $\langle h\rangle \cap \langle k\rangle \neq 1$. Then $G = \langle A, t : t^{-1}ht = k\rangle$ is π_c if and only if $h^\delta = k^{\pm\delta}$ for some $\delta > 0$.*

Corollary 2.11 *Let A be a finitely generated nilpotent group. Let $\bar{h}, \bar{k} \in Z(\bar{A})$ be of infinite order, where $\bar{A} = A/Z_i$. If $\langle \bar{h}\rangle \cap \langle \bar{k}\rangle = 1$ or $\bar{h}^\delta = \bar{k}^{\pm\delta}$ for some $\delta > 0$, then $G = \langle A, t : t^{-1}ht = k\rangle$ is π_c.*

Corollary 2.12 *Let A be a finitely generated nilpotent group. Let $h, k \in Z(A)$ be of infinite order. Then $G = \langle A, t : t^{-1}ht = k\rangle$ is π_c if and only if $\langle h\rangle \cap \langle k\rangle = 1$ or $h^\delta = k^{\pm\delta}$ for some $\delta > 0$.*

We also have the following from Lemma 2.8.

Corollary 2.13 *Let A be a finitely generated nilpotent group. Let $h, k \in A$ be of infinite order and let $G = \langle A, t : t^{-1}ht = k\rangle$.*

(1) *If $h \sim_A k$ then G is π_c.*
(2) *Suppose $\langle h, k\rangle \lhd A$. Then G is π_c if and only if $\langle h\rangle \cap \langle k\rangle = 1$ or $h^\delta = k^{\pm\delta}$ for some $\delta > 0$.*

From now on, we consider the residual finiteness of HNN-extensions.

Lemma 2.14 *Suppose A is not a cyclic group and $Z(A) \neq 1$. Let $h \in A$ be of infinite order. Then $G = \langle A, t : t^{-1}h^\delta t = h\rangle$ is not residually finite if $|\delta| > 1$.*

Proof Let $|\delta| > 1$. If $\langle h\rangle$ is not isolated in A, say $h^i = w^\alpha$ for $w \in A\backslash\langle h\rangle$ and $|\alpha| > 1$, then the subgroup $\langle w, t : t^{-1}w^{\alpha\delta}t = w^\alpha\rangle$ of G is not residually finite by Theorem 2.2. Hence G is not residually finite. Thus we can assume that $\langle h\rangle$ is isolated in A for the rest of the proof.

Suppose there exists $w \in Z(A)\backslash\langle h\rangle$. Since $|\delta| > 1$, $g = [t^{-1}ht, w] \neq 1$. If G is residually finite then there exists a finite homomorphic image $\bar{G}$ of G with $\bar{g} \neq 1 \neq \bar{h}$. Let $n = |\bar{h}|$. Since $h^\delta \sim_G h$, $(\delta, n) = 1$. Therefore, there exist integers λ, μ such that $1 = \lambda\delta + \mu n$. Thus $\bar{t}^{-1}\bar{h}\bar{t} = \bar{t}^{-1}\bar{h}^{\lambda\delta+\mu n}\bar{t} = \bar{t}^{-1}\bar{h}^{\lambda\delta}\bar{t} = \bar{h}^\lambda$. It follows $\bar{g} = [\bar{t}^{-1}\bar{h}\bar{t}, \bar{w}] = [\bar{h}^\lambda, \bar{w}] = 1$, since $w \in Z(A)$. This contradicts the choice of $\bar{G}$, whence G is not residually finite.

Suppose there exists no $w \in Z(A)\backslash\langle h\rangle$. This means $1 \neq Z(A) \subset \langle h\rangle$. Let $Z(A) = \langle h^s\rangle$, where $s > 0$.

If $s = 1$ then $Z(A) = \langle h\rangle$. Since A is not cyclic and $|\delta| > 1$, we can choose $w \in A\backslash\langle h\rangle$ such that $g = [t^{-1}ht, w] \neq 1$. Then as before there is no finite homomorphic image $\bar{G}$ of G such that $\bar{g} \neq 1 \neq \bar{h}$. Hence G is not residually finite.

