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$\text{Pin}^{\pm}$-structures on non-oriented 4-manifolds via Lefschetz fibrations

Valentina Bais
Department of Mathematics, SISSA, via Bonomea 265, 34136 Trieste, Italy (vbais@sissa.it)

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We study necessary and sufficient conditions for a 4-dimensional Lefschetz fibration over the 2-disk to admit a $\text{Pin}^{\pm}$-structure, extending the work of A. Stipsicz in the orientable setting. As a corollary, we get existence results of Pin^{+} and Pin^{-}-structures on closed non-orientable 4-manifolds and on Lefschetz fibrations over the 2-sphere. In particular, we show via three explicit examples how to read-off $\text{Pin}^{\pm}$-structures from the Kirby diagram of a 4-manifold. We also provide a proof of the well-known fact that any closed 3-manifold M admits a Pin^{-}-structure and we find a criterion to check whether or not it admits a Pin^{+}-structure in terms of a handlebody decomposition. We conclude the paper with a characterization of Pin^{+}-structures on vector bundles.

Keywords: 4-manifold; Lefschetz fibration; non-orientable; Pin structure; η-invariant

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1. Introduction

$\text{Pin}^{\pm}$-structures can be thought of as the non-oriented analogue of Spin-structures. We refer the reader to [17] and [24, Section 2] for a precise definition of such Lie groups and for the background on $\text{Pin}^{\pm}(n)$-structures on low dimensional manifolds. In this note, we find necessary and sufficient conditions for a (possibly non-orientable) Lefschetz fibration over the 2-disk to support a Pin^{-} or a Pin^{+}-structure. In particular, such conditions are expressed in terms of the homology classes of the vanishing cycles of the Lefschetz fibration, with coefficients taken in $\mathbb{Z}_2$ and $\mathbb{Z}_4$ respectively. As an application of this criterion, we provide explicit necessary and sufficient conditions for a given Lefschetz fibration over the 2-sphere to support a $\text{Pin}^{\pm}$-structure. We refer the reader to Section 3 for a quick recap on basic facts about non-orientable Lefschetz fibrations. Our main reference for this subject is [19]. We also remark that in this note we consider $\text{Pin}^{\pm}$-structures on the tangent bundles of the manifolds under consideration, while in [13] they are defined on the normal bundles of embedded submanifolds.

In Section 4, we prove the following statement, extending the work of A. Stipsicz on Spin-structures over Lefschetz fibrations in [23, Theorem 1.1].

THEOREM 1. *Let X be a smooth 4-manifold and $f : X \to D^2$ a Lefschetz fibration with regular fibre Σ. There is no Pin$^-$-structure on X if and only if there are $k+1$ vanishing cycles $c_0, c_1, \ldots, c_k$ such that $[c_0] = \sum_{i=1}^{k} [c_i] \in H_1(\Sigma; \mathbb{Z}_2)$ and $k + \sum_{1 \le i < j \le k} c_i \cdot c_j \equiv 0 \mod 2$.*

As we will see in Section 4, Theorem 1 is a consequence of the fact that Pin$^-$-structures on a Lefschetz fibration over the 2-disk with regular fibre Σ naturally correspond to maps

$$q^- : H_1(\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_4$$

such that $q^-(x+y) = q^-(x) + q^-(y) + 2x \cdot y$ for every $x, y \in H_1(\Sigma; \mathbb{Z}_2)$ and $q^-([c]) = 2$ on every vanishing cycle $c \subset \Sigma$, see Lemma 1. Moreover, such correspondence is equivariant with respect to the action of $H^1(X; \mathbb{Z}_2)$.

Section 5 is devoted to the study of Pin$^+$-structures on Lefschetz fibrations. By [7], a Pin$^+$-structure on the regular fibre Σ corresponds to a map

$$q_0^+ : H_1(\Sigma; \mathbb{Z}_4) \to \mathbb{Z}_2$$

with the property that $q_0^+(x+y) = q_0^+(x) + q_0^+(y) + x \cdot y \in \mathbb{Z}_2$ for all $x, y \in H_1(\Sigma; \mathbb{Z}_4)$. We show that we can choose q_0^+ to be the restriction of a Pin$^+$-structure defined on the whole fibration if and only if the following holds.

THEOREM 2. *The total space of the Lefschetz fibration $f : X \to D^2$ with vanishing cycles $c_1, \ldots, c_n \subset \Sigma$ supports a Pin$^+$-structure if and only if Σ supports a Pin$^+$-structure and*

$$\mathit{rank}(C) = \mathit{rank}(C \mid A)$$

where C is the $\mathbb{Z}_2$-reduction of the $n \times r$ matrix (c_{ij}) whose rows are given by the components of $[c_1], \ldots, [c_n] \in H_1(\Sigma; \mathbb{Z}_4) \cong \mathbb{Z}^r$ with respect to a fixed basis $e_1, \ldots, e_r$ of the free $\mathbb{Z}_4$-module $H_1(\Sigma; \mathbb{Z}_4)$ and A is the column vector with entries $A_i = 1 + q_0^+([c_i]) \in \mathbb{Z}_2$ for $i = 1, \ldots, n$.

A key tool in the proof of the above result is Lemma 2, in which we show that Pin$^+$-structures on X correspond to maps

$$q^+ : H_1(\Sigma; \mathbb{Z}_4) \to \mathbb{Z}_2$$

such that $q^+(x+y) = q^+(x) + q^+(y) + x \cdot y$ for every $x, y \in H_1(\Sigma; \mathbb{Z}_4)$ and $q^+([c]) = 1$ on every vanishing cycle $c \subset \Sigma$. The correspondence is also in this case equivariant with respect to the $H^1(X; \mathbb{Z}_2)$-action.

We remark that the study of Pin$^+$ and Pin$^-$-structures on non-orientable 4-manifolds is essentially reduced to the case of non-orientable Lefschetz fibrations over the 2-disk due to the following result, which is a straightforward consequence of [19, Theorem 1.1].

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THEOREM 3 (Miller-Özbağcı [19]). *Let X be a closed non-orientable smooth 4-manifold. There is a decomposition*

$$X = L \cup_\partial H$$

where L is a non-orientable Lefschetz fibration over the 2-disk and H is a non-orientable 1-handlebody.

In particular, the proof of [19, Theorem 1.1] shows an explicit way of endowing a (possibly non-orientable) 2-handlebody with the structure of a Lefschetz fibration over the 2-disk, see also [14] and [8]. Moreover, a 4-manifold $X = L \cup_\partial H$ as in Theorem 3 admits a Pin$^\pm$-structure if and only if L does. This latter conditions can be checked using Theorem 1 and Theorem 2.

In Section 6 we provide explicit examples in which, starting from a handlebody decomposition of a non-orientable 4-manifold, we endow its 2-handlebody with the structure of a Lefschetz fibration over the 2-disk. We then apply our results in order to check whether such 4-manifolds support a Pin$^-$ or a Pin$^+$-structure. In the case they do, we find all the possible structures by looking at the $H^1(X, \mathbb{Z}_2)$-action on the associated quadratic enhancements on the regular fibres. The non-orientable handlebodies under consideration describe the three 4-manifolds $\mathbb{RP}^4$, $S^2 \widetilde{\times} \mathbb{RP}^2$ and $S^2 \times \mathbb{RP}^2$ and can be found in [2], [4], [18] and [25]. In particular, that there are two non-orientable S^2-bundles over $\mathbb{RP}^2$ and we denote by $S^2 \widetilde{\times} \mathbb{RP}^2$ the non-trivial one, see [4, Section 2.3].

