ON SOME PROPERTIES OF FUNCTIONS ANALYTIC IN A HALF-PLANE

P. G. ROONEY

1. Introduction. The spaces $\mathfrak{H}_p(\omega)$, ω real, $1 \leqslant p < \infty$, consist of those functions $f(s)$, analytic for Re $s > \omega$, and such that $\mu_p(f; x)$ is bounded for $x > \omega$, where

$$\mu_p(f; x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x + iy)|^p \, dy. \tag{1.1}$$

Doetsch **(1)** has shown that if $e^{-\omega t}\phi(t) \in L_p\ (0, \infty)$, $1 < p \leqslant 2$, and f is the Laplace transform of ϕ, that is,

$$f(s) = \int_0^{\infty} e^{-st}\, \phi(t) dt, \qquad Re\ s > \omega,$$

then $f \in \mathfrak{H}_q(\omega)$, where

$$p^{-1} + q^{-1} = 1, \tag{1.2}$$

and that conversely if $f \in \mathfrak{H}_p(\omega)$, $1 < p \leqslant 2$, then there is a function ϕ, with $e^{-\omega t}\phi(t) \in L_q\ (0, \infty)$, such that f is the Laplace transform of ϕ.

The proofs of Doetsch's theorems are based on a generalization of Plancherel's theorem due to Titchmarsh **(5)**. Titchmarsh's theorem states that if $F \in L_p(-\infty, \infty)$, $1 < p \leqslant 2$, then F has a Fourier transform $G \in L_q\ (-\infty, \infty)$.

However, there are other extensions of Plancherel's theorem due to Hardy and Littlewood **(3)**. They have shown that if $F \in L_p\ (-\infty, \infty)$, $1 < p \leqslant 2$, then F has a Fourier transform G such that $|x|^{1-2/p}G(x) \in L_p\ (-\infty, \infty)$, and that conversely if $|x|^{1-2/q}F(x) \in L_q\ (-\infty, \infty)$, $q \geqslant 2$, then F has a Fourier transform $G \in L_q\ (-\infty, \infty)$—for this form of Hardy and Littlewood's theorems see **(7**, Theorems 79 and 80**)**. One might expect that a theory similar to Doetsch's theory could be constructed from these theorems, and this we shall do here.

To this end we define spaces $\mathscr{H}_p(\omega)$, $1 < p < \infty$, to consist of those functions $f(s)$ such that $(s - \omega)^{1-2/p}f(s) \in \mathfrak{H}_p(\omega)$ (where $(s - \omega)^{1-2/p}$ takes on its principal value). This is equivalent to saying that $\nu_p(f; x, \omega)$ should be bounded for $x > \omega$, where

$$\nu_p(f; x, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |x - \omega + iy|^{p-2} |f(x + iy)|^p \, dy. \tag{1.3}$$

In § 3 we shall obtain theorems corresponding to Doetsch's results for these new spaces. It will be noticed that $\mathfrak{H}_2(\omega) = \mathscr{H}_2(\omega)$, so that one would expect

Received July 28, 1958. This paper was written while the author was a fellow at the 1958 Summer Research Institute of the Canadian Mathematical Congress.

that our new theorems should reduce, for $p = 2$, to Doetsch's theorems. This is actually the case.

In an earlier paper **(4)** we generalized Doetsch's theory. In order to obtain theorems dealing with the Laplace transformation of functions, of the form $t^\lambda\phi(t)$, where $e^{-\omega t}\phi(t) \in L_p\ (0, \infty)$ and $\lambda \geqslant 0$, we "generalized" the spaces $\mathfrak{H}_p(\omega)$ to spaces $\mathfrak{H}_{\lambda,p}(\omega)$. We can carry out a similar programme here, and to this end we define spaces $\mathscr{H}_{\lambda,p}(\omega)$ as follows. $\mathscr{H}_{0,p}(\omega) = \mathscr{H}_p(\omega)$; if $\lambda > 0$, $\mathscr{H}_{\lambda,p}(\omega)$ consists of those functions f in $\mathscr{H}_p(\omega')$ for every $\omega' > \omega$ such that $\nu_p^\lambda(f; \omega)$ is finite, where

$$(1.4) \qquad \nu_p^\lambda (f; \omega) = \int_\omega^\infty (x - \omega)^{p\lambda-1}\ \nu_p(f; x, \omega)\, dx.$$

The theorems corresponding to the results of **(4)** are obtained in § 4.