Let $s > 1$. Since A is not cyclic, there exists $b \in A\backslash\langle h\rangle$. If $|b| = n < \infty$ then $(bh^s)^n = h^{sn} \in \langle h\rangle$, where $\langle h\rangle$ is isolated. Hence $bh^s \in \langle h\rangle$ and $b \in \langle h\rangle$, which is impossible. Thus $|b| = \infty$. Consider the subgroup $B = \langle b, h^s, t : [b, h^s] = 1, t^{-1}h^{s\delta}t = h^s\rangle$ of G. By [3, Corollary 3], B is not residually finite. Thus G is not residually finite. This completes the proof. ■

Theorem 2.15 *Let A be a finitely generated nilpotent group. Let $h, k \in A$ be of infinite order such that $\langle h\rangle \cap \langle k\rangle \neq 1$. Then $G = \langle A, t : t^{-1}ht = k\rangle$ is residually finite if and only if one of the following holds:*

(i) *If $A = \langle b\rangle$, $h = b^\alpha$ and $k = b^\beta$, then $|\alpha| = 1$ or $|\beta| = 1$ or $\alpha \pm \beta = 0$.*
(ii) *If A is not cyclic then $h^\delta = k^{\pm\delta}$ for some $\delta > 0$.*

Proof By Theorem 2.2, if (i) holds then G is residually finite. For (ii), if $h^\delta = k^{\pm\delta}$ for some $\delta > 0$ then, by Corollary 2.10, G is π_c, hence G is residually finite.

Conversely suppose G is residually finite. Then (i) holds by Theorem 2.2. If A is not cyclic, then $h^\lambda = k^\delta$ for some $\lambda \neq 0 \neq \delta$. Thus $t^{-1}h^\delta t = k^\delta = h^\lambda$. Hence $\langle h, t\rangle = \langle h, t : t^{-1}h^\delta t = h^\lambda\rangle < G$. Since G is residually finite, $\langle h, t\rangle$ is also residually finite. Thus, by (i), $|\delta| = 1$ or $|\lambda| = 1$ or $\delta = \pm\lambda$. Suppose $\lambda = \pm 1$. Then $h = k^{\pm\delta}$. Hence $G = \langle A, t : t^{-1}k^{\pm\delta}t = k\rangle$. Since G is residually finite, by Lemma 2.14 we have $|\delta| = 1$. Similarly, if $|\delta| = 1$ then $|\lambda| = 1$. Hence, in any case, $\delta = \pm\lambda$, and thus $h^\delta = k^{\pm\delta}$ for some $\delta > 0$. ■

Theorem 2.16 *Let A be a finitely generated abelian group. Let $h, k \in A$ be of infinite order. The HNN-extension $G = \langle A, t : t^{-1}ht = k\rangle$ is residually finite if and only if one of the followings holds:*

(i) If $A = \langle b \rangle$, $h = b^{\alpha}$ and $k = b^{\beta}$, then $|\alpha| = 1$ or $|\beta| = 1$ or $\alpha \pm \beta = 0$.
(ii) If A is not cyclic then $\langle h \rangle \cap \langle k \rangle = 1$ or $h^{\delta} = k^{\pm\delta}$ for some $\delta > 0$.

Proof Suppose G is residually finite. Then, by Theorem 2.2, (i) holds if A is cyclic. If A is not cyclic and $\langle h \rangle \cap \langle k \rangle \neq 1$ then, by Theorem 2.15, $h^{\delta} = k^{\pm\delta}$ for some $\delta > 0$.

Conversely, by Theorem 2.2, (i) implies that G is residually finite. If $\langle h \rangle \cap \langle k \rangle = 1$ or $h^{\delta} = k^{\pm\delta}$ for $\delta > 0$ then, by Corollary 2.12, G is π_c. Hence G is residually finite. ■

Finally we give some examples of non-residually finite HNN-extensions of nilpotent groups which supplement Corollary 2.10, 2.11 and Theorem 2.15.