Section 7 is devoted of the study of necessary and sufficient conditions for the existence of a Pin$^\pm$-structure on a Lefschetz fibration over the 2-sphere. This is the non-orientable version of [23, Theorem 1.3].

THEOREM 4. *Let $f : X \to S^2$ be a (possibly non-orientable) Lefschetz fibration with regular fibre Σ. Then X supports a Pin$^-$-structure if and only if $X \setminus \nu(\Sigma)$ does and there exists a smoothly embedded surface $\sigma \subset X$ which is dual to Σ in $H_2(X; \mathbb{Z}_2)$ and such that*

$$[\sigma]^2 + (w_1(\sigma) \cup w_1(\nu(\sigma))([\sigma]) + w_1^2(\nu(\sigma)) = 0 \in \mathbb{Z}_2.$$

Analogously, X supports a Pin$^+$-structure if and only if $X \setminus \nu(\Sigma)$ does and there exists a smoothly embedded surface $\sigma \subset X$ which is dual to Σ in $H_2(X; \mathbb{Z}_2)$ and such that

$$\chi(\sigma) + [\sigma]^2 + (w_1(\sigma) \cup w_1(\nu(\sigma)))([\sigma]) = 0 \in \mathbb{Z}_2.$$

In analogy to the Spin case, one can define the Pin$^+$ and Pin$^-$ cobordism groups in each dimension, see [17, Section 1]. As sets, they consist of the equivalence classes of closed n-dimensional smooth manifolds with a fixed Pin$^+$ (resp. Pin$^-$)-structure, which are identified whenever they co-bound a Pin$^+$ (resp. Pin$^-$) $(n + 1)$-manifold. The group operation is the one induced by the disjoint union of manifolds. We remark that Pin$^+$ and Pin$^-$ cobordism groups are drastically different. In particular, the 4-dimensional Pin$^+$-cobordism group is

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$$\Omega_4^{\mathrm{Pin}^+} \cong \mathbb{Z}_{16}$$

(see [10, Section 2] and [17, Section 3]) and the η-invariant modulo $2\mathbb{Z}$ of the twisted Dirac operator associated to a Pin^+-structure on a 4-manifold is a complete invariant for Pin^+-cobordism classes, see [24]. In particular, there are instances in which such invariant can detect exotic behaviours in the non-orientable 4-dimensional realm, in the sense that there are examples of non-orientable Pin^+ 4-manifolds which share the same homeomorphism type but define two different classes in $\Omega_4^{\mathrm{Pin}^+}$ and hence cannot be diffeomorphic. To the best of our knowledge, following a suggestion in [11], this was shown for the first time in [24], where the η-invariant of Pin^+-structures is used to detect the exotic $\mathbb{RP}^4$ constructed in [5]. We remark that another exotic $\mathbb{RP}^4$ is constructed in [9] and its η-invariant is computed in [21]. An analogous approach has then been used also in [4] and [25]. On the other hand

$$\Omega_4^{\mathrm{Pin}^-} = 0$$

(see again [3, Theorem 5.1] and [17, Section 3]) and hence it is not possible to find Pin^- exotic 4-manifolds using this strategy. In particular, the study of Pin^+-structures on 4-manifolds could potentially lead to better understanding of exotic behaviours in the non-orientable realm and this fact is one of the main motivations for this note. In particular, in Section 5 we show that if $X = L \cup H$ is a closed 4-manifold with a decomposition as in the statement of Theorem 3, then the datum of a Pin^+-structure on X is equivalent to the datum of a map

$$q^+ : H_1(\Sigma; \mathbb{Z}_4) \to \mathbb{Z}_2$$

such that $q^+(x+y) = q^+(x) + q^+(y) + x \cdot y$ for all $x, y \in H_1(\Sigma; \mathbb{Z}_4)$ taking value $1 \in \mathbb{Z}_2$ on all the vanishing cycles of L, where Σ denotes a regular fibre. We leave the reader with the following question.

Question: Is there a formula for the η-invariant modulo $2\mathbb{Z}$ of a Pin^+-structure on X in terms of the corresponding map $q^+ : H_1(\Sigma; \mathbb{Z}_4) \to \mathbb{Z}_2$?

Section 2 contains a short discussion on how to check the vanishing of w_2 and w_1^2 in terms of the embeddings of homologically essential surfaces inside the ambient 4-manifold. In Section 7 we study Lefschetz fibrations over the 2-sphere, 8 is devoted to a brief discussion on the 3-dimensional case, while Section 9 contains a characterization of Pin^+-structures on vector bundle which resembles Milnor's characterization of Spin-structures, see [20, Alternative definitions 2].

All manifolds in this note are smooth. We will not assume our 4-manifolds to be orientable, unless otherwise stated.

2. $\mathrm{Pin}^{\pm}$-structures and embedded surfaces

For every $n \in \mathbb{N}$, there are two distinct central extensions

$$\mathrm{Pin}^{\pm}(n) \to O(n) \tag{1}$$

which are topologically a disjoint union $\mathrm{Spin}(n) \cup \mathrm{Spin}(n)$. In the orientable setting, it is known that the existence of a Spin-structure on a real vector bundle ξ on a

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manifold X is equivalent to the vanishing of $w_2(\xi) \in H^2(X, \mathbb{Z}_2)$, where w_i is the ith Stiefel-Whitney class of ξ. In the case in which X is 4-dimensional, this equivalent to having an even intersection form, provided that $H^2(X;\mathbb{Z})$ has no 2-torsion [12, Corollary 5.7.6]. This can be shown via the Wu formula, which states that for any $a \in H_2(X;\mathbb{Z}_2)$ one has

$$\langle w_2(X), a\rangle = a^2 \text{ modulo } 2 \tag{2}$$

where a^2 denotes the algebraic count of self-intersections of a surface in X representing a, see [12, Proposition 1.4.18]. In a similar way, there are cohomological obstructions to the existence of Pin$^+$ and Pin$^-$-structures on the tangent bundle of a manifold, namely the vanishing of $w_2(X)$ and of $(w_2 + w_1^2)(X)$ respectively, see [17, Lemma 1.3]. Moreover, both Pin$^+$ and Pin$^-$-structures (when they do exist) are torsors over $H^1(X, \mathbb{Z}_2)$, meaning that there is a free and transitive action of $H^1(X;\mathbb{Z}_2)$ over the sets of such geometric structures. However, the Wu formula does not hold when X is non-orientable, but one can still interpret the vanishing of $w_2(X)$ and $(w_2 + w_1^2)(X)$ in terms of properties of embedded surfaces. Indeed, for any 4-manifold X and $a \in H_2(X;\mathbb{Z}_2)$ there is a (possibly non-orientable) smoothly embedded surface $\Sigma \subset X$ representing a (see [12, Remark 1.2.4]) and one can show that

$$\langle w_2(X), a\rangle = \chi(\Sigma) + (w_1(\Sigma) \smile w_1(\nu(\Sigma)))([\Sigma]) + [\Sigma]^2 \text{ modulo } 2. \tag{3}$$

Since the tangent bundle of a 4-manifold supports a Pin$^+$-structure if and only if $w_2(X)$ vanishes, this is equivalent to checking that the evaluation (3) is trivial on any homology class $a \in H_2(X;\mathbb{Z}_2)$.