In § 2 we prove certain preliminary lemmas concerning the properties of functions in $\mathscr{H}_p(\omega)$.

2. Preliminary lemmas.

LEMMA 1. *If $f \in \mathscr{H}_p(\omega)$, $1 < p < \infty$, then*

$$f(\omega + iy) \equiv \lim_{x\to\omega+} f(x + iy)$$

exists for almost all y, and $|y|^{1-2/p}f(\omega + iy) \in L_p\ (-\infty, \infty)$. Further, $(x - \omega + iy)^{1-2/p}f(x + iy)$ converges in mean of order p to $(iy)^{1-2/p}f(\omega + iy)$ as $x \to \omega+$. Also, $\nu_p\ (f; x, \omega)$ tends steadily from below, as $x \to \omega+$, to

$$\int_{-\infty}^\infty |y|^{p-2}\, |f(\omega + iy)|^p\, dy.$$

Proof. The statement follows on applying **(1**, Lemma **7)** to $F(z) = (z - \omega)^{1-2/p}f(z)$.

LEMMA 2. *Let $f(s)$ be analytic for* Re $s > \omega$, *and suppose*

$$\int_{-\infty}^\infty |x - \omega + iy|^{p-2}|f(x + iy)|^p\, dy$$

is bounded for $x_1 \leqslant x \leqslant x_2$, where $p > 1$, $x_1 > \omega$. Then as $y \to \pm\infty$, $f(x + iy) = o(|y|^{1-2/q})$, uniformly in x for $x_1 + \delta \leqslant x \leqslant x_2 - \delta$, where $0 < \delta < \frac{1}{2}(x_2 - x_1)$.

Proof. Let $\Phi(\zeta) = (-i\zeta)^{1-2/p}f(\omega - i\zeta)$, where $\zeta = \xi + i\eta$, and $(-i\zeta)^{1-2/p}$ has its principal value. Then if $\eta > 0$,

$$\int_{-\infty}^\infty |\Phi(\xi + i\eta)|^p\, d\xi = \int_{-\infty}^\infty |\eta - i\xi|^{p-2}|f(\omega + \eta - i\xi)|^p\, d\xi$$
$$= \int_{-\infty}^\infty |\eta + i\xi|^{p-2}\, |f(\omega + \eta + i\xi)|^p\, d\xi$$

which is bounded for $x_1 - \omega \leqslant \eta \leqslant x_2 - \omega$. Hence by **(7**, Lemma, p. 125**)**,

$\lim_{\xi\to\pm\infty}\Phi(\xi+i\eta) = 0$ uniformly in η for $x_1-\omega+\delta \leqslant \eta \leqslant x_2-\omega-\delta$. Thus, setting $x=\omega+\eta$, $y=-\xi$,

$$\lim_{y\to\pm\infty}(x-\omega+iy)^{1-2/p}f(x+iy) = 0$$

uniformly in x for $x_1+\delta \leqslant x \leqslant x_2-\delta$. But clearly

$$(x-\omega+iy)^{1-2/p} = O(|y|^{1-2/p}) \quad \text{as} \quad y\to\pm\infty,$$

uniformly in x for x in the same interval. Hence

$$\lim_{y\to\pm\infty}|y|^{1-2/p}f(x+iy) = 0$$

uniformly in x for x in this interval; that is,

$$f(x+iy) = o(|y|^{-(1-2/p)}) = o(|y|^{1-2/q})$$

uniformly in x for $x_1+\delta \leqslant x \leqslant x_2-\delta$.