Example 1 Let $A = \langle a, b : [a, b, a], [a, b, b] \rangle$ be a free nilpotent group of class 2. Let $h = a^2[a, b]^2$ and $k = a^3[a, b]^2$. Then $\langle h \rangle \cap \langle k \rangle = 1$. Consider the HNN-extension $G = \langle A, t : t^{-1}ht = k \rangle$. Let $g = \big[t^{-1}a[a, b]t, a[a, b]\big]$. Then $g \neq 1$. Suppose G is residually finite. Then there exists $N \triangleleft_f G$ such that $g, h \notin N$. Let $N \cap \langle h \rangle = \langle h^n \rangle$ for some n. Thus $h^n = a^{2n}[a, b]^{2n} \in N$ and $k^n = a^{3n}[a, b]^{2n} \in N$. Hence $a^n \in N$. If $n = 2m$ then $h^m = (a^2[a, b]^2)^m = a^{2m}[a^{2m}, b] = a^n[a^n, b] \in N$. Therefore $(n, 2) = 1$. Thus there exist integers λ, μ such that $1 = \lambda n + \mu 2$, whence, in $\bar{G} = G/N$, $\overline{t^{-1}a[a, b]t} = \bar{t}^{-1}(\overline{a[a, b]})^{2\mu}\bar{t} = \bar{t}^{-1}\bar{h}^{\mu}\bar{t} = \bar{k}^{\mu}$. It follows that $\bar{g} = \big[\bar{k}^{\mu}, \overline{a[a, b]}\big] = 1$, a contradiction. This implies that G is not residually finite. We note that A is not quasi-regular at $\{h, k\}$, since there are no integer λ and $N \triangleleft_f A$ such that $N \cap \langle h \rangle = \langle h^{2\lambda} \rangle$ and $N \cap \langle k \rangle = \langle k^{2\lambda} \rangle$.

Example 2 Let $A = \langle a, b : [a, b, a], [a, b, b] \rangle$ be as above. Let $h = [a, b]^2$ and $k = a$. Then the HNN-extension $G = \langle A, t : t^{-1}ht = k \rangle$ is not residually finite as before. Here we consider $g = \big[t^{-1}[a, b]t, a[a, b]\big]$. Also A is not quasi-regular at $\{h, k\}$.

We note that the subgroup $\langle h, k, t : [h, k] = 1, t^{-1}ht = k \rangle$ of G in the above examples is π_c by Corollary 2.13.

3 Criteria for Quasi-Regularity

Example 1, Section 2, shows that, for a finitely generated torsion-free nilpotent group A of class 2, the HNN-extension $G = \langle A, t : t^{-1}ht = k \rangle$ need not be residually finite, whence not π_c, even if $\langle h \rangle \cap \langle k \rangle = 1$. In the example $\langle h, k \rangle$ and $\langle h \rangle$ are not isolated in A. In this section we derive some results for A to be quasi-regular at $\{h, k\}$ when A is a finitely generated torsion-free nilpotent group. Applying these results we derive some conditions for the HNN-extension G to be residually finite or π_c.

Lemma 3.1 *Let A be a finitely generated torsion-free nilpotent group. Let $z \in Z$ such that $\langle z \rangle$ is isolated and let $x \in Z_2 \backslash Z$. Then for every $\epsilon > 0$, $\langle x^{\epsilon} \rangle^A \cap \langle z \rangle \subset \langle z^{\epsilon} \rangle$.*

Proof Suppose $z^l \in \langle x^{\epsilon} \rangle^A$. Let $g_1, \dots, g_m \in A$ such that

$$z^l = g_1^{-1}x^{k_1\epsilon}g_1 \cdots g_m^{-1}x^{k_m\epsilon}g_m = x^{k_1\epsilon}[x^{k_1\epsilon}, g_1] \cdots x^{k_m\epsilon}[x^{k_m\epsilon}, g_m].$$