On the other hand, the obstruction for Pin$^-$-structures to exist is $(w_2 + w_1^2)(X)$ [17, Lemma 1.3] so, in order to understand when such structures exist, we need to compute $w_1^2(X)$ and one can easily show that

$$\langle w_1^2(X), [\Sigma]\rangle = w_1^2(\Sigma) + w_1^2(\nu(\Sigma)). \tag{4}$$

Example 1 ($\mathbb{RP}^4$). The 4-dimensional real projective space $\mathbb{RP}^4$ is an example of non-orientable 4-manifold supporting two distinct Pin$^+$-structures but no Pin$^-$-structure. We have that $H_2(\mathbb{RP}^4;\mathbb{Z}_2) \cong \mathbb{Z}_2$ is generated by the homology class of $\mathbb{RP}^2$. Moreover, the tubular neighbourhood $\nu(\mathbb{RP}^2) \subset \mathbb{RP}^4$ is diffeomorphic to the twisted 2-disk bundle $D^2 \widetilde{\times} \mathbb{RP}^2$ defined as the quotient of $D^2 \times S^2$ via the involution

$$\begin{aligned} D^2 \times S^2 &\to D^2 \times S^2 \\ (x, y) &\mapsto (\rho_\pi(x), -y) \end{aligned}$$

where ρ_π denotes the rotation of S^2 of π radians about a fixed axis. One can show that such a quotient is diffeomorphic to

$$D^2 \widetilde{\times} \mathbb{RP}^2 \cong (D^2 \times D^2) \cup_\varphi (D^2 \times \mathrm{Mb}) \tag{5}$$

where Mb denotes the Möbius strip and φ is the map

$$\varphi : D^2 \times S^1 \to D^2 \times S^1, \quad (x, \theta) \mapsto (\rho_\theta(x), \theta)$$

and $\rho_\theta : D^2 \to D^2$ denotes the rotation of the 2-disk of angle θ with respect to the origin, see [1, Section 0] and [4, Example 7].

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By (3) we have that

$$\langle w_2(\mathbb{RP}^4), [\mathbb{RP}^2]\rangle = \chi(\mathbb{RP})^2 + (w_1(\mathbb{RP}^2) \cup w_1(\nu(\mathbb{RP}^2)))([\mathbb{RP}^2]) + [\mathbb{RP}^2]^2 = 0 \in \mathbb{Z}_2$$

since $\chi(\mathbb{RP}^2) = 1 = [\mathbb{RP}^2]^2$ and $w_1(\nu(\mathbb{RP}^2)) = 0$, while (4) implies that

$$\langle w_1(\mathbb{RP}^4)^2, [\mathbb{RP}^2]\rangle = w_1^2(\mathbb{RP}^2) + w_1^2(\nu(\mathbb{RP}^2)) = 1 + 0 = 1 \in \mathbb{Z}_2$$

where $w_1^2(\mathbb{RP}^2)$ is computed as the self-intersection of a loop in $\mathbb{RP}^2$ whose $\mathbb{Z}_2$-homology class generates $H_1(\mathbb{RP}^2; \mathbb{Z}_2)$.

3. Non-orientable Lefschetz fibrations

Since the conditions 3 and 4 are not immediate to check in the non-orientable setting and depend heavily on the embeddings of the homologically essential surfaces, in the following sections we develop another way to combinatorially understand when a 4-manifold is Pin^+ or Pin^- by means of a specific kind of decomposition. To do this, we will need the notion of non-orientable Lefschetz fibration over the 2-disk.

We start by recalling the definition of Lefschetz fibration. Our main reference for this topic in the non-orientable realm is [19].

Definition 1. *Let X be a compact, connected 4-manifold and let B be a compact, connected surface, both with possibly non-empty boundary. A Lefschetz fibration is a smooth submersion*

$$f : X \to B$$

away from finitely many points in the interior of B such that each fibre contains at most one critical point and f is a fibre bundle with surface fibre on the complement of the critical values. Moreover, around each critical point, we require f to conform to the local complex model

$$(z_1, z_2) \to z_1 z_2.$$

Remark 1. If X is non-orientable and B is oriented, each regular fibre is a non-orientable surface. In this case it makes no sense to ask for the orientation of the local model around a critical point to be compatible to the one of X.

In the following, we will just consider the case in which $B = D^2$ and Σ is the regular fibre. It is possible to show that every Lefschetz fibration

$$f : X \to D^2$$

is obtained from the product one

$$\Sigma \times D^2 \to D^2$$

by gluing 4-dimensional 2-handles along $\Sigma \times \partial D^2$ with framing ± 1 with respect to the fibre framing. The attaching curves of such 2-handles are push-offs in distinct fibres of simple closed curves inside Σ, which are called the vanishing cycles of the fibration and are denoted in the following by $c_1, \ldots, c_n \subset \Sigma$. Moreover,

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$\langle w_1(X), [c_i]\rangle = 0$ for all $i = 1, \dots, n$ and hence the tubular neighbourhood of these attaching regions is necessarily diffeomorphic to a product of S^1 with the unit interval. We will call *two-sided* all curves with this property.

4. Pin$^-$-structures on 4-manifolds via Lefschetz fibrations

It is well known that the datum of a Spin-structure over an orientable surface is equivalent to the one of a quadratic enhancement

$$s : H_1(\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_2$$

satisfying the condition

$$s(x + y) = s(x) + s(y) + x \cdot y$$

for all $x, y \in H_1(\Sigma; \mathbb{Z}_2)$, where $\cdot$ denotes the $\mathbb{Z}_2$-intersection number between cycles, see [16] and [23]. A geometric interpretation of this algebraic object can be given as follows. The 1-dimensional Spin-cobordism group is $\Omega_1^{\mathrm{Spin}} \cong \mathbb{Z}_2$ [20] and every closed simple curve $\gamma \subset \Sigma$ in a Spin surface inherits a Spin-structure. In particular, if s is the quadratic enhancement associated to the fixed Spin-structure on Σ, then $s([\gamma]) = 0$ if and only if the induced Spin-structure on γ is the one bounding the unique Spin-structure on the 2-disk.

It is possible to give a similar description for Pin$^-$ structures on (not necessarily orientable) surfaces, as shown by the following result.

Theorem 5 (Kirby-Taylor, [17]). *There is a 1:1 correspondence between Pin$^-$-structures on a surface Σ and quadratic enhancements*

$$q^- : H_1(\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_4$$

with the property that

$$q^-(x + y) = q^-(x) + q^-(y) + 2\ x \cdot y$$

for any $x, y \in H_1(\Sigma; \mathbb{Z}_2)$.

In particular, if $\gamma \subset \Sigma$ is a simple closed curve, then $q^-([\gamma])$ is even if and only if γ is two-sided and this is the case whenever γ is a vanishing cycle for a Lefschetz fibration. Moreover, recall that the 1-dimensional Pin$^-$-cobordism group is $\Omega_1^{\mathrm{Pin}^-} \cong \mathbb{Z}_2$ (see [3, Theorem 5.1] and [17, Section 0]) and every closed simple curve $\gamma \subset \Sigma$ in a Pin$^-$ surface inherits a Pin$^-$ structure. We have that $q^-([\gamma]) = 0$ if and only if γ inherits the bounding Pin$^-$-structure.