LEMMA 3. *If* $f\in\mathscr{H}_q(\omega)$, $q\geqslant 2$, *and* $\omega\leqslant\xi<\operatorname{Re} s$, *then*

$$f(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(\xi+i\eta)}{s-(\xi+i\eta)}\,d\eta.$$

Proof. Suppose first $\omega<\xi<\operatorname{Re} s$. Let $s=x+iy$, and choose R and ρ so that $\rho>x$, and $R>|y|$. Then

$$f(s) = \frac{1}{2\pi i}\int\frac{f(\zeta)}{\zeta-s}\,d\zeta,$$

the integral being taken around the rectangle with vertices $\xi\pm iR$ and $\rho\pm iR$. The integral along the upper side of the rectangle is given by

$$\frac{1}{2\pi i}\int_{\xi}^{\rho}\frac{f(\alpha+iR)}{s-(\alpha+iR)}\,d\alpha.$$

But by Lemma 2, $f(\alpha+iR)=o(R^{1-2/p})$ as $R\to\infty$, uniformly in α for $\xi\leqslant\alpha\leqslant\rho$. Hence the integral along the upper side is $o(R^{-2/p})$ and consequently tends to zero as $R\to\infty$. Similarly, the integral along the lower side of the rectangle tends to zero as $R\to\infty$. Hence letting $R\to\infty$,

$$f(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(\xi+i\eta)}{s-(\xi+i\eta)}\,d\eta - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(\rho+i\eta)}{s-(\rho+i\eta)}\,d\eta.$$

Now the second of these integrals tends to zero as $\rho\to\infty$. For from Hölder's inequality it is smaller in modulus than

$$(\nu_q(f;\rho,\omega))^{1/q}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\rho-\omega+i\eta|^{p-2}}{|s-(\rho+i\eta)|^{p}}\,d\eta\right\}^{1/p}.$$

The first term of this expression is bounded by hypothesis; since $1<p\leqslant 2$, the second term is smaller than

$$\left\{\frac{(\rho-\omega)^{p-2}}{2\pi}\int_{-\infty}^{\infty}\frac{d\eta}{((\rho-x)^2+(\eta-y)^2)^{1/2p}}\right\}^{1/p}$$

$$=\left\{\frac{(\rho-\omega)^{p-2}}{(\rho-x)^p}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\eta}{(1+\eta^2)^{1/2p}}\right\}^{1/p}=O(\rho^{-2/p})$$

as $\rho\to\infty$. Hence letting $\rho\to\infty$

$$f(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(\xi+i\eta)}{s-(\xi+i\eta)}\,d\eta,\qquad \omega<\xi<Re\,s.$$

It remains to show that this equation remains true when $\xi=\omega$. For this we write the equation in the form

$$f(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\{(\xi-\omega+i\eta)^{1-2/q}f(\xi+i\eta)\}\left\{\frac{(\xi-\omega+i\eta)^{1-2/p}}{s-(\xi+i\eta)}\right\}d\eta.$$

The first term of the integrand of this last integral converges in mean of order q to $(i\eta)^{1-2/q}f(\omega+i\eta)$ as $\xi\to\omega+$. We shall show that the second term of the integrand converges in mean of order p to $(i\eta)^{1-2/p}/(s-(\omega+i\eta))$ as $\xi\to\omega+$. Clearly it tends to this limit pointwise. Further, since $1<p\leqslant 2$, we have if $\xi<\gamma<x$,

$$\left|\frac{(\xi-\omega+i\eta)^{1-2/p}}{s-(\xi+i\eta)}-\frac{(i\eta)^{1-2/p}}{s-(\omega+i\eta)}\right|^p$$

$$\leqslant 2^p\left\{\frac{((\xi-\omega)^2+\eta^2)^{\frac{1}{2}(p-2)}}{((x-\xi)^2+(\eta-y)^2)^{\frac{1}{2}p}}+\frac{|\eta|^{p-2}}{((x-\omega)^2+(\eta-y)^2)^{1/2p}}\right\}$$

$$\leqslant 2^{p+1}\cdot\frac{|\eta|^{p-2}}{((x-\gamma)^2+(\eta-y)^2)^{1/2p}}$$

which is in $L_1(-\infty,\infty)$ as a function of η. Hence by Lebesgue's theorem of dominated convergence,

$$\lim_{\xi\to\omega+}\int_{-\infty}^{\infty}\left|\frac{(\xi-\omega+i\eta)^{1-2/p}}{s-(\xi+i\eta)}-\frac{(i\eta)^{1-2/p}}{s-(\omega+i\eta)}\right|^p d\eta=0.$$