Since $x \in Z_2$, $[x, g_i] \in Z$. This implies

$$z^l = x^{(k_1+\cdots+k_m)\epsilon}\big([x, g_1]^{k_1} \cdots [x, g_m]^{k_m}\big)^{\epsilon}.$$

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Now A is torsion-free and $x \in Z_2 \backslash Z$. This implies $k_1 + \cdots + k_m = 0$. Moreover $\langle z \rangle$ is isolated. This means $[x, g_1]^{k_1} \cdots [x, g_m]^{k_m} \in \langle z \rangle$, whence $z^l \in \langle z^\epsilon \rangle$. This proves the lemma. ∎

Lemma 3.2 *Let A be a finitely generated torsion-free nilpotent group. Let $h \in Z$ and $k \in Z_2 \backslash Z$ such that $\langle h \rangle$ is isolated. Then for every $\epsilon > 0$, there exists $N_\epsilon \lhd_f A$ such that $N_\epsilon \cap \langle h \rangle = \langle h^\epsilon \rangle$ and $N_\epsilon \cap \langle k \rangle = \langle k^\epsilon \rangle$.*

Proof Let $M = \langle h^\epsilon \rangle \langle k^\epsilon \rangle^A$. Clearly $\langle h^\epsilon \rangle, \langle k^\epsilon \rangle^A \subset M$. By Lemma 3.1, $\langle k^\epsilon \rangle^A \cap \langle h \rangle \subset \langle h^\epsilon \rangle$. Hence $M \cap \langle h \rangle = \langle h^\epsilon \rangle$. Suppose $k^i \in M \cap \langle k \rangle$. Then,

$$\begin{aligned} k^i &= h^{s\epsilon} \cdot g_1^{-1} k^{t_1 \epsilon} g_1 \cdots g_m^{-1} k^{t_m \epsilon} g_m \\ &= k^{(t_1 + \cdots + t_m)\epsilon} \cdot h^{s\epsilon} [k^{t_1 \epsilon}, g_1] \cdots [k^{t_m \epsilon}, g_m], \end{aligned}$$

where $g_1, \ldots, g_m \in A$. This implies, in $\bar{A} = A/Z$, $\bar{k}^i = \bar{k}^{(t_1 + \cdots + t_m)\epsilon}$. Since $\bar{A}$ is torsion-free nilpotent, $i = (t_1 + \cdots + t_m)\epsilon$. It follows that $k^i \in \langle k^\epsilon \rangle$, whence $M \cap \langle k \rangle = \langle k^\epsilon \rangle$.

Let $\tilde{A} = A/M$. Then $|\tilde{h}| = |\tilde{k}| = \epsilon$. Since $\tilde{A}$ is residually finite, there exists $\tilde{N} \lhd_f \tilde{A}$ such that $\tilde{N} \cap \langle \tilde{h} \rangle \langle \tilde{k} \rangle = 1$. Let N_ϵ be the preimage of $\tilde{N}$ in A. Then $N_\epsilon \cap \langle h \rangle = \langle h^\epsilon \rangle$ and $N_\epsilon \cap \langle k \rangle = \langle k^\epsilon \rangle$. ∎

By Example 2, if $\langle h \rangle$ is not isolated then the above lemma fails.

Lemma 3.3 *Let A be a finitely generated torsion-free nilpotent group. Let $h, k \in Z_2 \backslash Z$ and $1 \neq z \in Z$ such that $\langle z \rangle$ is isolated. Suppose $h^\alpha = k^\beta z^\delta$, where $(\alpha, \beta, \delta) = 1$. Then for every integer $n > 0$, there exists $N_n \lhd_f A$ such that $N_n \cap \langle h \rangle = \langle h^n \rangle$ and $N_n \cap \langle k \rangle = \langle k^n \rangle$ if (1) $\delta = \pm 1$, or (2) $\langle h \rangle, \langle k \rangle$ are isolated.*

Proof We first note that $[h, k] = 1$. This is because $h^\alpha = k^\beta z^\delta$ with $z \in Z$ implies $[h^\alpha, k^\beta] = 1$. Now $h, k \in Z_2$ implies $1 = [h^\alpha, k^\beta] = [h, k]^{\alpha\beta}$. Since A is torsion-free, $[h, k] = 1$.