In order to prove Theorem 1, we will need the following Lemma.

Lemma 1. *Let $f : X \to D^2$ be a Lefschetz fibration with regular fibre Σ and vanishing cycles $c_1, \dots, c_n \subset \Sigma$. There is a natural one to one correspondence between the set of Pin$^-$-structures on X and the set of quadratic enhancements*

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$$q^- : H_1(\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_4$$

such that $q^-(x+y) = q^-(x)+q^-(y)+2x\cdot y$ *for all* $x, y \in H_1(\Sigma; \mathbb{Z}_2)$ *and* $q^-([c_i]) = 2$ *for all* $i = 1, \dots, n$. *Moreover, such correspondence is equivariant with respect to the free and transitive action of* $H^1(X; \mathbb{Z}_2)$.

Proof. The existence of a Pin$^-$-structure on X is equivalent to the existence of a Pin$^-$-structure on Σ that extends to the 2-handles. This follows from the fact that any Pin$^-$-structure on X induces by restriction a Pin$^-$ structure on a (trivial) tubular neighbourhood $\nu(\Sigma) \cong \Sigma \times D^2$ and all the Pin$^-$-structures on $\Sigma \times D^2$ are pull-backs of the ones on Σ via the projection map $\Sigma \times D^2 \to \Sigma$, see also the proof of [23, Theorem 1.1]. Since such 2-handles are attached to $\Sigma \times \partial D^2$ with relative odd framing, this corresponds to a Pin$^-$-structure on Σ restricting to the non-bounding one on every vanishing cycle. The conclusion follows from Theorem 5.

The free and transitive action of $H^1(X; \mathbb{Z}_2)$ on the set of Pin$^-$-structures on X can be seen as follows. In [17, Section 3], it is shown that $H^1(\Sigma; \mathbb{Z}_2)$ acts on the set of Pin$^-$-structures of the surface Σ by

$$q^-_\gamma(x) = q^-(x) + 2 \cdot \gamma(x) \tag{6}$$

for all $\gamma \in H^1(\Sigma; \mathbb{Z}_2)$ and $x \in H_1(\Sigma; \mathbb{Z}_2)$. In particular, Pin$^-$-structures on Σ equivariantly correspond to quadratic enhancements and the action (6) is well defined on $H^1(X; \mathbb{Z}_2) \subset H^1(\Sigma; \mathbb{Z}_2) \cong H^1(X; \mathbb{Z}_2) \oplus K$, where $K \subset H^1(\Sigma; \mathbb{Z}_2)$ is the kernel of the map induced by the inclusion $\Sigma \subset X$. □

In light of this fact, it is possible to characterize the existence of Pin$^-$-structures on non-orientable Lefschetz fibrations over the 2-disk in terms of the $\mathbb{Z}_2$-homology classes of the vanishing cycles.

Proof of Theorem 1. By Lemma 1, X does not support any Pin$^-$-structure if and only if it is not possible to build a map

$$q^- : H_1(\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_4$$

such that $q^-(x+y) = q^-(x) + q^-(y) + 2x \cdot y$ for all $x, y \in H_1(\Sigma; \mathbb{Z}_2)$ and such that $q^+([c_i]) = 2$ for all $i = 1, \dots, n$. The remaining part of the proof is essentially the same as the proof of [23, Theorem 1.1]. We will now sketch it for the convenience of the reader. Let $V \subset H_1(\Sigma; \mathbb{Z}_2)$ be the vector subspace generated by the classes of the vanishing cycles. Notice that any given enhancement q^- on V can be extended to the whole $H_1(\Sigma; \mathbb{Z}_2)$ by setting $q^- = 0$ on a basis $u_1, \dots, u_k$ of an orthogonal vector subspace and using the relation (6) to define it on the vectors of the form $\sum_j u_{i_j} + v$, where $v \in V$. In particular, in order to define a Pin$^-$-structure on X, one needs to find a quadratic enhancement $q^- : V \to \mathbb{Z}_4$ which takes value 2 on the class of any vanishing cycle. Let $v_1, \dots, v_r$ be a basis of V represented by vanishing cycles and let $V_i = \langle v_1, \dots, v_i \rangle$ for $i = 1, \dots, r$. We then want to define q^- inductively on each V_i as in the proof of [23, Theorem 1.1], setting $q^-(v_i) = 2$ at each step. One then notices that things go wrong if and only if there are k classes $v_{i_1}, \dots, v_{i_k}$ such that their sum $v_0 = \sum_{j=1}^k v_{i_j} \in H_1(\Sigma; \mathbb{Z}_2)$ is again represented by

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a vanishing cycle and $q^-(v_0) \neq 2 \in \mathbb{Z}_4$. This implies that $q^-(v_0) = 0 \in \mathbb{Z}_4$, being v_0 is represented by a curve with trivial tubular neighbourhood. Since by construction we had that $q^-(v_{i_j}) = 2$ for $j = 1, \ldots, k$, using (6) the condition $q^-(q_0) = 0 \in Z_4$ translates into

$$q^-(v_0) = \sum_{j=1}^{k} q^-(v_{i_j}) + 2\cdot\Big(\sum_{1\leq i_n \leq i_m \leq k} v_{i_n}\cdot v_{i_m}\Big) = 2\cdot\Big(k + \sum_{1\leq i_n\leq i_m\leq k} v_{i_n}\cdot v_{i_m}\Big) = 0 \in \mathbb{Z}_4.$$

The conclusion then follows by noticing that $2 \cdot x = 0 \in \mathbb{Z}_4$ if and only if $x = 0 \in \mathbb{Z}_2$. □

Note that, if we restrict to orientable surfaces, there is a natural map

$$\Omega_1^{\mathrm{Spin}} \to \Omega_1^{\mathrm{Pin}^-}$$

giving a group isomorphism, see [17, Theorem 2.1]. At the level of the associated quadratic forms, the enhancement

$$s : H_1(\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_2$$

corresponds to

$$q^- : H_1(\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_4$$

given by

$$q^-(x) = 2 \cdot s(x)$$

for every $x \in H_1(\Sigma; \mathbb{Z}_2)$, where $2\cdot$ denotes the inclusion $\mathbb{Z}_2 \subset \mathbb{Z}_4$. In particular, the condition on the vanishing cycles we found in Theorem 1 coincides with the one in [23, Theorem 1.1] and this is due to the fact that, when restricting to orientable Lesfschetz fibrations over D^2, being Spin coincides with being Pin^-.

5. Pin^+-structures on Lefschetz fibrations over the 2-disk

As a consequence of [7, Theorem A], there is a canonical affine bijective correspondence between Pin^+-structures on a surface Σ and the set of maps

$$q^+ : H_1(\Sigma; \mathbb{Z}_4) \to \mathbb{Z}_2 \tag{7}$$

with the property that

$$q^+(x + y) = q^+(x) + q^+(y) + x \cdot y \tag{8}$$

for every $x, y \in H_1(\Sigma; \mathbb{Z}_4)$, where $x \cdot y \in \mathbb{Z}_2$ denotes the algebraic intersection modulo 2 between representatives of the $\mathbb{Z}_2$-reductions of the classes $x, y \in H_1(\Sigma; \mathbb{Z}_4)$.

