§ 6. Euler's Line.

*The circumcentre, the centroid, and the orthocentre of a triangle are collinear, and the distance between the first two is half the distance between the last two.**

Figure 61.

First Demonstration.†

Let A′, B′ be the mid points of BC, CA, and let the perpendiculars to BC, CA at A′, B′ meet at O;
then O is the circumcentre.
Let the perpendiculars AX, BY meet at H;
then H is the orthocentre.

Join OH and let AA′ meet it at G. Join A′B′.

Because triangles HAB, OA′B′ have their sides respectively parallel to each other, they are similar;
therefore HA : OA′ = AB : A′B′ = 2 : 1.
Again triangles HAG, OA′G are similar;
therefore HG : OG = HA : OA′ = 2 : 1
that is, AA′ cuts OH so that HG = twice OG.
Hence also the medians from B and C cut OH
so that HG = twice OG;
therefore G is the centroid, and H, G, O are collinear.

This is also a proof that the medians are concurrent.

Second Demonstration.

Let AA′ be the median from A, G the centroid, and O the circumcentre.

Join OA′, OG, and let AX the perpendicular from A to BC meet OG produced at H.

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* Proved by Euler in 1765. His proof will be found in *Novi Commentarii Academiae ... Petropolitanae*, XI. 114. An abstract of this paper of Euler's is printed in the *Proceedings of the Edinburgh Mathematical Society*, IV. 51-55 (1886).

† This method of proof is given in Carnot's *Géométrie de Position*, § 131 (1803). The second and third methods are imitations of it.

Then triangles HAG, OA′G are similar;

therefore $\qquad HG : OG = AG : A'G = 2 : 1,$

that is, AX cuts OG produced so that HG = twice OG.

Hence also the perpendiculars from B and C cut OG produced so that HG = twice OG;

therefore H is the orthocentre, and H, G, O are collinear.

This is also a proof that the perpendiculars to the sides from the vertices are concurrent.

Third Demonstration.

Let AX be the perpendicular, AA′ the median, from A to BC; and let H be the orthocentre, G the centroid.

Join HG, and let the perpendicular from A′ to BC meet HG produced at O.

Then triangles HAG, OA′G are similar;

therefore $\qquad HG : OG = AG : A'G = 2 : 1,$

that is, the perpendicular to BC from its mid point cuts HG produced so that HG = twice OG.

Hence also the perpendiculars to CA, AB from their mid points cut HG produced so that HG = twice OG;

therefore O is the circumcentre, and H, G, O are collinear.

This is also a proof that the perpendiculars to the sides from their mid points are concurrent.

Fourth Demonstration.*

Figure 62.

Let H be the orthocentre, determined by drawing AX, BY perpendicular to BC, CA; O the circumcentre, determined by drawing A′O, B′O perpendicular to BC, CA from their mid points A′, B′. Join HO and let it meet the median AA′ at G.

Bisect HA, HB at U, V, and GA, GH at P, Q; join UV, PQ, A′B′.

* This mode of proof assumes only the first book of Euclid's *Elements* and its immediate consequences.

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Then $A'B'$ is parallel to AB and equal to $\frac{1}{2}AB$,
and UV is parallel to AB and equal to $\frac{1}{2}AB$;
therefore $A'B'$ is parallel to UV and equal to UV.
Because OA' and HU are both perpendicular to BC;
therefore OA' is parallel to HU.
Similarly OB' is parallel to HV.
Hence the triangles $OA'B'$, HUV are mutually equiangular, and, since $A'B' = UV$, congruent.
Therefore $$OA' = HU = \tfrac{1}{2}AH.$$

Again PQ is parallel to AH and equal to $\frac{1}{2}AH$;
therefore PQ is parallel to OA' and equal to OA'.
Hence the triangles $A'GO$, PGQ are congruent;
therefore $A'G = PG = \frac{1}{2}AG$;
therefore G is the centroid, and $OG = QG = \frac{1}{2}HG$.

The straight line HGO is frequently called *Euler's line*.

(1) *The twelve radii drawn from the incentre and the excentres of a triangle perpendicular to the sides of the triangle meet by threes in four points, and these four points are the circumcentres of the triangles* *

$$I_1I_2I_3,\quad II_3I_2,\quad I_3II_1,\quad I_2I_1I.$$

FIGURE 63.