Next we show that $(\alpha, \beta) = 1$. Suppose $(\alpha, \beta) = d$. Let $\alpha = d\alpha'$ and $\beta = d\beta'$. This implies $z^\delta = (h^{\alpha'} k^{-\beta'})^d$. Since $\langle z \rangle$ is isolated, $h^{\alpha'} k^{-\beta'} = z^\lambda$. Thus $\delta = \lambda d$, which implies $d \mid (\alpha, \beta, \delta) = 1$. Hence $d = 1$, *i.e.*, $(\alpha, \beta) = 1$. Therefore, there exist integers s, t such that $s\alpha + t\beta = 1$.

To prove the lemma we first consider the case when $n = p^\epsilon$, where p is a prime. Let $\bar{A} = A/Z^{p^\epsilon}$. Then $\bar{h}^{\alpha p^\epsilon} = \bar{k}^{\beta p^\epsilon}$. Since $h \in Z_2$, $[h^{p^\epsilon}, g] = [h, g]^{p^\epsilon} \in Z^{p^\epsilon}$ for each $g \in A$. This implies $\bar{h}^{p^\epsilon} \in Z(\bar{A})$. In the same way $\bar{k}^{p^\epsilon} \in Z(\bar{A})$. Since $[h, k] = 1$, we have

$$\bar{h}^{p^\epsilon} = \bar{h}^{p^\epsilon(s\alpha + t\beta)} = \bar{k}^{s\beta p^\epsilon} \bar{h}^{t\beta p^\epsilon} = (\bar{k}^s \bar{h}^t)^{\beta p^\epsilon}.$$

Similarly $\bar{k}^{p^\epsilon} = (\bar{k}^s \bar{h}^t)^{\alpha p^\epsilon}$. Let $\bar{u} = (\bar{k}^s \bar{h}^t)^{p^\epsilon}$ and $\bar{U} = \langle \bar{u} \rangle$. Then $\langle \bar{h}^{p^\epsilon} \rangle \subset \bar{U} \cap \langle \bar{h} \rangle$ and $\langle \bar{k}^{p^\epsilon} \rangle \subset \bar{U} \cap \langle \bar{k} \rangle$. We shall show that equality holds if either $\delta = \pm 1$ or if $\langle h \rangle, \langle k \rangle$ are isolated.

Suppose $\bar{h}^i = \bar{u}^j$. This means $\bar{h}^i = (\bar{k}^s \bar{h}^t)^{j p^\epsilon}$. Thus, $\bar{h}^{(i - j t p^\epsilon)\beta} = \bar{k}^{j s \beta p^\epsilon} = \bar{h}^{j s \alpha p^\epsilon}$. Since $|\bar{h}| = \infty$, $(i - j t p^\epsilon)\beta = j s \alpha p^\epsilon$. This implies $i\beta = j p^\epsilon (s\alpha + t\beta) = j p^\epsilon$. Therefore, $p^\epsilon \mid i\beta$ and $\bar{h}^i = (\bar{k}^s \bar{h}^t)^{j p^\epsilon} = (\bar{k}^s \bar{h}^t)^{i\beta}$. Hence $\bar{k}^{is\beta} = \bar{h}^{i(1 - t\beta)} = \bar{h}^{is\alpha} = \bar{k}^{is\beta} \bar{z}^{is\delta}$. This implies that $\bar{z}^{is\delta} = 1$, whence $z^{is\delta} \in Z^{p^\epsilon} \cap \langle z \rangle$. Since $\langle z \rangle$ is isolated, $Z^{p^\epsilon} \cap \langle z \rangle = \langle z^{p^\epsilon} \rangle$. Hence $p^\epsilon \mid is\delta$.