Recall that the 1-dimensional Pin^+-cobordism group is

$$\Omega_1^{\mathrm{Pin}^+} \cong \{0\}$$

see [17, Section 3]. However, S^1 has two distinct Pin^+-structures. These correspond to two different trivializations of the direct sum $TS^1 \oplus \varepsilon$ of the tangent bundle of

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S^1 with a trivial line bundle, where the first one is given by the restriction to S^1 of a trivialization of the tangent bundle TD^2 of the 2-disk and the second one comes from the Lie group framing of S^1. In particular, one can show that such structures have the property that they respectively bound the 2-disk and the Möbius strip, see [17, Section 1]. A close look at the construction of q^+ starting from a Pin$^+$-structure on Σ shows that $q^+([\gamma]) = 0$ if the induced Pin$^+$-structure on γ is the one bounding a 2-disk, while $q^+([\gamma]) = 1$ if it is the one bounding the Möbius strip, see [17]. Note that, given any two maps q_1^+, q_2^+ corresponding to Pin$^+$-structures on Σ, their difference can be regarded as an element

$$q_1^+ - q_2^+ \in \mathrm{Hom}(H_1(\Sigma;\mathbb{Z}_4), \mathbb{Z}_2).$$

The proof of Theorem 2 is based on the following Lemma.

LEMMA 2. *Let $f : X \to D^2$ be a Lefschetz fibration with regular fibre Σ and vanishing cycles $c_1, \ldots, c_n \subset \Sigma$. There is a natural bijection between the set of Pin$^+$-structures on X and the set of quadratic enhancements*

$$q^+ : H_1(\Sigma;\mathbb{Z}_4) \to \mathbb{Z}_2$$

such that $q^+(x+y) = q^+(x) + q^+(y) + x \cdot y$ for all $x, y \in H_1(\Sigma;\mathbb{Z}_4)$ and $q^+([c_i]) = 1$ for all $i = 1, \ldots, n$. Moreover, such correspondence is equivariant with respect to the free and transitive action of $H^1(X;\mathbb{Z}_2)$.

Proof. As is the Pin$^-$ case, the existence of a Pin$^+$-structure on X is equivalent to the existence of a Pin$^+$-structure on Σ that extends to the 2-handles. Since such 2-handles are attached to $\Sigma \times \partial D^2$ with relative odd framing, this corresponds to a Pin$^+$-structure on Σ restricting to the one bounding the Möbius strip on every vanishing cycle. The conclusion follows from [7].

In the Pin$^+$-case, $H^1(\Sigma;\mathbb{Z}_2)$ acts on the set of Pin$^+$-structures of the surface Σ by

$$q_\gamma^+(x) = q^+(x) + \gamma(x) \tag{9}$$

for all $\gamma \in H^1(\Sigma;\mathbb{Z}_2)$ and $x \in H_1(\Sigma;\mathbb{Z}_4)$, see [7]. In particular, Pin$^+$-structures on Σ equivariantly correspond to quadratic enhancements and the action (9) is well defined on $H^1(X;\mathbb{Z}_2) \subset H^1(\Sigma;\mathbb{Z}_2) \cong H^1(X;\mathbb{Z}_2) \oplus K$, where $K \subset H^1(\Sigma;\mathbb{Z}_2)$ is the kernel of the map induced by the inclusion $\Sigma \subset X$. □

Let $f : X \to D^2$ be a Lefschetz fibration over the 2-disk with surface fibre Σ. Let $c_1, \ldots, c_n \subset \Sigma$ be its vanishing cycles. Note that one can find simple closed curves $e_1, \ldots, e_g$ inducing a basis of $H_1(\Sigma;\mathbb{Z}_2) \cong \mathbb{Z}_2^g$. From now on, we will assume that Σ has a Pin$^+$-structure, since this is a necessary condition for X to be a Pin$^+$-manifold. In particular, this means that Σ is not a closed non-orientable surface of odd Euler characteristic. Let

$$q_0^+ : H_1(\Sigma;\mathbb{Z}_4) \to \mathbb{Z}_2 \tag{10}$$

be the quadratic enhancement associated to a fixed Pin$^+$-structure on Σ and let $c_j = \sum_{i=1}^r c_{ji}[e_i]$. This gives the setting of Theorem 2, of which we now provide a proof.

Proof of Theorem 2. Lemma 2 implies that X has a Pin$^+$-structure if and only if there is a map

$$q^+ : H_1(\Sigma, \mathbb{Z}_4) \to \mathbb{Z}_2$$

such that $q^+(x+y) = q^+(x)+q^+(y)+x\cdot y$ for all $x, y \in H_1(\Sigma; \mathbb{Z}_4)$ and $q^+([c_i]) = 1$ for all $i = 1, \dots, n$. In particular, every such q^+ is necessarily of the form $q^+ = q_0^+ + l$ for some linear map $l = \sum_{i=1}^r \lambda_i [e^i]$, where $[e^i] \in H^1(\Sigma; \mathbb{Z}_2)$ denotes the dual element to $[e_i] \in H_1(\Sigma; \mathbb{Z}_2)$. In particular, this reduces the problem to finding $l = \sum_{i=1}^r \lambda_i e^i$ such that for every $j = 1, \dots, n$ we have

$$q^+([c_j]) = q_0^+([c_j]) + l([c_j]) = q_0^+([c_j]) + \sum_{i=1}^r c_{ji}\lambda_i = 1.$$

The conclusion follows from Rouché-Capelli's Theorem [22, Theorem 2.38]. □

Remark 2. The proof of Theorem 1 as well as the one of [23, Theorem 1.1] rely on the fact that $H_1(\Sigma; \mathbb{Z}_2)$ is a vector space. This is why we cannot adopt the same strategy for Theorem 2, since $H_1(\Sigma; \mathbb{Z}_4)$ is just a $\mathbb{Z}_4$-module.

6. Some examples

In this section, we show how to apply the results proven so far. In particular, starting from the handlebody decomposition of some small non-orientable 4-manifold M, we endow the union of handles up to index 2 with the structure of a Lefschetz fibration over the 2-disk, following the procedure explained in the proof of [19, Theorem 1.1]. At this point, we apply Theorem 1 and Theorem 2 to understand whether or not M admits a Pin$^+$ or a Pin$^-$-structure, describing the action of $H^1(M; \mathbb{Z}_2)$ in terms of the action of a subgroup of $H^1(\Sigma; \mathbb{Z}_2)$ on the quadratic enhancements on the non-singular fibre Σ of the Lefschetz fibration. We refer the reader to [1, Section 1.5] for the conventions and backgound on Kirby diagrams of non-orientable 4-manifolds, see also [2], [4, Section 2.1], [6] and [18]. The procedure we describe in this section applies to any closed non-orientable 4-manifold represented by means of a Kirby diagram.

Example 2 ($\mathbb{RP}^4$). Figure 1 represents a Kirby diagram for the standard handlebody decomposition of $\mathbb{RP}^4$ with a single k-handle for each $k = 0, \dots, 4$. In particular, the union of all handles up to index 2 corresponds to a tubular neighbourhood $\nu(\mathbb{RP}^2) \cong D^2 \tilde{\times} \mathbb{RP}^2$ of $\mathbb{RP}^2$ (see [2, Section 0] and [4, Section 2.5]) and $\mathbb{RP}^4$ is the result of attaching a 3-handle and a 4-handle to this handlebody. Recall that 3-handles and 4-handles need not be drawn by [6] and [18], see [4, Example 7].