Thus, letting $\xi\to\omega+$ we obtain from (**6**, § 12.5, example (iv)))

$$f(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\{(i\eta)^{1-2/q}f(\omega+i\eta)\}\left\{\frac{(i\eta)^{1-2/p}}{s-(\omega+i\eta)}\right\}d\eta$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(\omega+i\eta)}{s-(\omega+i\eta)}\,d\eta.$$

LEMMA 4. *If* $f\in\mathscr{H}_q(\omega)$, $q\geqslant 2$, *and if* $\xi\geqslant\omega$ *and* Re $s<\xi$, *then*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(\xi+i\eta)}{s-(\xi+i\eta)}\,d\eta=0.$$

Proof. The statement follows much as in the previous lemma.

3. The spaces $\mathscr{H}_p(\omega)$. Theorems 1 and 2 correspond to Theorems 2 and 3 respectively of Doetsch **(1)**.

THEOREM 1. *If $e^{-\omega t}\phi(t) \in L_p\ (0, \infty)$, $1 < p \leqslant 2$, and*

$$f(s) = \int_0^\infty e^{-st}\,\phi(t)\,dt, \qquad Re\, s > \omega,$$

then $f \in \mathscr{H}_p(\omega)$ and if $x > \omega$,

$$\nu_p(f; x, \omega) \leqslant K \int_0^\infty e^{-pxt}\,|\phi(t)|^p\,dt,$$

where K depends on p alone.

Proof. If $x > \omega$,

$$f(x - iy) = \int_0^\infty e^{iyt}(e^{-xt}\phi(t))\,dt;$$

that is, for each fixed $x > \omega$, $f(x - iy)$ is the Fourier transform of a function in $L_p\ (0, \infty)$. Hence by **(7**, Theorem 80**)**, since $1 < p \leqslant 2$,

$$\begin{aligned}\nu_p(f; x, \omega) &= \frac{1}{2\pi}\int_{-\infty}^\infty |x - \omega + iy|^{p-2}\,|f(x + iy)|^p\,dy\\ &\leqslant \frac{1}{2\pi}\int_{-\infty}^\infty |y|^{p-2}|f(x - iy)|^p\,dy\\ &\leqslant \frac{K(p)}{2\pi}\int_0^\infty e^{-pxt}\,|\phi(t)|^p\,dt \leqslant \frac{K(p)}{2\pi}\int_0^\infty e^{-p\omega t}\,|\phi(t)|^p\,dt,\end{aligned}$$

so that $f \in \mathscr{H}_p(\omega)$ and the stated inequality holds with $K = K(p)/2\pi$.

THEOREM 2. *If $f \in \mathscr{H}_q(\omega)$, $q \geqslant 2$, then there is a function ϕ, with $e^{-\omega t}\phi(t) \in L_q(0, \infty)$, such that*

$$f(s) = \int_0^\infty e^{-st}\,\phi(t)\,dt, \qquad Re\, s > \omega.$$

Further, if $x > \omega$,

$$\int_0^\infty e^{-qxt}\,|\phi(t)|^q dt \leqslant K\,\nu_q(f; x, \omega),$$

where K depends on q alone.

Also for $x \geqslant \omega$ and for almost all t,

$$e^{xt}\,\underset{a\to\infty}{\mathfrak{L}_q}\,\frac{1}{2\pi}\int_{-a}^a e^{it\eta} f(x + i\eta)\,d\eta = \begin{cases}\phi(t), & t > 0\\ 0, & t < 0\end{cases}$$

(where $\mathfrak{L}_q$ denotes the limit in mean of order q).