The triads of concurrent radii are

$$I_1D_1,\ I_2E_2,\ I_3F_3 \qquad ID,\ I_3E_3,\ I_2F_2$$
$$I_3D_3,\ IE,\ I_1F_1 \qquad I_2D_2,\ I_1E_1,\ IF$$

and the theorem follows at once from the converse of the first part of § 5, (32).

A second proof of the concurrency of these four triads may be derived from Oppel's theorem in § 2 and the expressions in § 4, (5).

* The results (1)–(7) are given by T. S. Davies in the *Philosophical Magazine*, II. 26-34 (1827). The concurrency of the first triad at the circumcentre of triangle $I_1I_2I_3$, and the length of the radius, $2R$, of that triangle were pointed out by Benjamin Bevan in Leybourn's ***Mathematical Repository***, new series, I. 18 (pagination of questions), 143 (pagination of Part I.) in 1804. Compare the subscripts in the designations of the four I triangles with the subscripts of the radii which meet at their circumcentres.

A third proof may be got from a theorem of Steiner*:

If the three perpendiculars from the vertices of one triangle on the sides of another are concurrent, the three corresponding perpendiculars from the vertices of the latter on the sides of the former are also concurrent.

The following proof is due to Mr W. J. C. Miller †:

Because

$$\angle E_2I_2A = \tfrac{1}{2}CAB = \angle F_3I_3A$$
$$\angle F_3I_3B = \tfrac{1}{2}ABC = \angle D_1I_1B$$
$$\angle D_1I_1C = \tfrac{1}{2}BCA = \angle E_2I_2C;$$

therefore I_1D_1, I_2E_2, I_3F_3 will meet in a point O_0 such that

$$O_0I_1 = O_0I_2 = O_0I_3;$$

hence O_0 is the circumcentre of $I_1I_2I_3$.

Similarly for the other triads, which meet at the points

$$O_1, \; O_2, \; O_3.$$

Def. Mr Lemoine has proposed ‡ to call triangles such as those of Steiner's theorem *orthologous*, and the points of concurrency of the perpendiculars *centres of orthology*.

Hence ABC is orthologous with each of the triangles

$$I_1I_2I_3, \quad II_3I_2, \quad I_3II_1, \quad I_2I_1I$$

and the respective centres of orthology are

$$I, O_0; \quad I_1, O_1; \quad I_2, O_2; \quad I_3, O_3.$$

(2) *The figures $O_0I_2O_1I_3$, $O_0I_3O_2I_1$, $O_0I_1O_3I_2$ are rhombi.*

For O_0I_2, O_1I_3 are perpendicular to AC

O_0I_3, O_1I_2 ,, ,, ,, AB;

and $O_0I_2 = O_0I_3$.

Hence O_0O_1, O_0O_2, O_0O_3

are respectively perpendicular to

$$I_2I_3, \quad I_3I_1, \quad I_1I_2.$$

* Crelle's *Journal*, II. 287 (1827), or Steiner's *Gesammelte Werke*, I. 157 (1881).

† *Lady's and Gentleman's Diary* for 1863, p. 54.

‡ *Journal de Mathématiques Spéciales*, 3rd series, III. 63 (1889), and the memoir *Sur les triangles orthologiques* read at the Limoges meeting (1890) of the *Association Française pour l'avancement des Sciences*.

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(3) O_2O_3, O_3O_1, O_1O_2

are respectively parallel to

I_2I_3, I_3I_1, I_1I_2.

For O_2I_3, O_3I_2 are perpendicular to BC;
and they are equal, since the radii of the circumcircles of the four I triangles are equal;
therefore $O_2O_3I_2I_3$ is a parallelogram.

(4) The four O triangles are congruent to their respective four I triangles, and their corresponding sides are parallel.

(5) The points O_0, O_1, O_2, O_3 are the orthocentres of the four O triangles taken in order.

(6) *The figures $IO_2I_1O_3$, $IO_3I_2O_1$, $IO_1I_3O_2$ are rhombi.*

For IO_2, I_1O_3 are perpendicular to AC

IO_3, I_1O_2 ,, ,, ,, AB;

and $IO_2 = IO_3$,

since the radii of the circumcircles of the four I triangles are equal.

(7) The points I, I_1, I_2, I_3 are the circumcentres of the four O triangles taken in order.

(8) IO_0, I_1O_1, I_2O_2, I_3O_3

are the Euler's lines of the four I triangles, and of the four O triangles, and the circumcentre O of ABC is the mid point of each of them.

