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The conjugacy problem for Out(F_3)

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Abstract
We present a solution to the conjugacy problem in the group of outer automorphisms of F_3, a free group of rank 3. We distinguish according to several computable invariants, such as irreducibility, subgroups of polynomial growth and subgroups carrying the attracting lamination. We establish, by considerations on train tracks, that the conjugacy problem is decidable for the outer automorphisms of F_3 that preserve a given rank 2 free factor. Then we establish, by consideration on mapping tori, that it is decidable for outer automorphisms of F_3 whose maximal polynomial growth subgroups are cyclic. This covers all the cases left by the state of the art.

Contents

1. Introduction

In this paper, we solve Dehn's conjugacy problem in Out(F_3), the group of outer automorphisms of the free group F_3 of rank 3:

Theorem 1.1. *There exists an algorithm which takes two automorphisms $\phi, \psi \in Aut(F_3)$ and correctly outputs **yes** or **no** whether their outer classes are conjugate in* Out(F_3).

The conjugacy problem has already been solved within certain classes of outer automorphisms of free groups: the atoroidal fully irreducible ones by Sela [Sel95], all the irreducible ones by Los [Los96], all the atoroidal ones by [Dah16], the linearly growing ones by Kristić, Lustig and Vogtmann [KLV01], and the unipotent polynomially growing ones by Feighn and Handel [FH19].

In the case of a free group of rank 2, $\mathrm{Out}(F_2) \simeq \mathrm{GL}(2, \mathbb{Z})$ is well-known and virtually free, and there is an algorithm to solve its conjugacy problem.

In the case of a free group of rank 3, $\mathrm{Out}(F_3)$ is more complicated, and our conjugacy decision algorithm operates by classifying the pair of input automorphisms by means of invariants, in subclasses, in which specific methods can be applied.

Let us mention key conjugacy invariants that are relevant in these classifications and are computable. They give a first frame of reference.

○ Irreducibility (whether there is an invariant conjugacy class of free factor system, or not), [Kap14], [FM22];
○ exponential growth (whether there is a conjugacy class whose iterated images grow exponentially fast, or not), [BFH05];
○ rank of the maximal polynomially growing subgroups (the maximal subgroups whose elements have iterated images that grow at most polynomially in conjugacy length), and polynomial degree of growth in these subgroups (Proposition 2.7);
○ arrangement of these subgroups (their orbit under the automorphism group of the ambient free group) (Gersten's algorithm);
○ ranks of the free factors carrying the so-called attracting lamination (Proposition 2.9).

Example 1.2. The map $a \mapsto a$, $b \mapsto ba$, $c \mapsto cb$ extends to the free group of basis $\{a, b, c\}$ as an automorphism that is reducible ($\langle a, b \rangle$ is an invariant free factor of rank 2), polynomially growing of degree 2 (the growth of c is quadratic).

Example 1.3. A pseudo-Anosov mapping class on a twice punctured torus gives an automorphism that is reducible (each puncture corresponds to a free factor of rank one that is preserved), of exponential growth, with two conjugacy classes of maximal polynomially growing subgroups (corresponding to the punctures), both cyclic, on which the automorphism has polynomial degree 0. They are both rank 1 free factors, but not in a same free factor system. The attracting lamination is supported by the entire group.

Example 1.4. Take T a torus with one hole ∂T, with a base point on its boundary, and a circle C with a base point, and take the wedge of these spaces, identifying the two base points: its fundamental group is F_3. Consider a pseudo-Anosov mapping class on T, fixing the boundary pointwise, and a map that sends C on the concatenation $C \cdot \ell$ for a chosen loop ℓ in T. The defined map on $T \wedge C$ induces an automorphism of F_3 that is reducible, of exponential growth, with an invariant free factor of rank 2 (the group of T). The cyclic group of the boundary ∂T of T is polynomially growing of degree 0. Depending on the choice of ℓ, it might or might not be a maximal polynomially growing subgroup: if ℓ is a power of the loop ∂T, the maximal polynomially growing subgroup containing $\pi_1(\partial T)$ is actually of rank 2, on which the automorphism has polynomial degree 0 or 1, and it is not a free factor of F_3 but rather a factor of a decomposition of F_3 as some amalgamated free product

$$\pi_1(T) *_{\pi_1(\partial T)} \pi_1(C \wedge \partial T).$$

The attracting lamination is carried by the rank 2 free factor $\pi_1(T)$. We show in Proposition 6.5 that any exponentially growing automorphism of F_3 with a nontrivial and noncyclic polynomially growing subgroup must be of this form.

It is nevertheless not sufficient to collect these invariants to have characterised the conjugacy class of an outer automorphism. For instance, knowing that the automorphisms are irreducible, with pure exponential growth (i.e., the only polynomially growing subgroup is the trivial one) still requires stamina to decide the conjugation between such outer automorphisms. Actually, all the solved cases listed in the beginning of this introduction all correspond to some particular situation in this frame of reference.
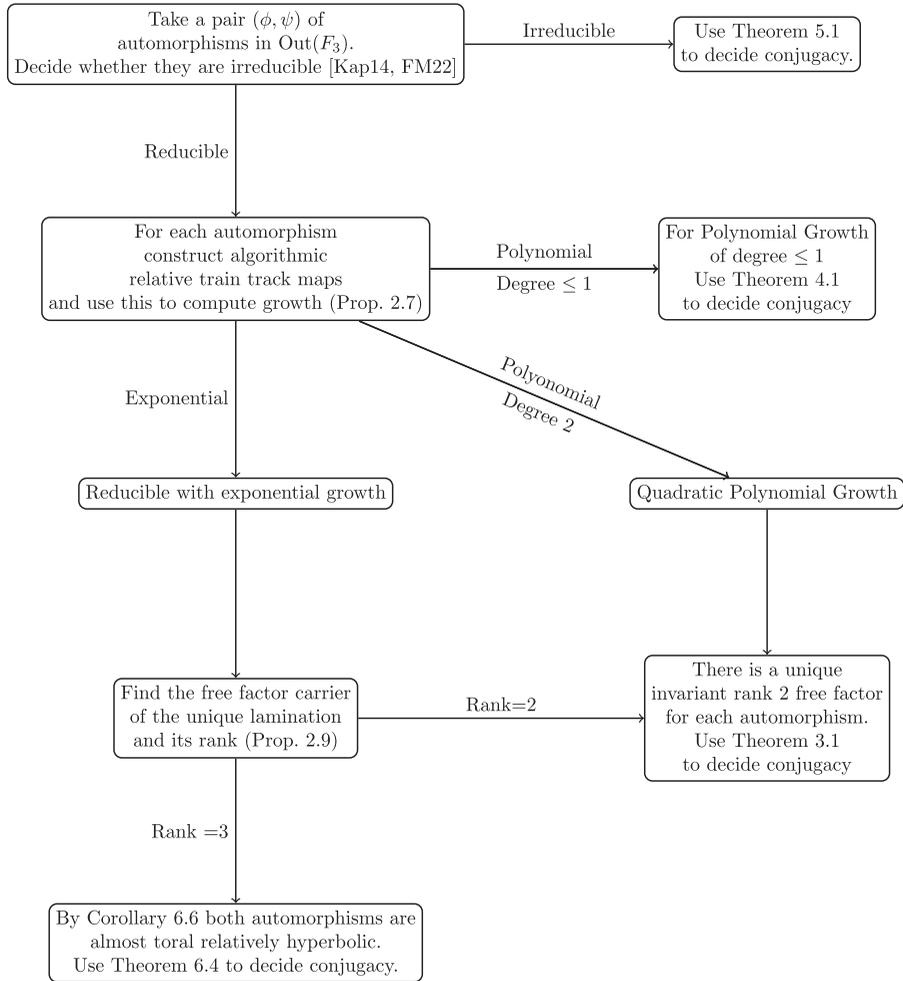
Figure 1. *Our algorithm to solve the conjugacy problem in* $\mathrm{Out}(F_3)$. *If at any time the given automorphisms ϕ and ψ follow different arrows, they are not conjugate.*