Proof. By Lemma 1, $|y|^{1-2/q}f(\omega + iy) \in L_q\,(-\infty, \infty)$. Hence by **(7**, Theorem 79**)**, $f(\omega + iy)$ has a Fourier transform $F \in L_q\ (-\infty, \infty)$, given by the formula

https://doi.org/10.4153/CJM-1959-044-4 Published online by Cambridge University Press

$$F(t) = \underset{a\to\infty}{\mathfrak{L}_q}\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-a}^{a} e^{it\eta} f(\omega + i\eta)\, d\eta.$$

Let $\phi(t) = (2\pi)^{-\frac{1}{2}} e^{\omega t} F(t)$. Clearly $e^{-\omega t}\phi(t) \in L_q\ (-\infty, \infty)$.

Now for each s with $\operatorname{Re} s \neq \omega$, $(s - (\omega + i\eta))^{-1} \in L_p\ (-\infty, \infty)$ as a function of η. Also a straightforward calculation shows that if $\operatorname{Re} s > \omega$,

$$\frac{1}{(2\pi)^{\frac{1}{2}}}(P)\int_{-\infty}^{\infty}\frac{e^{it\eta}}{s - (\omega + i\eta)}\, d\eta = \begin{cases} -\ (2\pi)^{\frac{1}{2}}\, e^{t(s-\omega)},\ t > 0,\ Re\ s < \omega \\ (2\pi)^{\frac{1}{2}}\, e^{t(s-\omega)},\ t < 0,\ Re\ s > \omega, \\ 0, \qquad (Re\ s - \omega)\, t > 0, \end{cases}$$

so that the Fourier transform of $((s - (\omega + i\eta))^{-1}$ is given by this expression. Hence from Lemma 3 and (**7**, Theorem 81), if $\operatorname{Re} s > \omega$,

$$f(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(\omega + i\eta)}{s - (\omega + i\eta)}\, d\eta = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{0} e^{t(s-\omega)}\, F(-t)\, dt$$
$$= \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{0}^{\infty} e^{-st}\, e^{\omega t}\, F(t)\, dt = \int_{0}^{\infty} e^{-st}\, \phi(t)\, dt,$$

so that f is the Laplace transform of a function ϕ with $e^{-\omega t}\phi(t) \in L_q\ (0, \infty)$. Also from Lemma 4 and (**7**, Theorem 81), if $\operatorname{Re} s < \omega$,

$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(\omega + i\eta)}{s - (\omega + i\eta)}\, d\eta = -\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{0}^{\infty} e^{t(s-\omega)}\, F(-t)\, dt$$
$$= -\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{0}^{\infty} e^{st}\, \phi(-t)\, dt,$$

that is, the Laplace transform of $\phi(-t)$, with variable $-s$, vanishes. Hence by (**2**, chapter 2, § 9, Theorem 4) $\phi(-t) = 0$ a.e. for $t > 0$, or equivalently $\phi(t) = 0$ a.e. for $t < 0$,

Further, from (**7**, Theorem 79),

$$(3.1)\quad \int_{0}^{\infty} e^{-q\omega t}|\phi(t)|^q\, dt = \frac{1}{(2\pi)^{\frac{1}{2}q}}\int_{-\infty}^{\infty}|F(t)|^q\, dt$$
$$\leqslant \frac{K(q)}{2\pi}\int_{-\infty}^{\infty}|y|^{q-2}\, |f(\omega + iy)|^p\, dy.$$

Now since $q \geqslant 2$, if $\omega < \omega' < x$ and $g \in \mathscr{H}_q(\omega)$, then $\nu_q(g; x, \omega') \leqslant \nu_q(g; x, \omega)$, so that $g \in \mathscr{H}_q(\omega')$. Hence if $x > \omega$, $f \in \mathscr{H}_q(x)$ so that by what we have just proved there is a function ϕ_x with $e^{-xt}\phi_x(t) \in L_q\ (0, \infty)$, satisfying (3.1) with ω replaced by x, such that for $\operatorname{Re} s > x$,

$$f(s) = \int_{0}^{\infty} e^{-st}\, \phi_x(t)\, dt, \qquad Re\ s > x,$$

and so that for almost all t

$$e^{xt}\underset{a\to\infty}{\mathfrak{L}_q}\frac{1}{2\pi}\int_{-a}^{a} e^{it\eta} f(x + i\eta)\, d\eta = \begin{cases}\phi_x(t),\ t > 0 \\ 0, \qquad t < 0\end{cases}$$

https://doi.org/10.4153/CJM-1959-044-4 Published online by Cambridge University Press