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Case 1 $\delta = \pm 1$. Then $p^\epsilon \mid is$. This implies $p^\epsilon \mid is\alpha + it\beta = i$. Hence $\bar{U} \cap \langle \bar{h} \rangle = \langle \bar{h}^{p^\epsilon} \rangle$. In the same way, we can show $\bar{U} \cap \langle \bar{k} \rangle = \langle \bar{k}^{p^\epsilon} \rangle$.

Case 2 $\langle h \rangle, \langle k \rangle$ are isolated. If $p \nmid \delta$, then $z^{is\delta} \in \langle z^{p^\epsilon} \rangle$ implies $p^\epsilon \mid is$. Thus, as in Case 1, we have $\bar{U} \cap \langle \bar{h} \rangle = \langle \bar{h}^{p^\epsilon} \rangle$ and $\bar{U} \cap \langle \bar{k} \rangle = \langle \bar{k}^{p^\epsilon} \rangle$. So, we can assume $p \mid \delta$. Let $\delta = p\delta'$. Suppose $p \mid \beta$ and $\beta = p\beta'$. This implies $(k^{\beta'} z^{\delta'})^p = k^\beta z^\delta = h^\alpha$. Since $\langle h \rangle$ is isolated, $k^{\beta'} z^{\delta'} \in \langle h \rangle$. Let $k^{\beta'} z^{\delta'} = h^\lambda$. Then $h^{\lambda p} = h^\alpha$, whence $\lambda p = \alpha$. This implies $p \mid (\alpha, \beta, \delta) = 1$, which is impossible. Therefore, we can assume $p \nmid \beta$ if $p \mid \delta$. Thus $p^\epsilon \mid i\beta$ implies $p^\epsilon \mid i$. Hence, as in Case 1, $\bar{U} \cap \langle \bar{h} \rangle = \langle \bar{h}^{p^\epsilon} \rangle$ and $\bar{U} \cap \langle \bar{k} \rangle = \langle \bar{k}^{p^\epsilon} \rangle$.

Since $\bar{h}^{p^\epsilon}, \bar{k}^{p^\epsilon} \in Z(\bar{A})$, $\bar{U} \lhd \bar{A}$. Let $\tilde{A} = \bar{A}/\bar{U}$. Then $\tilde{A}$ is residually finite with $|\tilde{h}| = |\tilde{k}| = p^\epsilon$. Therefore, there exists $\tilde{M} \lhd_f \tilde{A}$ such that $\tilde{M} \cap \langle \tilde{h} \rangle \langle \tilde{k} \rangle = 1$. Let M be the preimage of $\tilde{M}$ in A. Then $M \lhd_f A$ and $M \cap \langle h \rangle = \langle h^{p^\epsilon} \rangle$ and $M \cap \langle k \rangle = \langle k^{p^\epsilon} \rangle$.

Let $n = p_1^{n_1} \cdots p_l^{n_l}$, where the p_i's are distinct primes. By above, for each i, there exists $N_i \lhd_f A$ such that $N_i \cap \langle h \rangle = \langle h^{p_i^{n_i}} \rangle$ and $N_i \cap \langle k \rangle = \langle k^{p_i^{n_i}} \rangle$. Let $N_n = \bigcap_{i=1}^{l} N_i$. Then $N_n \lhd_f A$, $N_n \cap \langle h \rangle = \langle h^n \rangle$ and $N_n \cap \langle k \rangle = \langle k^n \rangle$. This completes the proof. ∎

We can now prove the following theorem:

Theorem 3.4 *Let A be a finitely generated torsion-free nilpotent group. Let $h, k \in Z_2$ such that $\langle h, k \rangle$ is isolated. Then A is quasi-regular at $\{h, k\}$ if and only if $\langle h \rangle \cap \langle k \rangle = 1$ or $h = k^{\pm 1}$.*