In our study of $\mathrm{Out}(F_3)$, we will observe that, given the state of the art, only three configurations in this frame of reference need to be treated. We will cast two of them in the point of view of automorphisms preserving a specific rank 2 free factor. We will cast the last one in the point of view of toral relatively hyperbolic mapping tori. The first configurations is that of the polynomial growth of degree 2 on the group F_3 (well illustrated by Example 1.2). The second configuration is the case of exponential growth in which a rank 2 free factor is invariant and 'attracts' all the growth. In that case (that is well illustrated by our example 1.4), there is only one class of maximal polynomially growing subgroup, it is not a free factor, and it is either cyclic (generated by the commutator of a basis in the invariant rank 2 free factor), or of rank 2 (attached to this free factor). The last case (illustrated by Example 1.3) is when the growth is exponential, with a rank 1 free factor preserved, and the maximal polynomial growth subgroups are of rank 1 (there might be two of them up to conjugacy).

Our algorithm is shown in Figure 1, and its correctness is derived from the correctness of every classification step as well as every terminal step. There are several ways to separate cases toward the use of Theorem 3.1 or the use of Theorem 6.4 or of methods in Section 6 since they overlap, but considering complexity, it seems reasonable to leave the later for the smaller number of cases.

In Section 3, we show how to solve the conjugacy problem in $\text{Out}(F_3)$ among all outer automorphisms with a given invariant rank 2 free factor. This result, in a sense, is a type of induction step from $\text{Out}(F_2)$ to $\text{Out}(F_3)$, but as will be explained in Remark 3.11, it will not generalize similarly to higher ranks. This case is critically important as we will show that most problematic situations (which actually fall in the case where the given automorphism induces a polynomial growth automorphism on a noncyclic subgroup) lead to an invariant free factor of rank 2.

In Section 4, we show that quadratically growing outer automorphisms have an invariant rank 2 free factor. In Section 5, we use train track methods to study the types of attracting laminations that arise. One possibility gives rise (again) to an an invariant rank 2 free factor. In Section 6, we show that the other possibility implies that the outer automorphisms are so-called almost toral relatively hyperbolic and the conjugacy problem for this class of outer automorphisms is established by the main result of [DT21] in its particular case where the so-called peripheral subgroups are $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z} \rtimes \mathbb{Z}$, the fundamental group of the Klein bottle.

Our examples

In order to explain our flowchart, we describe how to distinguish the conjugacy classes of the three automorphisms given at the start of this section.

To start with, each of the automorphisms is reducible. In Example 1.2, there are invariant free factors of rank one and two – namely, $\langle a \rangle$ and $\langle a, b \rangle$. In Example 1.3, each puncture corresponds to an invariant rank one free factor. Finally, in Example 1.4. the torus with one hole is an invariant rank 2 free factor.

(However, in Example 1.3, if the punctures are permuted, then it is irreducible but not fully irreducible, which would entail that it could not be conjugate to the other two).

The next stage of the flowchart would construct a relative train track to determine growth. The automorphisms are essentially already presented in this way with Example 1.2 having quadratic growth, whereas Examples 1.3 and 1.4 have exponential growth. This would already mean that the first is not conjugate to the other two.

One can then find the free factor carrier of the attracting lamination for each of the exponential growth automorphisms, and one finds that in Example 1.3, it has rank three, whereas for Example 1.4, it is carried by the invariant rank two free factor.

Hence, these three automorphisms are not conjugate in $\text{Out}(F_3)$.

We note that the flowchart is presented 'efficiently' by using a common algorithm for both quadratic polynomial growth automorphisms and for exponential growth automorphisms whose attracting lamination is carried by a rank two free factor. This algorithm – given by Theorem 3.1 – will indeed distinguish the conjugacy classes of those automorphisms, but our global algorithm already determines growth which is a conjugacy invariant.

Remark. We would like to comment that the methods used in this paper are somewhat broad but roughly break down into using mapping torus techniques and relative hyperbolicity versus train track methods and Culler-Vogtmann space.

In choosing which methods to use, we have tried to take the shortest routes that served our proof strategy rather than the most coherent ones. This unfortunately makes the background reading quite extensive. However, in producing this paper, we often produced multiple versions of the argument, and so we are aware that one could rewrite the proofs leaning more towards mapping tori techniques rather than train track techniques and vice versa. For instance, one could solve the conjugacy problem for the exponentially growing case using purely mapping torus techniques or using purely train track techniques. We have opted to mix and match and rely on existing results as far as possible.

The main place where these main techniques are not entirely applicable is in the quadratic growth case, Theorem 3.1, which is somehow a mapping torus situation in the absence of relative hyperbolicity.

2. Computability of a few invariants

Notations

We begin by some general notations and definitions.

Notation 2.1.