The union of handles up to index 2 can be seen as a Lefschetz fibration over the 2-disk with the Möbius band Mb as fibre and a single vanishing cycle c_1, given by the projection of the $(1, 0)$-framed 2-handle, see Figure 2.

A Pin$^+$-structure on Mb extending to $\mathbb{RP}^4$ is given by the quadratic enhancement

$$q^+ : H_1(\mathrm{Mb}; \mathbb{Z}_4) \to \mathbb{Z}_2 \tag{11}$$

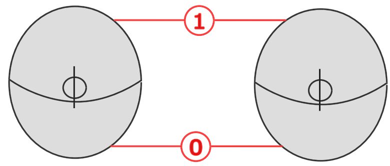

Figure 1. A Kirby diagram of $\mathbb{RP}^4$.

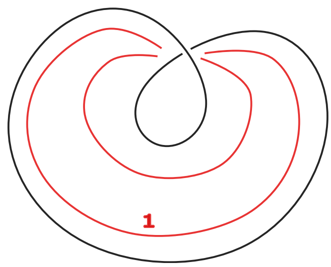

Figure 2. The Lefschetz fibration associated to the 2-handlebody of $\mathbb{RP}^4$.

defined by the condition $q^+([e_1]) = 1$, where $[e_1]$ is the generator of $H_1(\mathrm{Mb};\mathbb{Z}_4) \cong \mathbb{Z}_4$ represented by the core of the band. Indeed, $[c_1] = 2[e_1]$ and hence

$$q^+([c_1]) = q^+(2[e_1]) = q^+([e_1]) + q^+([e_1]) + e_1^2 = 1 \in \mathbb{Z}_2.$$

The other Pin^+-structure

$$q^+_x(y) = q^+(y) + x \cdot y \quad \text{for all } y \in H_1(\mathrm{Mb};\mathbb{Z}_4)$$

is obtained by acting on q^+ by the cohomology class dual to $[e_1]$, where $x \cdot y$ indicates the intersection number between x and the $\mathbb{Z}_2$-reduction of y. Note that the condition $q^+_x([c_1]) = 1$ is still satisfied.

On the other hand, if $\mathbb{RP}^4$ supported a Pin^--structure, then there would be a quadratic enhancement

$$q^- : H_1(\mathrm{Mb};\mathbb{Z}_2) \to \mathbb{Z}_4$$

satisfying $q^-([c_1]) = 2$. But this is impossible, since it would imply that

$$2 = q^-([c_1]) = q^-(2[e_1]) = q^-(0) = 0 \in \mathbb{Z}_4.$$

Example 3 ($S^2 \mathbin{\tilde{\times}} \mathbb{RP}^2$). The non-orientable 4-manifold $S^2 \mathbin{\tilde{\times}} \mathbb{RP}^2$ is obtained as a quotient of $S^2 \times S^2$ under the orientation-reversing involution

$$\begin{aligned} \iota : S^2 \times S^2 &\to S^2 \times S^2 \\ (x, y) &\mapsto (-x, \rho_\pi(y)) \end{aligned}$$

where $\rho_\theta : S^2 \to S^2$ is the rotation of S^2 of angle θ about a fixed axis for any $\theta \in S^1$, see [15, Chapter 12]. Moreover, it can also be seen as the result of a Gluck twist on

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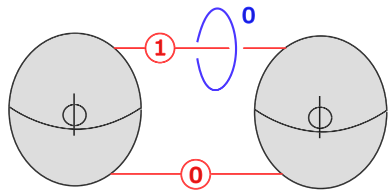

Figure 3. A Kirby diagram of $S^2 \mathbin{\widetilde{\times}} \mathbb{RP}^2$.

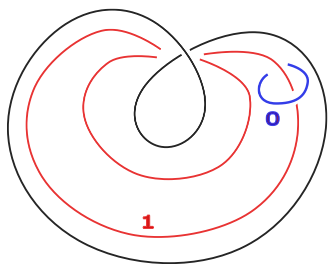

Figure 4. Lefschetz fibration associated to the 2-handlebody of $S^2 \mathbin{\widetilde{\times}} \mathbb{RP}^2$.

the product $S^2 \times \mathbb{RP}^2$ along a 2-sphere fibre, see [4, Lemma 1]. In particular, there is a decomposition

$$S^2 \mathbin{\widetilde{\times}} \mathbb{RP}^2 = (S^2 \times D^2) \cup_{\varphi} (S^2 \times \mathrm{Mb})$$

where the gluing map is

$$\begin{aligned} \varphi : S^1 \times S^1 &\to S^2 \times S^2 \\ (x, \theta) &\mapsto (\rho_\theta(x), \theta). \end{aligned}$$

A Kirby diagram of this manifold is given in Figure 3, see [2, Section 0], [4, Figure 1], [18] and [25, Section 4.1]. It is obtained by considering $S^2 \mathbin{\widetilde{\times}} \mathbb{RP}^2$ as the double of $D^2 \mathbin{\widetilde{\times}} \mathbb{RP}^2$. Figure 4 represents the projection of the attaching circles of the 2-handles onto the page of the trivial Lefschetz fibration given by the union of handles up to index one. In order to endow the 2-handlebody of $S^2 \mathbin{\widetilde{\times}} \mathbb{RP}^2$ with the structure of a Lefschetz fibration over D^2, we need to stabilize the page and add vanishing cycles as in Figure 5, so that the attaching spheres of the 2-handles can be isotoped into different pages and have the right framing. This is Harer's trick and is explained in the non-orientable setting in [19, Proof of Theorem 1.1], see also [14] and [8, Theorem 2.1]. At the level of Kirby diagrams, this corresponds to adding canceling pairs of 1- and 2-handles.

In Figure 6 we fix a set of simple closed loops $e_1, \ldots, e_6$ inducing a homology basis of the page Σ of this new fibration. The quadratic enhancement

$$q^+ : H_1(\Sigma; \mathbb{Z}_4) \to \mathbb{Z}_2$$

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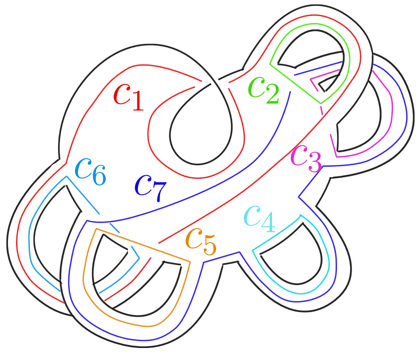

Figure 5. The 2-handlebody of $S^2 \widetilde{\times} \mathbb{RP}^2$ as a Lefschetz fibration over D^2.

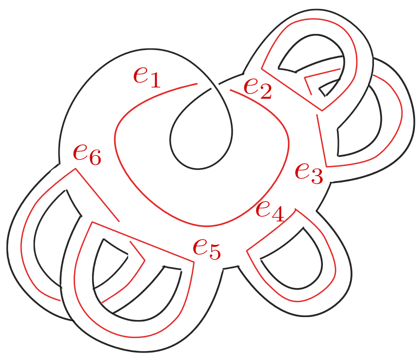

Figure 6. Curves representing a basis of $H_1(\Sigma)$.

defined by the condition $q^+([e_i]) = 1$ for every $i = 1, \ldots, 6$ has the property that $q^+([c_i]) = 1$ for all the vanishing cycles $c_1, \ldots, c_7$ and hence it determines a Pin$^+$-structure on $S^2 \widetilde{\times} \mathbb{RP}^2$. The other Pin$^+$-structure can be found by acting on q^+ with $[e_1] \in H_1(\Sigma; \mathbb{Z}_2)$.