But by **(2**, chapter 2, § 9, Theorem 4**)**, $\phi_x(t) = \phi(t)$ a.e. for $t > 0$. Hence for any $x \geqslant \omega$ and almost all t

$$e^{xt} \underset{a\to\infty}{\mathfrak{L}_q} \frac{1}{2\pi}\int_{-a}^{a} e^{it\eta} f(x+i\eta)\,d\eta = \begin{cases}\phi(t), & t>0,\\ 0, & t<0.\end{cases}$$

Finally from (3.1), with ω replaced by x, we obtain, since $q \geqslant 2$,

$$\int_0^\infty e^{-qxt}|\phi(t)|^q\,dt = \int_0^\infty e^{-qxt}|\phi_x(t)|^q\,dt \leqslant \frac{K(q)}{2\pi}\int_{-\infty}^{\infty}|y|^{q-2}|f(x+iy)|^q\,dy$$
$$\leqslant K\,\nu_q(f;x,\omega),$$

where $K = K(q)/2\pi$.

4. The spaces $\mathscr{H}_{\lambda,p}(\omega)$. Theorems 3 and 4 correspond to Theorems 1 and 2 of **(4)**.

THEOREM 3. *If $e^{-\omega t}\phi(t) \in L_p\ (0,\infty)$, $1 < p \leqslant 2$, $\lambda \geqslant 0$, and*

$$f(s) = \int_0^\infty e^{-st}\,t^\lambda\,\phi(t)\,dt, \qquad Re\, s > \omega,$$

then $f \in \mathscr{H}_{\lambda,p}(\omega)$.

Proof. If $\lambda = 0$ the statement reduces to that of Theorem 1. Hence we may assume $\lambda > 0$. If $\omega' > \omega$, then since $t^\lambda e^{-(\omega'-\omega)t}$ is bounded for $t \geqslant 0$, $e^{-\omega' t}t^\lambda\phi(t) \in L_p\ (0,\infty)$, and hence by Theorem 1 $f \in \mathscr{H}_p(\omega')$, and if $x > \omega'$

$$\nu_p(f;x,\omega') \leqslant K\int_0^\infty e^{-pxt}\,t^{p\lambda}\,|\phi(t)|^p\,dt.$$

Let $x > \omega$, and choose ω' so that $\omega < \omega' < x$. Then since $1 < p \leqslant 2$,

$$\nu_p(f;x,\omega) \leqslant \nu_p(f;x,\omega') \leqslant K\int_0^\infty e^{-pxt}\,t^{p\lambda}\,|\phi(t)|^p\,dt.$$

Hence

$$\begin{aligned}\nu_p^\lambda(f;\omega) &= \int_\omega^\infty (x-\omega)^{p\lambda-1}\,\nu_p(f;x,\omega)dx\\ &\leqslant K\int_\omega^\infty (x-\omega)^{p\lambda-1}\,dx\int_0^\infty e^{-pxt}\,t^{p\lambda}\,|\phi(t)|^p\,dt\\ &= K\int_0^\infty t^{p\lambda}\,|\phi(t)|^p\,dt\int_\omega^\infty (x-\omega)^{p\lambda-1}\,e^{-pxt}dt = \frac{K\Gamma(p\lambda)}{p^{p\lambda}}\int_0^\infty e^{-p\omega t}|\phi(t)|^p\,dt,\end{aligned}$$

and $f \in \mathscr{H}_{\lambda,p}(\omega)$.