- In a group G, the conjugation of g by h is $g^h := h^{-1}gh$.
- $\mathrm{Aut}(G)$ acts on the right: for $\phi \in \mathrm{Aut}(G)$, $g \in G$, the image of g under ϕ is $g\phi$.
- $\mathrm{Ad}(g)$ denotes the automorphism of G that is conjugation by g: $x\,\mathrm{Ad}(g) = x^g$. Hence, $\mathrm{Ad}(g)\,\mathrm{Ad}(h) = \mathrm{Ad}(gh)$.
- $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$, for $\mathrm{Inn}(G) = \{\mathrm{Ad}(g), g \in G\}$.
- $F_3 = \langle a, b, c \rangle$ denotes a free group of rank 3.
- We will prefer the notation K for free factors, and H for subgroups.
- We write outer automorphisms with capital greek letters, and automorphisms with small greek letter: if Φ is an outer automorphism, $\phi \in \Phi$ is an automorphism in its class. In this convention, (according to context) X reads *Chi*, and $\chi \in \mathrm{X}$ is an automorphism.

For $\Phi \in \mathrm{Out}(F)$, we say a subgroup $H \leq F$ is Φ-*invariant* if for any $\phi \in \Phi$, we have that H is conjugate to $H\phi$.

A *free factor system* of F is a set of conjugacy classes of subgroups of F, $\{[K_1], \dots, [K_m]\}$ such that there exists a free subgroup $F_r < F$ for which $F = K_1 * \cdots * K_m * F_r$. It is proper if the K_i are neither $\{1\}$ nor F. It is Φ invariant if all K_i (not necessarily F_r) are Φ invariants.

A free factor system $\{[Y_1], \dots, [Y_s]\}$ is *smaller* than $\{[K_1], \dots, [K_m]\}$ if for each i, there exists j such that Y_i has conjugate inside K_j.

Irreducibility

A first conjugacy invariant, and decidable, property is the irreducibility of outer automorphisms.

Definition 2.2. An outer automorphism Φ is called *reducible* if it admits an invariant proper, nontrivial, free factor system; see [BFH00]. Otherwise, it is said to be *irreducible*.

An automorphism is *fully irreducible* if every positive power is irreducible.

Remark 2.3. An automorphism is fully irreducible if and only if the only *periodic* free factors preserved up to conjugacy are given by either the trivial subgroup or the whole group. Every fully irreducible automorphism is irreducible, but not conversely.

An example of an irreducible automorphism which is not fully irreducible is as follows: take a surface with $p > 1$ punctures. Consider a pseudo-Anosov map on the surface which cyclically permutes the punctures. Then the outer automorphism induced on the fundamental group is irreducible but not fully irreducible. In fact, any example of an exponential growth automorphism which is irreducible but not fully irreducible arises in this way; see [Mut21] Corollary A.5.

Fully irreducible is often referred to as 'iwip' – irreducible with irreducible powers – in the literature.

There is an algorithm to detect whether $\Phi \in \mathrm{Out}(F_n)$ is fully irreducible given in [Kap14], and whether it is irreducible in [FM22]. A more recent algorithm given in [KB19] decides whether Φ is fully irreducible in polynomial time.

Growth

A second set of conjugacy invariant, and decidable, properties is related to growth.

Definition 2.4. Given $\phi \in \Phi \in \mathrm{Out}(F_n)$ and $g \in F_n$ and a fixed basis X of F_n, we define the growth rate of g as follows:

○ The element g has polynomial growth of degree at most d under Φ if $\|g\phi^n\|_X = O(n^d)$.
○ The element g has exponential growth if there exists $\lambda > 1$ for which $\lambda^n = O(\|g\phi^n\|_X)$.[1]

This growth rate is independent of the choice of X and $\phi \in \Phi$. Moreover, by [BH92], the growth rate of every $g \in F_n$ is either at least exponential or at most polynomial.

Definition 2.5. An outer automorphism Φ of F_n is said to have

○ *polynomial growth* if all elements $g \in F_n$ have polynomial growth rate,
○ *exponential growth* if *some* element $g \in F_n$ has exponential growth rate.

In the case of a polynomial growth outer automorphism of F_n, there exists $d \geq 0$ for which all elements of F have polynomial growth of degree at most d, and moreover, the smallest such d satisfies $d \leq n - 1$; see [BFH05], [Lev09].

Definition 2.6. A polynomial growth outer automorphism of F_n is *unipotent* if it induces an automorphism of the abelianisation $\mathbb{Z}^n$ that is represented by a unipotent matrix.

Such a matrix of the abelianisation of an automorphism of F_n of polynomial growth only has eigenvalues of modulus 1, and in $GL(n, \mathbb{Z})$, so its $|GL(n, \mathbb{Z}/3)|$-th power is unipotent; see [BFH00], Corollary 5.7.6.

The following is known to specialists and is important to us. We explain a proof, using the theory of CT maps [FH18], that are certain homotopy equivalences on graphs, representing outer automorphisms on their fundamental groups; the interested reader is referred to this reference.

Proposition 2.7 [FH11, FH18]. *Given an automorphism,* $\Phi \in \mathrm{Out}(F_n)$, *there is an algorithm to decide*

○ *if Φ has polynomial or exponential growth and,*
○ *if Φ has polynomial growth, decide the degree of polynomial growth.*

Proof. By [FH18], there is an algorithm to construct a CT map (which is, in particular, a relative train track map) representing a rotationless power of Φ. Since the properties needed here are invariant under taking positive powers, we may as well assume that Φ itself is so-called rotationless (the power needed can be determined in advance). In particular, a rotationless automorphism of polynomial growth would be unipotent; see [FH11, Lemma 4.2.2].

So take a CT map, $f : G \to G$ representing Φ.

It then follows immediately that if f admits an exponential stratum, then Φ will have exponential growth. So we can algorithmically distinguish between exponential and polynomial growth.

So if Φ has polynomial growth, then all strata will be NEG (non-exponentially growing); note that by [FH11], Theorem 2.19, there will be no zero strata in the absence of exponential strata. Then Lemma 4.2 of [FH11] shows that each such NEG stratum consists of a single edge, E, such that $f(E) = E \cdot u$, where the dot denotes a *splitting*. In particular, the growth of $f^k(E)$ is polynomial of one degree higher than that of u. Hence, by induction, we can determine the growth of every edge.

This almost determines the growth of the automorphism, Φ, in the sense that Φ has polynomial growth of degree $d > 1$ if and only if some edge grows polynomially with degree d. However, it is possible to represent the identity map as a CT map where some edge grows linearly, and this is the only exception. See [Mac02, Lemma 2.16] and [AHK22, Lemma 2.3].