On the other hand, $S^2 \widetilde{\times} \mathbb{RP}^2$ does not support a Pin$^-$-structure. Indeed, if this was the case, we could find a quadratic enhancement

$$q^- : H_1(\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_4$$

such that $q^-([e_i]) = 2$ for all $i = 1, \ldots, 7$ and this would imply that

$$\begin{aligned} 2 &= q^-([c_1]) = q^-([e_2 + e_6]) = q^-([e_2]) + q^-([e_6]) = q^-([c_2]) + q^-([c_6]) = 2 + 2 \\ &= 0 \in \mathbb{Z}_4 \end{aligned}$$

a contradiction.

Example 4 ($S^2 \times \mathbb{RP}^2$). A Kirby diagram of the product 4-manifold $S^2 \times \mathbb{RP}^2$ is given in Figure 7, see [2, Section 0], [4, Figure 1], [18] and [25, Section 4.1]. It is obtained by considering $S^2 \times \mathbb{RP}^2$ as the double of $D^2 \times \mathbb{RP}^2$. Figure 8 shows

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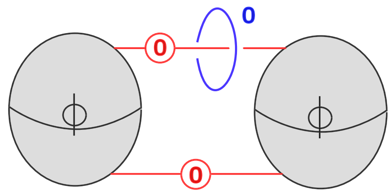

Figure 7. A Kirby diagram of $S^2 \times \mathbb{RP}^2$.

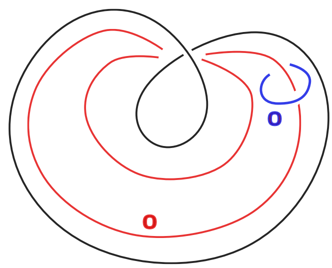

Figure 8. Lefschetz fibration associated to the 2-handlebody of $S^2 \times \mathbb{RP}^2$.

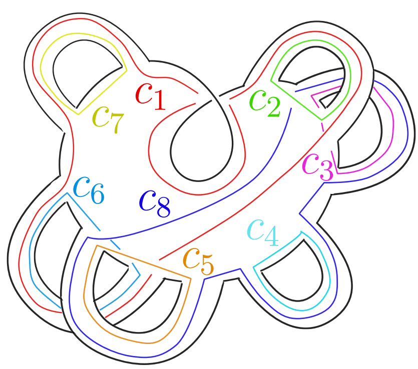

Figure 9. The 2-handlebody of $S^2 \times \mathbb{RP}^2$ as a Lefschetz fibration over D^2.

the projection of the attaching circles of the 2-handles onto the page of the trivial Lefschetz fibration given by the union of handles up to index one.

We use Harer's trick again (see [19, Proof of Theorem 1.1], [14] and [8, Theorem 2.1]) and show that the union of the handles up to index 2 in the fixed decomposition of $S^2 \times \mathbb{RP}^2$ is given by the surface Σ and by the vanishing cycles $c_1, \ldots, c_8$ in Figure 9. Note that, with respect to the case of $S^2 \widetilde{\times} \mathbb{RP}^2$, we need to stabilize the page one additional time in order to adjust the framing of the red coloured 2-handle.

In Figure 10 we again fix a set of loops $e_1, \ldots, e_7 \subset \Sigma$ inducing a first homology basis of the page.

The quadratic enhancement

$$q^- : H_1(\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_4$$

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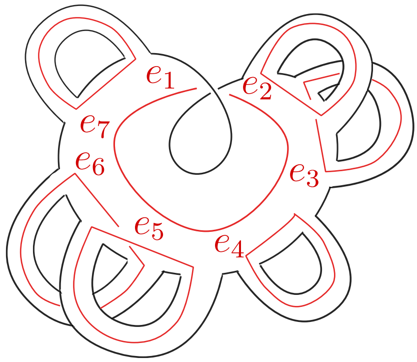

Figure 10. Curves representing a basis of $H_1(\Sigma)$.

defined by the condition $q^-([e_i]) = 2$ for every $i = 1, \ldots, 7$ has the property that $q^-([c_i]) = 2$ for all the vanishing cycles $c_1, \ldots, c_8$ and hence it determines a Pin$^-$-structure on $S^2 \times \mathbb{RP}^2$. The other Pin$^-$-structure can be found by acting on q^- with $[e_1] \in H_1(\Sigma; \mathbb{Z}_2)$.

On the other hand, $S^2 \times \mathbb{RP}^2$ can not support a Pin$^+$-structure. Indeed, if this was the case we could find a quadratic enhancement

$$q^+ : H_1(\Sigma; \mathbb{Z}_4) \to \mathbb{Z}_2$$

such that $q^+([c_i]) = 1$ for all $i = 1, \ldots, 8$ and this would imply that

$$\begin{aligned} 1 = q^+([c_1]) = q^+([2e_1 + e_2 + e_6 + e_7]) = \\ 2q^+([e_1]) + e_1^2 + q^+([e_2]) + q^+([e_6]) + q^+([e_7]) = \\ 1 + q^+([c_2]) + q^+([c_6]) + q^+([c_7]) = 1 + 1 + 1 + 1 = 0 \in \mathbb{Z}_2 \end{aligned}$$

a contradiction.

7. Pin$^\pm$-structures on Lefschetz fibrations over S^2

In this section, we provide a proof of Theorem 4.

Proof. Proof of Theorem 4 The proof is a $\mathbb{Z}_2$-coefficients version of the one of [23, Theorem 1.3] using the two formulas (4) and (3) in Section 2. We hence leave it to the interested reader as an exercise. □

Remark 3. One can check whether $X \setminus \nu(\Sigma)$ supports a Pin$^\pm$-structure by applying Theorem 1 and Theorem 2 respectively.

8. Pin$^+$ and Pin$^-$-structures on 3-manifolds

The following well-known fact can be easily proven by adapting to the Pin$^-$ case the one of [23, Theorem 1.4], see [17].

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THEOREM 6. *Any closed* 3*-manifold* M *admits a Pin*$^-$*-structure.*

Sketch of Proof. We reduce to the case in which M is non-orientable, since if M is orientable then it is automatically Spin and hence also Pin$^+$ and Pin$^-$. The idea is to write M as the union of a non-orientable 3-dimensional handlebody H of genus g with g 2-handles and a single 3-handle. Then [17, Corollary 1.12] implies that M admits a Pin$^-$-structure if and only if the closed surface ∂H supports a Pin$^-$-structure which restricts to the one bounding the 2-disk on all the attaching spheres of the 2-handles and on all the belt spheres of the 1-handles. Such a Pin$^-$-structure can always be constructed by means of a quadratic enhancement $q^- : H_1(\partial H; \mathbb{Z}_2) \to \mathbb{Z}_4$ which vanishes on all such curves. The proof follows the same lines of the one of [23, Theorem 1.4] and we leave the details to the interested reader. $\square$

On the other hand, it is not true that any closed 3-manifold admits a Pin$^+$-structure, since the condition $w_2(M) = 0$ is not always satisfied in the non-orientable setting. $\mathbb{RP}^2 \times S^1 \# N$ for any 3-manifold N is such an example. However, we now show that it is still possible to check this condition by considering a handlebody decomposition of M.