THEOREM 4. *If $f \in \mathscr{H}_{\lambda,q}(\omega)$, $q \geqslant 2$, $\lambda \geqslant 0$, then there is a function ϕ, with $e^{-\omega t}\phi(t) \in L_q\ (0,\infty)$, such that*

$$f(s) = \int_0^\infty e^{-st}\,t^\lambda\,\phi(t)\,dt.$$

Proof. Since $f \in \mathscr{H}_q(\omega')$ for every $\omega' > \omega$, by Theorem 2 if $\omega' > \omega$ there is a function $\phi_{\omega'}$, with

$$e^{-\omega' t}\,\phi_{\omega'}(t) \in L_q(0,\infty),$$

such that

$$f(s) = \int_0^\infty e^{-st}\, \phi_{\omega'}(t)dt, \qquad Re\, s > \omega'.$$

But by (**2**, chapter 2, § 9, Theorem 4), if ω' and ω'' are larger than ω, $\phi_{\omega'}(t) = \phi_{\omega''}(t)$ a.e. for $t \geqslant 0$. Hence if ϕ_0 is any one of these functions and Re $s > \omega$, then choosing ω' so that $\omega < \omega' <$ Re s we obtain

$$f(s) = \int_0^\infty e^{-st}\, \phi_{\omega'}(t)dt = \int_0^\infty e^{-st}\, \phi_0(t)dt.$$

Also from Theorem 2, since $q \geqslant 2$, if $x > \omega$ and ω' is chosen so that $\omega < \omega' < x$

$$\int_0^\infty e^{-qxt}\, |\phi_0(t)|^p dt = \int_0^\infty e^{-qxt}\, |\phi_{\omega'}(t)|^p dt \leqslant K\, \nu_q(f; x, \omega') \leqslant K\, \nu_q(f; x, \omega).$$

Hence, if we multiply this inequality by $(x - \omega)^{q\lambda-1}$ and integrate from ω to ∞, we obtain

$$\int_\omega^\infty (x-\omega)^{q\lambda-1}dx \int_0^\infty e^{-qxt}\, |\phi_0(t)|^q dt \leqslant K \int_\omega^\infty (x-\omega)^{q\lambda-1}\, \nu_q(f; x, \omega)dx$$
$$= K\nu_q^\lambda(f; \omega).$$

But the integral on the left-hand side of this inequality is equal to

$$\int_\omega^\infty (x-\omega)^{q\lambda-1}dx \int_0^\infty e^{-qxt}\, |\phi_0(t)|^q dt = \int_0^\infty |\phi_0(t)|^q dt \int_\omega^\infty (x-\omega)^{q\lambda-1} e^{-qxt} dx$$
$$= \frac{\Gamma(q\lambda)}{q^{q\lambda}} \int_0^\infty e^{-q\omega t}\, t^{-q\lambda}\, |\phi_0(t)|^q\, dt,$$

so that

$$\int_0^\infty e^{-q\omega t}\, t^{-q\lambda}\, |\phi_0(t)|^q dt \leqslant \frac{q^{q\lambda} K \nu_q(f, \omega)}{\Gamma(q\lambda)} < \infty.$$

Hence if we let $\phi(t) = t^{-\lambda}\phi_0(t)$, then $e^{-\omega t}\phi(t) \in L_q\ (0, \infty)$, and if Re $s > \omega$

$$f(s) = \int_0^\infty e^{-st}\, t^\lambda\, \phi(t)dt.$$

References

1. G. Doetsch, *Bedingungen für die Darstellbarkeit einer Funktion als Laplace–Integral und eine Umkehrformel für die Laplace–Transformation*, Math. Zeit., *42* (1937), 263–86.

2. ———, *Handbuch der Laplace–Transformation. I* (Basel, 1950).

3. G. H. Hardy and J. E. Littlewood, *Some new properties of Fourier constants*, Math. Annal., *97* (1926), 159–209.

4. P. G. Rooney, *On some theorems of Doetsch*, Can. J. Math., *10* (1958), 421–30.

5. E. C. Titchmarsh, *A contribution to the theory of Fourier transforms*, Proc. Lond. Math. Soc. (2), *23* (1923), 279–89.

6. ———, *The theory of functions* (2nd ed.; Oxford, 1939).

7. ———, *Introduction to the theory of Fourier integrals* (2nd ed.; Oxford, 1948).

University of Toronto