However, since we have already passed to a rotationless power (in particular, it will be UPG), our automorphism will have growth of degree 0 if and only if it represents the identity and this is immediately checked.   □

For an outer automorphism of F_n of exponential growth, there exists a canonical finite collection of conjugacy classes of finitely generated subgroups of F_n, such that an element $g \in F_n$ has polynomial

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[1]Note that (for nonnegative functions), $f(n) = O(g(n))$ means $f(n) \leq M g(n)$ for some constant M. The reason for the asymmetric definition of growth is the idea to bound polynomial growth from above and exponential growth from below. It is easy to see that there is always an exponential upper bound.

growth rate if and only if it is conjugate in one of these subgroups [Lev09]. Those are the maximal polynomially growing subgroups.

Given H such a maximal subgroup, and $\phi \in \Phi$, there is an integer $m > 0$ and $g \in F_n$ for which $H(\phi^m \operatorname{Ad}(g)) = H$. The automorphism $\phi^m \operatorname{Ad}(g)$ of H is then a polynomial growth automorphism. We again refer to [Lev09]. Through standard arguments of quasi-isometry, its polynomial degree of growth rate does not depend on $\phi \in \Phi$, m and g satisfying the above. We thus call it the degree of growth of Φ on H.

Proposition 2.8. *There is an algorithm that, given $\Phi \in \operatorname{Out}(F_n)$ of exponential growth, computes a basis for each conjugacy-representative of maximal polynomially growing subgroup of F_n, and the degree of growth of Φ on it.*

Proposition 2.8 follows from the relative hyperbolicity of $F_n \rtimes \mathbb{Z}$ given by Φ [Gho23, DL20], and the computability of its peripheral subgroups [DG13]. This detects the maximal polynomially growing subgroups of F_n. Then, Proposition 2.7 determines the induced growth on each of them.

Lamination carriers are computable

A more involved conjugacy invariant for outer automorphisms of free groups that have exponential growth is that of the carrier of the attracting laminations [BFH00, Section 3]. This is a specific conjugacy class of free factor systems of F_n that in some sense, attracts all the exponential growth of the automorphism.

More concretely, let ∂F_n, denote Gromov boundary of F_n, which for a free group is the same as the set of ends. Let $\widetilde{\mathcal{B}} = (\partial F_n \times \partial F_n - \Delta)/\mathbb{Z}_2$, where Δ is the diagonal subset of $\partial F_n \times \partial F_n$, and the $\mathbb{Z}_2$ action is via interchanging the coordinates. That is, $\widetilde{\mathcal{B}}$ can be thought of as the set of unordered pairs of distinct points on the boundary of F_n; morally, this is the set of lines in F_n. The action of F_n on its boundary extends to an action on $\widetilde{\mathcal{B}}$.

We then let $\mathcal{B} = \widetilde{\mathcal{B}}/F_n$. We say that $\beta \in \mathcal{B}$ is *carried* by (the conjugacy class of) a free factor K if β lies in the image of $\partial K \times \partial K - \Delta \to \mathcal{B}$. A subset S of $\mathcal{B}$ is carried by a free factor system, $\mathcal{F}$, if every element of S is carried by some free factor in $\mathcal{F}$.

An *attracting lamination* for Φ is then a closed subset of $\mathcal{B}$ which is the closure of a single point, β, and has some extra properties (β is birecurrent, admits an attracting neighbourhood for some positive iterate of Φ, and is not carried by a Φ-periodic free factor of rank one; see [BFH00, Definition 3.1.5]).

Any automorphism, Φ, then admits a finite set of attracting laminations, $\mathcal{L}(\Phi)$, which is carried by a Φ-invariant free factor system ([BFH00, Lemmas 3.1.6 and 3.1.13]). Morally, one has one attracting lamination for every exponential stratum for a relative train track representative for Φ (or some power), and moreover, $\mathcal{L}(\Phi)$ is canonical.

For instance, a fully irreducible automorphism will have a single attracting lamination which is only carried by the whole group. A polyonomially growing automorphism will have no attracting laminations.

Again, the theory of CT maps allows to compute this invariant. We would like to thank Mark Feign and Michael Handel for the proof of the following Proposition.

Proposition 2.9. *Given an automorphism, $\Phi \in \operatorname{Out}(F_n)$ of exponential growth, there is an algorithm to find the smallest free factor system which carries the set of attracting laminations.*

Proof. Since the set of attracting laminations is stable under taking positive powers, we are free to replace Φ with a (rotationless) power and use [FH18] to algorithmically produce a CT map representing this power of Φ. Since we are dealing with a rotationless automorphism, each exponential stratum produces exactly one attracting lamination. It is therefore sufficient to produce an algorithm which, given this CT map and a specific exponential stratum, H_k of it, finds the smallest free factor carrying Λ, the associated attracting lamination.

Let $f : G \mapsto G$ be the CT map for the power of Φ and let H_k be an exponential stratum with corresponding attracting lamination Λ. Construct a leaf L of Λ by choosing a point fixed by f in the

interior of an edge e of H_k and considering

$$e \subset f(e) \subset f^2(e) \subset \ldots$$

Note that L is f-invariant.

Choose a segment of L that starts and ends with a copy of the same oriented edge of H_k and with e interior to the segment. Glue the initial and terminal edges of the segment to form a loop γ immersing to G. For $i \geq 0$, denote by γ_i the immersed loop (or conjugacy class) $[f^i(\gamma)]$. Note that γ_i is obtained from L by gluing segments of the form $[f^i(edge)]$. In particular, these segments are long if i is big.

<u>Claim:</u> The free factor support $F(\Lambda)$ of Λ is equal to the free factor support, $F(\cup_{i \geq 0}\{\gamma_i\})$ of

$$\cup_{i \geq 0}\{\gamma_i\}.$$

Comment: The free factor support of a finite collection of elements can be found algorithmically; see [FH18, Lemma 4.2]. Moreover, the sequence of free factor supports of $\{\gamma_1\}, \{\gamma_1, \gamma_2\}, \ldots$ stabilises.

This is because each free factor system has a well-defined complexity (consisting of the ranks of the components of the free factors which comprise it, ordered lexicographically; see [BFH00], page 531). There are only finitely many complexities possible, and if one free factor system is strictly contained inside another, then it will have strictly smaller complexity; see [BFH00], Lemma 2.6.3. This is exactly our situation since if we label the free factor support of $\{\gamma_1, \ldots, \gamma_r\}$, $\mathcal{F}_r$, then we clearly get that $\mathcal{F}_r \sqsubseteq \mathcal{F}_{r+1}$. Moreover, we know that $\mathcal{F}_r \sqsubseteq F(\Lambda)$ for all r and $\mathcal{F}_r = F(\Lambda)$ as long as $\mathcal{F}_r$ is Φ-invariant, which is algorithmically decidable.