Proof If $h = k^{\pm 1}$, then A is quasi-regular at $\{h, k\}$ by Lemma 2.5. So we suppose $\langle h \rangle \cap \langle k \rangle = 1$. If $h, k \in Z$ then A is quasi-regular at $\{h, k\}$ by Corollary 2.7. Hence let $h \in Z$ and $k \notin Z$ (or $h \notin Z$ and $k \in Z$). Since $\langle h, k \rangle$ and Z are isolated and $h \in Z$, $\langle h, k \rangle \cap Z = \langle h \rangle$ is isolated. Thus A is quasi-regular at $\{h, k\}$ by Lemma 3.2. Finally we suppose that $h, k \notin Z$. If $\langle Zh \rangle \cap \langle Zk \rangle = Z$ or $Zh = Zk^{\pm 1}$ then A is quasi-regular at $\{h, k\}$ by Lemma 2.6. So let $Zh^\alpha = Zk^\beta$. Then, by the unique root property, $(\alpha, \beta) = 1$. This implies $h^\alpha = k^\beta z^\delta$, where $1 \neq z^\delta \in Z$ and $\langle z \rangle$ is isolated. Moreover $(\alpha, \beta, \delta) = 1$. Thus A is quasi-regular at $\{h, k\}$ by Lemma 3.3.

Conversely, suppose A is quasi-regular at $\{h, k\}$. If $\langle h \rangle \cap \langle k \rangle \neq 1$ then, by Lemma 2.5, $h^\delta = k^{\pm\delta}$ for some $\delta > 0$. It follows from the unique root property that $h = k^{\pm 1}$. ∎

The proof of the next result is very similar to above.

Theorem 3.5 *Let A be a finitely generated torsion-free nilpotent group. Let $h, k \in Z_2$ such that $\langle h \rangle, \langle k \rangle$ are isolated. Then A is quasi-regular at $\{h, k\}$ if and only if $\langle h \rangle \cap \langle k \rangle = 1$ or $h = k^{\pm 1}$.*

Applying Theorem 2.9, we immediately have the following:

Corollary 3.6 *Let A be a finitely generated torsion-free nilpotent group. Let $h, k \in Z_2$ such that either $\langle h, k \rangle$ is isolated or $\langle h \rangle$, $\langle k \rangle$ are isolated. Then the HNN-extension $G = \langle A, t : t^{-1}ht = k \rangle$ is π_c if and only if $\langle h \rangle \cap \langle k \rangle = 1$ or $h = k^{\pm 1}$.*

Corollary 3.7 *Let A be a finitely generated torsion-free nilpotent group. Let $h, k \in Z_2$ such that either $\langle h, k \rangle$ is isolated or $\langle h \rangle$, $\langle k \rangle$ are isolated. Then the HNN-extension $G = \langle A, t : t^{-1}ht = k \rangle$ is residually finite if and only if $\langle h \rangle \cap \langle k \rangle = 1$ or $h = k^{\pm 1}$.*

Proof Clearly if $\langle h\rangle \cap \langle k\rangle = 1$ or $h = k^{\pm 1}$ then G is residually finite by the above result. For the converse, suppose that G is residually finite. If $\langle h\rangle\cap\langle k\rangle \neq 1$ then, by Theorem 2.15, $h^{\delta} = k^{\pm\delta}$ for some $\delta > 0$. Hence, by the unique root property of A, $h = k^{\pm 1}$ as required. ∎

Above two results characterize the residual finiteness and cyclic subgroup separability of the HNN-extension $\langle A, t : t^{-1}ht = k\rangle$ of the finitely generated torsion-free nilpotent group A of class 2 when $\langle h, k\rangle$ is isolated or $\langle h\rangle$ and $\langle k\rangle$ are isolated. We note that, in Example 1, $\langle h, k\rangle = \langle a, [a, b]^2\rangle$ is not isolated in A and $\langle h\rangle = \langle a^2[a, b]^2\rangle$ is not isolated.

Problem Let A be a finitely generated nilpotent group. Characterize all $h, k \in A$ such that A is quasi-regular at $\{h, k\}$.

Acknowledgement The authors thank the referee for several helpful suggestions.

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