THEOREM 7. *Let* $M = H \cup H'$ *be a closed non-orientable* 3*-manifold obtained by adding* g 2*-handles and a* 3*-handle to a genus-*g *non-orientable handlebody* H *and let* $q_0^+ : H_1(\partial H; \mathbb{Z}_4) \to \mathbb{Z}_2$ *be a quadratic enhancement corresponding to a Pin*$^+$ *structure on* ∂H*. Let* $a_1, \dots, a_g \in H_1(\partial H; \mathbb{Z}_4)$ *and* $b_1, \dots, b_g \in H_1(\partial H; \mathbb{Z}_4)$ *be the homology classes of the attaching circles of the* 2*-handles and of the belt circles of the* 1*-handles of* H *respectively and set*

$$q_0^+([a_j]) = \alpha_j \quad \textit{and} \quad q_0^+([b_j]) = \beta_j$$

for $j = 1, \dots, g$*. If for a fixed basis* $e_1, \dots, e_{2g}$ *of* $H_1(\partial H; \mathbb{Z}_2)$ *we have*

$$\tilde{a}_j = \sum_{i=1}^{2g} a_{j,i} e_i \quad \textit{and} \quad \tilde{b}_j = \sum_{i=1}^{2g} b_{j,i} e_i$$

for $j = 1, \dots, g$*, where* $\tilde{a}_j$ *and* $\tilde{b}_j$ *are the* $\mathbb{Z}_2$*-reduction of* a_j *and* b_j *respectively, then* M *has a Pin*$^+$*-structure if and only if*

$$\mathit{rank}(C|A) = \mathit{rank}(C)$$

where C *and* A *are respectively the* $2g \times 2g$ *matrix and the column vector*

$$C = \begin{pmatrix} a_{1,1} & \dots & a_{1,2g} \\ \vdots & \ddots & \vdots \\ a_{g,1} & \dots & a_{g,2g} \\ b_{1,1} & \dots & b_{1,2g} \\ \vdots & \ddots & \vdots \\ b_{2g,1} & \dots & b_{g,2g} \end{pmatrix}, \qquad A = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_g \\ \beta_1 \\ \vdots \\ \beta_g \end{pmatrix}. \tag{12}$$

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Proof. There is a Pin^+-structure on M if and only if the non-orientable closed surface ∂H has a Pin^+-structure which extends to the attaching circles of the 2-handles and to the belt spheres of the 1-handles in H. In particular, this is equivalent to check whether or not there exists a map

$$q^+ : H_1(\partial H; \mathbb{Z}_4) \to \mathbb{Z}_2 \tag{13}$$

satisfying the condition $q^+(x+y) = q^+(x) + q^+(y) + x \cdot y$ for all $x, y \in H_1(\partial H; \mathbb{Z}_4)$ and which vanishes on $a_1, \dots, a_g$ and on $b_1, \dots, b_g$.

If we fix a quadratic enhancement

$$q_0^+ : H_1(\partial H; \mathbb{Z}_4) \to \mathbb{Z}_2 \tag{14}$$

corresponding to a Pin^+-structure on ∂H, then all the other Pin^+-structures on ∂H will be of the form $q^+ = q_0^+ + l$ for some linear map $l \in \mathrm{Hom}(H_1(\partial H; \mathbb{Z}_4), \mathbb{Z}_2)$. It follows that M supports a Pin^+-structure if and only if there is $l = \sum_{i=1}^{2g} x_i e^i \in \mathrm{Hom}(H_1(\partial H; \mathbb{Z}_4), \mathbb{Z}_2)$ such that

$$\sum_{i=1}^{2g} a_{j,i} x_i = \alpha_j \quad \text{and} \quad \sum_{i=1}^{2g} b_{j,i} x_i = \beta_j$$

in $\mathbb{Z}_2$ for $j = 1, \dots, g$, where $e^1, \dots, e^{2g}$ is the dual basis to $e_1, \dots, e_{2g} \in H_1(\partial H; \mathbb{Z}_2)$. The conclusion follows from Rouché–Capelli's theorem. □

9. Interpretation of Pin^+-structures à la Milnor

Kirby and Taylor showed in [17] that the sets of Pin^+ and Pin^--structures on a fixed vector bundle ξ are in one to one correspondence with Spin-structures on $\xi \oplus 3 \cdot \det(\xi)$ and $\xi \oplus \det(\xi)$ respectively. In this section, we remark that for Pin^+-structures one can get another equivalent characterization, which recalls the following one of Spin-structures by Milnor.

Theorem 8 (Milnor [20]). *There is a canonical bijection between the set of Spin-structures on a real rank-*k *vector bundle $\xi : E \to M$ and the set of homotopy classes of trivializations of $\xi|_{M^1}$ extending to trivializations of $\xi|_{M^2}$, where M^i denotes the ith skeleton of M. Moreover, such correspondence is equivariant with respect to the action of $H^1(M; \mathbb{Z}_2)$.*

Recall that $x \in H^1(M; \mathbb{Z}_2)$ acts on a trivialization of $\xi|_{M^1}$ by flipping the framing of any loop $\gamma \subset M^1$ such that $\langle x, [\gamma] \rangle = 1 \in \mathbb{Z}_2$.

In a similar fashion, we prove the following result. We remark that this follows essentially from the discussion in [17].

Theorem 9. *Let $\xi : E \to M$ be a real rank-k vector bundle. There is a canonical bijection between the set of Pin^+-structures on ξ and the set of homotopy classes of $(k-1)$-tuples of everywhere linearly independent sections of $\xi|_{M^1}$ extending to $\xi|_{M^2}$, where M^i denotes the ith skeleton of M. Moreover, such correspondence is equivariant with respect to the action of $H^1(M; \mathbb{Z}_2)$.*

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We remark that the definition of the action of $H^1(M;\mathbb{Z}_2)$ on the set of homotopy classes of linearly independent sections of $\xi|_{M^1}$ is completely analogous to the one described after the statement of Theorem 8.

Proof. The vanishing of $w_2(\xi)$ is a necessary and sufficient condition for the existence of both a $(k-1)$-tuple of linearly independent sections of $\xi|_{M^2}$ and a Pin$^+$-structure on ξ. We now suppose that $w_2(\xi)$ is trivial and we endow $\det(\xi)$ with its canonical Pin$^+$-structure, see [17, Addendum to 1.2]. Let $v_1,\dots,v_{k-1}$ be a $(k-1)$-tuple of linearly independent sections of $\xi|_{M^1}$ extending to $\xi|_{M^2}$. We are now going to associate to such a $(k-1)$-tuple a Pin$^+$-structure on $\xi|_{M^2}$. This will uniquely determine a Pin$^+$-structure on ξ, as in the case of Spin-structures. Using $v_1,\dots,v_{k-1}$ one can define a vector bundle isomorphism

$$\xi|_{M^2} \cong \varepsilon^{k-1} \oplus \det(\xi) \tag{15}$$

where ε^{k-1} denotes the trivial rank-$(k-1)$ real vector bundle. The associated Pin$^+$-structure on $\xi|_{M^2}$ is given by the pull-back via (15) of the Pin$^+$-structure on $\varepsilon^{k-1}\oplus\det(\xi)$ given by the direct sum of the trivial Spin-structure on ε^{k-1} with the fixed Pin$^+$-structure on $\det(\xi)$. It is now easy to verify that this correspondence satisfies all the desired properties. □

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