Hence, for an algorithm, it is enough to prove the claim.

Proof of Claim: That $F(\Lambda) \subset F(\cup_{i \geq 0}\{\gamma_i\})$ is clear.

For the other direction, we will show that γ_i is contained in every Φ-invariant free factor F' that carries Λ. It is enough to assume that i is large.

Let G' be a marked graph with sub-graph K' representing F'. Let $\tilde{K}' \subset \tilde{G}'$ and $\tilde{G}$ denote universal covers. View L (defined above) as a subset of $\tilde{G}$ such that its representation in $\tilde{G}'$ is a subset of $\tilde{K}'$. Let T denote the covering translation of G that identifies the segments of L that give γ_i.

If lines in $\tilde{G}$ have long (depending on G') overlap, then their representations in G' intersect – this is a consequence of the bounded cancellation lemma. Let L' denote the representation in $\tilde{G}'$ of L and T' denote the covering translation of $\tilde{G}'$ corresponding to T. In particular, L' and $T'(L')$ intersect. Since L' is in $\tilde{K}'$ and intersects $T'(L')$, the union of L' and $T'(L')$ is contained in $\tilde{K}'$ – this follows since translates of $\tilde{K}'$ are either equal or disjoint.

More generally, the union as j ranges over the set of integers of $T'j(L')$ is connected, contained in $\tilde{K}'$, and T'-invariant. Hence, γ_i is carried by F'.   □

3. The case of a given invariant rank 2 free factor

For a free factor $K \leq F$, we define

$$\text{Out}(F, K) = \{\Phi \in \text{Out}(F) : K \text{ is } \Phi\text{-invariant}\}.$$

For any $\Phi \in \text{Out}(F, K)$, the restriction to K is not well-defined as an automorphism, but it is well-defined as an outer automorphism because K is its own normaliser in F.

This section is devoted to the proof of the following.

Theorem 3.1. *Let $K \leq F_3$ be a free factor of rank 2. Then the conjugacy problem in $\text{Out}(F_3, K)$ for pairs elements that induce infinite order outer automorphisms of K is decidable.*

It will be enough to consider the case where $K = \langle a, b \rangle < F_3 = \langle a, b, c \rangle$. Before the proof of Theorem 3.1, we establish some needed results.

Lemma 3.2. *Let* $G = \mathrm{Out}(F_3, K)$, *and* $X \in G$. *Then there exists a unique* $\chi \in X$ *such that there exists* $\epsilon = \pm 1, g \in \langle a, b \rangle$ *and* $\chi_0 \in \mathrm{Aut}(\langle a, b \rangle)$, *satisfying*

$$\chi : \begin{cases} c \mapsto gc^\epsilon \\ b \mapsto b\chi_0 \\ a \mapsto a\chi_0. \end{cases} \tag{1}$$

In particular, G *is isomorphic to an iterated semi-direct product,* $C_2 \ltimes (\mathrm{Aut}(F_2) \ltimes F_2)$, *where the* $\mathrm{Aut}(F_2)$ *action on* F_2 *is the natural one, and the* C_2 *action on* $\mathrm{Aut}(F_2) \ltimes F_2$ *is given by the involution which maps* (χ_0, g) *to* $(\chi_0 \mathrm{Ad}(g), g^{-1})$.

In the light of the decomposition of the lemma, we can represent any $\mathrm{Out}(F_3, K)$ uniquely as an ordered triple, (ϵ, χ_0, g).

Proof. The outer automorphism X preserves the conjugacy class of $K = \langle a, b \rangle$; hence, an element of X preserves $\langle a, b \rangle$. Note that this element is only well-defined up to the normaliser of $\langle a, b \rangle$, which is again $\langle a, b \rangle$.

<u>Claim</u>: If $\chi' \in X$ sends a, b into $\langle a, b \rangle$, then it sends c into $\langle a, b \rangle c^{\pm 1} \langle a, b \rangle$.

Suppose this was not the case. Then the reduced word for $c\chi'$ would have more than one occurrence of letters from the set $\{c, c^{-1}\}$. Since χ' restricts to an automorphism of $\langle a, b \rangle$, there is $\chi'_0 \in \mathrm{Aut}(F_3, K)$ that maps (a, b, c) to $(a\chi', b\chi', c)$. Then $\psi = (\chi'_0)^{-1}\chi' \in \mathrm{Aut}(F_3, K)$ maps the triple $(a, b, c) \mapsto (a, b, c\chi')$. We apply Nielsen moves on this triple to cancel out the $\langle a, b \rangle$-prefix and suffix of $c\chi'$ to get the tuple $\vec{t} = (a, b, c^{\epsilon_1} \ldots c^{\epsilon_r})$ with $\epsilon_1, \epsilon_r \in \{\pm 1\}$. This tuple is Nielsen-reduced and clearly not Nielsen-equivalent to (a, b, c), contradicting the fact that $\vec{t}$ is a basis of F_3 (see [LS01, §I.2].) The claim now follows.

Thus, X must send c into $\langle a, b \rangle c^{\pm 1} \langle a, b \rangle$, and therefore (after composing with another inner automorphism by an element of $\langle a, b \rangle$), an element of $\chi \in X$ sends c to $gc^\epsilon \in \langle a, b \rangle c^\epsilon$, for $\epsilon \in \{-1, 1\}$. An inner automorphism preserving $\langle a, b \rangle$ and sending c in $\langle a, b \rangle c^{\pm 1}$ has to be trivial, and therefore, we obtain the first part of the statement. The set of $X \in G$ for which $\epsilon = 1$ clearly forms a normal subgroup, G_1. In G_1, the set of elements for which $\chi_0 = Id$ is again a normal subgroup, G_2, isomorphic to $\langle a, b \rangle$. It is clear that the quotient G_1/G_2 (isomorphic to $\mathrm{Aut}(F_2)$) lifts in G_1, making $G_1 \simeq (\mathrm{Aut}(F_2) \ltimes F_2)$. Finally, $G/G_1 \simeq \mathbb{Z}/2$ also clearly lifts in G, as the automorphism given by $\chi_0 = 1, g = 1, \epsilon = -1$, making G an iterated semi-direct product. One verifies that the involution on G_1 given by the later lift is the one given in the statement. $\square$

Notation 3.3.

○ Let G be a group and H a subgroup. For $x, y \in G$, we write $x \sim_H y$ if there exists an $h \in H$ such that $x^h = y$. We do not require x, y to be in H.