(9) By referring to the Section* on the nine-point circle, it will be seen that the circumcircle of ABC is the nine-point circle of the eight I and O triangles, and that the radii of the circumcircles of these eight triangles are each 2R.

It will also be seen that the circumcircle of ABC bisects each of the six straight lines

II_1, II_2, II_3, I_2I_3, I_3I_1, I_1I_2

* *Proceedings of the Edinburgh Mathematical Society*, XI. 19-57 (1893).

at U , V , W , U' , V' , W' ;

and that UU', VV', WW'

are the diameters of the circumcircle ABC perpendicular to

BC , CA , AB.

(10) $U'V'W'$, $U'VW$, $UV'W$, UVW'

are the complementary triangles of the four I triangles taken in order.

(11) O_0O_1 , O_0O_2 , O_0O_3 , O_2O_3 , O_3O_1 , O_1O_2

pass respectively through the points

U' , V' , W' , U , V , W .

(12) The following pairs of straight lines intersect on the circumcircle of ABC:

O_0O_1 , O_2O_3 ; O_0O_2 , O_3O_1 ; O_0O_3 , O_1O_2

at U_1 , V_1 , W_1 .

(13) Triangle $U_1V_1W_1$ bears to $O_0O_1O_2$ exactly the same relations that ABC does to $I_1I_2I_3$.

FIGURE 64.

(14) Of the four I triangles taken in order let

G_0 , G_1 , G_2 , G_3

be the centroids; then the concurrency of

I_1U', I_2V', I_3W'	determines	G_0
IU', I_3V, I_2W	,,	G_1
I_3U, IV', I_1W	,,	G_2
I_2U, I_1V, IW'	,,	G_3.

(15)

G_0	lies on	IO_0	and	$IG_0 = 2O_0G_0$
G_1	,, ,,	I_1O_1	,,	$I_1G_1 = 2O_1G_1$
G_2	,, ,,	I_2O_2	,,	$I_2G_2 = 2O_2G_2$
G_3	,, ,,	I_3O_3	,,	$I_3G_3 = 2O_3G_3$.

(16) Through O pass *

IG_0 , I_1G_1 , I_2G_2 , I_3G_3 and

$OI = 3OG_0$, $OI_1 = 3OG_1$, $OI_2 = 3OG_2$, $OI_3 = 3OG_3$.

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* Thomas Weddle in the *Lady's and Gentleman's Diary* for 1849, p. 76.

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(17) If through G_0 parallels be drawn to

$$O_0U', \quad O_0V', \quad O_0W'$$

these parallels will meet

$$IU', \quad IV', \quad IW'$$

at $$G_1, \quad G_2, \quad G_3 .$$

(18) $G_1G_2G_3$ is directly similar to $U'V'W'$ and $I_1I_2I_3$, the ratio of similitude in the first case being $2:3$, and in the second $1:3$.

(19) I, U, V, W; O_0, U', V', W'; G_0, G_1, G_2, G_3 form orthic tetrastigms.

FIGURE 63.

(20) The areas of the six rhombi *

$$O_0I_2O_1I_3, \quad O_0I_3O_2I_1, \quad O_0I_1O_3I_2$$
$$IO_2I_1O_3, \quad IO_3I_2O_1, \quad IO_1I_3O_2$$

are $$2Ra, \quad 2Rb, \quad 2Rc .$$

(21) The areas of the three parallelograms †

$$I_2I_3O_2O_3, \quad I_3I_1O_3O_1, \quad I_1I_2O_1O_2$$

are $$2R(b+c), \quad 2R(c+a), \quad 2R(a+b) .$$

(22) The figure $I_1O_3I_2O_1I_3O_2$ is an equilateral hexagon ‡; its opposite sides are parallel ‡, and equal to the diameter of the circumcircle ‡ of ABC; its angles are the supplements§ of the angles of ABC; and its area § is equal to the sum of the areas of $I_1I_2I_3$ and $O_1O_2O_3$, that is equal to $4Rs$.

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* The last three are given by Rev. William Mason of Normanton in the *Lady's and Gentleman's Diary* for 1863, p. 53.

† Mr S. Constable in the *Educational Times*, XXXI. 113 (1878).

‡ T. S. Davies in the *Philosophical Magazine*, II. 32 (1827).

§ Rev. William Mason in the *Lady's and Gentleman's Diary* for 1863, p. 54.

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