○ We say that $\sim_H$ is decidable if we can construct an algorithm which decides on inputs $x, y \in G$, whether $x \sim_H y$. That is, $\sim_H$ is decidable means that it is a recursive subset of G^2, and we have an explicit algorithm to decide membership.

Lemma 3.4. *Let* G *be a group,* H *a subgroup of* G *and* H_0 *a finite index subgroup of* H. *Then if* $\sim_{H_0}$ *is decidable and we can compute a complete set* $h_1, \ldots, h_k$ *of coset representatives of* H/H_0, *then so is* $\sim_H$.

Proof. Let $h_1, \ldots, h_k$ be a computed set of coset representatives of H_0 in H. Then $x \sim_H y$ if and only if there exists an i such that $x^{h_i} \sim_{H_0} y$.

Hence, we can concurrently decide each problem $x^{h_i} \sim_{H_0} y$ and therefore decide whether $x \sim_H y$. $\square$

Remark 3.5. Note that the finiteness of the index is crucial here. If one had an infinite recursive set of coset representatives, h_i, then it would still be true that $x \sim_H y$ if and only if there exists an i such that $x^{h_i} \sim_{H_0} y$. However, we would also require a termination criterion for when to stop checking each conjugacy, $x^{h_i} \sim_{H_0} y$, since after a finite time, only finitely many of these checks can have been made.

To put it another way, suppose for some N that $x^{h_i} \not\sim_{H_0} y$, for $1 \leq i \leq N$. Then is that because $x \not\sim_H y$, or is it because $x^{h_{N+1}} \sim_{H_0} y$?

Remark 3.6. Some care needs to be taken here. It is false – there are counterexamples – that if the conjugacy problem is solvable in a finite index subgroup, then it is solvable in the group. That is, there are examples of groups G with finite index subgroups H such that

○ $\sim_H \cap H^2$ is recursive,
○ $\sim_G$ is not recursive.

Hence, conjugacy is solvable in H but not in G. This does not contradict the Lemma above since we have the stronger hypothesis that $\sim_H$ is decidable.

The main tool we will use to solve conjugacy in G is the twisted conjugacy problem.

Definition 3.7. Let $\phi \in \mathrm{Aut}(F_n)$. Then $x, y \in F_n$ are said to be twisted ϕ conjugate, written $x \sim_\phi y$ if there exists a $w \in F_n$ such that

$$x = (w\phi)yw^{-1}.$$

We need the following later:

Lemma 3.8. *Let $\phi \in \mathrm{Aut}(F_n)$ and $x \in F_n$. Then for any $k \in \mathbb{Z}$, $x \sim_\phi x\phi^k$.*

Proof. Simply notice that $x\phi = (x\phi)xx^{-1}$ and $x\phi^{-1} = ((x^{-1}\phi^{-1})\phi)x(x\phi^{-1})$ and that $\sim_\phi$ is an equivalence relation. □

Theorem 3.9 (Theorem 1.5 of [BMMV06]). *Let $\phi \in \mathrm{Aut}(F_n)$. Then the twisted conjugacy problem for ϕ is solvable. That is, $\sim_\phi$ is a recursive subset of F_n^2.*

We will also need the following:

Theorem 3.10. *The conjugacy problem is solvable in $\mathrm{Out}(F_2)$ and $\mathrm{Aut}(F_2)$. Moreover, for any $\Phi \in \mathrm{Out}(F_2)$ of infinite order and for any $\phi \in \Phi$,*

○ *the centraliser of Φ in $\mathrm{Out}(F_2)$ is virtually cyclic;*
○ *there is an algorithm that computes the coset representatives of $\langle \Phi \rangle$ in its centraliser;*
○ *the centraliser of ϕ in $\mathrm{Aut}(F_2)$ has a finite index subgroup*

$$C_0 = \{\phi^k \mathrm{Ad}(g) \; : \; k \in \mathbb{Z}, \; g \in Fix(\phi)\} = \langle \phi \rangle \times \langle \mathrm{Ad}(g) \; : \; g \in Fix(\phi) \rangle;$$

○ *the group $D = \{\chi \in \mathrm{Aut}(F_2) \; : \; \phi^\chi = \phi \mathrm{Ad}(h), h \in F_2\}$ has a finite index subgroup $D_0 = \{\phi^k \mathrm{Ad}(x) \; : \; k \in \mathbb{Z}, x \in F_2\}$.*

Proof. We note that $\mathrm{Out}(F_2) \cong GL_2(\mathbb{Z})$ is virtually a free group of rank 2, and therefore word hyperbolic. Therefore, the conjugacy problem is solvable, and centralisers of infinite order elements are virtually cyclic (see [BH99, Γ-2.12, and Γ-3 Coro. 3.10]). For any element of infinite order, g, in a hyperbolic group, it is known to be algorithmic to find the coset representatives of $\langle g \rangle$ in its centraliser. This is folklore, but a detailed proof appears in Proposition 4.11 of [BMV10].

The solution for the conjugacy problem for $\mathrm{Aut}(F_2)$ appears in [Bog00] and also as Corollary 5.2 of [BMV10].

The rest of the statements about $\mathrm{Aut}(F_2)$ are then just translations of the corresponding statements about $\mathrm{Out}(F_2)$. □

We now address the conjugacy problem in $\mathrm{Out}(F_3, K)$ for elements that induce infinite order outer automorphisms on K.

Proof of Theorem 3.1. Let $G = \mathrm{Out}(F_3, K)$, and $\Phi, \Psi \in G$, given by data, as in Lemma 3.2, as

$$\Phi = (\epsilon_1, \phi_0, u) \quad \text{and} \quad \Psi = (\epsilon_2, \psi_0, v).$$

We assume that ϕ_0, ψ_0 define infinite order outer automorphisms of K. We must decide whether there exists a conjugator, $\chi = (\epsilon_3, \chi_0, h) \in G$, such that $\Phi^{\chi} = \Psi$. By Lemma 3.4, we may assume that $\epsilon_3 = 1$; namely, we take $H = G$ and H_0 to be the index two subgroup of G where $\epsilon = 1$. We calculate the possibilities for Φ^{χ}, listed as a triple:

$$\text{If } \epsilon_1 = 1 \quad \Phi^{\chi} = (1, \phi_0^{\chi_0}, (h^{-1}\phi_0^{\chi_0})(u\chi_0)h)$$

$$\text{If } \epsilon_1 = -1 \quad \Phi^{\chi} = (-1, \phi_0^{\chi_0} \operatorname{Ad}(h), h^{-1}(h^{-1}\phi_0^{\chi_0})(u\chi_0)). \tag{2}$$

Hence, if $\epsilon_1 = 1$, then for Φ and Ψ to be conjugate, we would require that ϕ_0 and ψ_0 are conjugate. If $\epsilon_1 = -1$, they would need to be conjugate as outer automorphisms.

We next observe that if $\epsilon_1 = \epsilon_2 = 1$ and ϕ_0 and ψ_0 are conjugate in $\operatorname{Aut}(F_2)$, then there is a conjugator $\chi \in \operatorname{Out}(F_3)$ so that the restriction of Φ^{χ} to F_2 equals the restriction of Ψ to F_2. This is clear from the equation above.

We also want to say that same in the case when $\epsilon_1 = \epsilon_2 = -1$, and we argue as follows. Suppose ϕ_0, ψ_0 are conjugate in $\operatorname{Out}(F_2)$ (more correctly, the outer automorphisms they represent are conjugate). Then there is a $\chi_0 \in \operatorname{Aut}(F_2)$ and $h \in F_2$ such that $\phi_0^{\chi_0} = \psi_0 \operatorname{Ad}(h^{-1})$. But then if we set $\chi = (1, \chi_0, h)$, we get that

$$\Phi^{\chi} = (-1, \phi_0^{\chi_0} \operatorname{Ad}(h), h^{-1}(h^{-1}\phi_0^{\chi_0})(u\chi_0))$$

$$= (-1, \psi_0, h^{-1}(h^{-1}\phi_0^{\chi_0})(u\chi_0)).$$

Hence, if Φ, Ψ are to be conjugate, we would require $\epsilon_1 = \epsilon_2$ (otherwise, we know that they are not conjugate). Moreover, by solving conjugacy in $\operatorname{Aut}(F_2)$ or $\operatorname{Out}(F_2)$, we may assume that $\phi_0 = \psi_0$ as automorphisms. To summarise, we have reduced the problem to the situation where

○ $\epsilon_1 = \epsilon_2$,
○ $\phi_0 = \psi_0$ in $\operatorname{Aut}(F_2)$.

Case 1: $\epsilon_1 = \epsilon_2 = 1$.:
In this case, we must have that $\phi_0^{\chi_0} = \psi_0 = \phi_0$. Hence, χ_0 centralises ϕ_0. Let $C = C_{\operatorname{Aut}(F_2)}(\phi_0)$ be the centraliser of ϕ_0 in $\operatorname{Aut}(F_2)$. Then χ lies in the subgroup, $C \ltimes F_2$ of G. By Lemma 3.4, it will be enough to solve the problem where we look at conjugators that lie in the subgroup, $C_0 \ltimes F_2$, where C_0 is the finite index subgroup of C given by Theorem 3.10. Hence, we may assume that $\chi_0 = \phi_0^k \operatorname{Ad}(x)$, where $k \in \mathbb{Z}$ and $x \in Fix(\phi_0)$.

But then we get that

$$\Phi^{\chi} = (1, \phi_0, (h^{-1}\phi_0)(u\phi_0^k \operatorname{Ad}(x))h) = (1, \phi_0, ((h^{-1}x^{-1})\phi_0)(u\phi_0^k)xh),$$

using the fact that $x\phi_0 = x$. Since $\Psi = (1, \phi_0, v)$, that means we are trying to decide whether there exist $k \in \mathbb{Z}$, $h \in F_2$, $x \in Fix(\phi_0)$ such that

$$(h^{-1}x^{-1})\phi_0(u\phi_0^k)xh = v.$$

Putting $h' = xh$, this is equivalent to deciding

$$(h'^{-1})\phi_0(u\phi_0^k)h' = v.$$

But by Lemma 3.8, this is equivalent to saying that $u \sim_{\phi_0} v$, which is decidable by Theorem 3.9.

Case 2: $\epsilon_1 = \epsilon_2 = -1$.:
As before, we have that $\phi_0 = \psi_0$. This time, by Equation 2, we get that χ_0 belongs to the subgroup D given by Theorem 3.10. Hence, by Lemma 3.4, we may assume that $\chi_0 \in D_0$. Hence, $\chi_0 = \phi_0^k \operatorname{Ad}(x)$

for some $x \in F_2$. Therefore, using Equation 2 and simplifying,

$$\Phi^{\chi} = (-1, \phi_0 \operatorname{Ad}(w), h^{-1}(h^{-1}\phi_0 \operatorname{Ad}(wh^{-1}))(u\chi_0)),$$

where $w = (x^{-1}\phi_0)xh$. However, since we have reduced to the case where $\phi_0 = \psi_0$, we get the extra restriction that $w = 1$ is equivalent to $h = x^{-1}(x\phi_0)$. And hence for Φ and Ψ to be conjugate, we would need to simply equate the third entry of the triple:

$$(h^{-1}\phi_0)h^{-1}x^{-1}(u\phi_0^k)x = v.$$

Putting $h = x^{-1}(x\phi_0)$, this gives

$$(x^{-1}\phi_0^2)(x\phi_0)(x^{-1}\phi_0)(u\phi_0^k)x = (x^{-1}\phi_0^2)(u\phi_0^k)x = v.$$

Hence, this problem is equivalent by Lemma 3.8 to $u \sim_{\phi_0^2} v$ or $u\phi_0 \sim_{\phi_0^2} v$, both of which are solvable by Theorem 3.9. □

Remark 3.11. A natural question to ask is whether the strategy above can be generalized to higher rank free groups. We note that ingredients used in the proof above are

○ *the conjugacy problem in* $\operatorname{Aut}(F_2)$. The conjugacy problem in $\operatorname{Aut}(F_n)$ appears to be harder than the conjugacy problem in $\operatorname{Out}(F_n)$.
○ *explicit computations of centralizers in* $\operatorname{Out}(F_2)$. In our case, we were able to exploit the fact that $\operatorname{Out}(F_2)$ is virtually free. It was shown in [BFH97] that centralizers of irreducible automorphisms are virtually cyclic. As for more general automorphisms, the current state of the art for polynomially growing automorphisms [Rod13, RW15, AM22a] amounts to establishing finiteness properties.

For these reasons, the argument above does not obviously generalize.

4. Polynomially growing automorphisms

Before turning our attention to the quadratic growth case in F_3 (i.e., the largest polynomial growth permitted there), let us record the combination of classical results on the conjugacy problem for low degree polynomial growth outer automorphisms of F_n.

Theorem 4.1 [Krs89, KLV01]. *The conjugacy problem in* $\operatorname{Out}(F_n)$ *is decidable among outer automorphisms with growth of polynomial degree 0 or 1.*

Recall also that the conjugacy problem among unipotent polynomially growing outer automorphisms of F_n has been solved by [FH19].

Quadratic polynomial growth in F_3.

Every $\Phi \in \operatorname{Out}(F_n)$ of polynomial growth has a power Φ^n that is unipotent (see [FH18, Coro. 3.14]). In fact, one can take this power to be uniform for every n; the exponent (or order) of $GL_n(\mathbb{Z}_3)$ suffices, as one can define a UPG automorphism as one of polynomial growth which induces the trivial map in $\mathbb{Z}_3$ homology (see [BFH05, Prop. 3.5]).

Definition 4.2. Two automorphisms $\phi, \psi \in \operatorname{Aut}(F_n)$ are said to be *isogredient* if they are conjugate by an inner automorphism.

Recall that any outer automorphism of F_n is a coset of $\operatorname{Inn}(F_n)$ in $\operatorname{Aut}(F_n)$. On each such outer automorphism, the relation of isogredience is an equivalence relation.

Theorem 4.3 (Bestvina-Handel Theorem, [BH92]). *Let* $\Phi \in \text{Out}(F_n)$. *Then,*

$$\sum \max\{\text{rank}(\text{Fix}\phi) - 1, 0\} \le n - 1,$$

where the sum is taken over representatives, ϕ, *of isogredience classes in* Φ.

Theorem 4.4. *Let* $\Phi \in \text{Out}(F_n)$ *have quadratic growth. Then,*

$$1 \le \sum \max\{\text{rank}(\text{Fix}\phi) - 1, 0\} \le n - 2.$$

If $n = 3$, *this sum is exactly equal to 1 and exactly one isogredience class has a nonzero contribution to this sum.*

Proof. This follows from results in either [Lev09] or [Mar02]. A full discussion of this is also given in [AM22b].

More specifically, as argued in [AM22b, Theorems 2.4.6 and 2.4.8], any outer automorphism, Φ of F_n, which satisfies the equality from 4.3, namely,

$$\sum \max\{\text{rank}(\text{Fix}\phi) - 1, 0\} = n - 1,$$

must have linear growth. □

Lemma 4.5. *Let* $\Phi \in \text{Out}(F_2)$ *be UPG. Then some automorphism in* Φ *has fixed subgroup of rank 2.*

Proof. It is easy to show that there is a basis of F_2, $\langle a, b \rangle$ where (some automorphism of) Φ acts as: $a \mapsto a, b \mapsto ba^k$ for some integer k. Hence, the fixed subgroup is $\langle a, bab^{-1} \rangle$ or $\langle a, b \rangle$ when $k = 0$. □

Theorem 4.6. *Let* $\Phi \in \text{Out}(F_3)$ *be a UPG automorphism of quadratic growth. Then* Φ *admits a rank 2 invariant free factor, K which is algorithmically computable. Moreover, K is unique in the sense that if K_1 is any other invariant rank 2 free factor, then K_1 is conjugate to K.*

Proof. By [BFH05], there is a basis, a, b, c for Φ such that some $\phi \in \Phi$ has the following representation. In fact, by [FH18], there is an algorithm to produce the following basis:

$$
\begin{array}{c}
\phi \\
a \mapsto a \\
b \mapsto ba^k \\
c \mapsto ucv,
\end{array}
$$

where $k \in \mathbb{Z}$ and $u, v \in \langle a, b \rangle$ and $\phi \in \Phi$.

Notice that $k = 0$ implies that Φ has linear growth, and hence, we deduce that $k \ne 0$. In particular, $\text{Fix}\phi = \langle a, bab^{-1} \rangle$ (it cannot have higher rank due to Theorems 4.3 and 4.4). Hence, $\langle a, b \rangle$ is the smallest free factor containing $\text{Fix}\phi$.

Thus, we have algorithmically produced a Φ-invariant rank 2 free factor, $\langle a, b \rangle$, and all that remains is to show is that it is unique.

If K_1 were another Φ-invariant rank 2 free factor, then the restriction of Φ to K_1 would again be UPG, and Lemma 4.5 would imply that some $\psi \in \Phi$ would have a fixed subgroup of rank 2, contained in K_1. But Theorem 4.4 then implies that ϕ and ψ are isogredient and hence have conjugate fixed subgroups implying that $\langle a, b \rangle$ and K_1 are also conjugate. □

Corollary 4.7. *Let* $\Phi \in \text{Out}(F_3)$ *be an automorphism of quadratic growth. Then* Φ *admits a unique invariant rank 2 free factor. Moreover, there is an algorithm to determine this free factor.*

Proof. Some positive power, Φ^k of Φ, is UPG. By Theorem 4.6, Φ^k admits a unique invariant rank 2 free factor, K. But $K\Phi$ is also Φ^k invariant, and hence, the uniqueness of K implies that K is conjugate to $K\Phi$.

Finally, if K_1 is any Φ-invariant rank 2 free factor, it must also be Φ^k-invariant, and hence, by Theorem 4.6 again, K_1 is conjugate to K.

This free factor is produced algorithmically as in Theorem 4.6.   □

Conjugacy problem in polynomial growth for Out F_3.

We may conclude for this section.

Recall (Proposition 2.7) that given Φ and Ψ two outer automorphisms of F_3, we may decide whether they are of polynomial growth, and we may compute their degree.

Assume first that both are polynomial growth of degree 2. We may compute by 4.7 the unique conjugacy classes $[K_\Phi]$, $[K_\Psi]$ of invariant free factor of rank 2 of F_3. After conjugating Ψ by a (computable) automorphism that sends K_Φ to K_Ψ, we may assume that these two groups are equal. We may then apply Theorem 3.1 in order to decide whether Φ and Ψ are conjugate in Out (F_3, K).

Since by Corollary 4.7, the rank 2 invariant free factor of Φ is unique, the two outer automorphisms are conjugate in Out (F_3, K) if and only if they are conjugate in Out (F_3).

This, together with Theorem 4.1, solves the conjugacy problem in Out F_3 for all polynomially growing outer automorphisms.

5. Exponential growth: laminations

Recall (references in Section 2, Proposition 2.7) that one can decide whether an outer automorphism is of exponential growth and whether it is irreducible. In that case, we recall the following.

Theorem 5.1 [Los96, Lus07, Kap14, FM22]. *The conjugacy problem is decidable among irreducible elements of* Out(F_n).

We will now focus on the reducible case.

Invariant free factor systems in F_3

We will now work towards describing reducible exponentially growing automorphisms. Following [BFH00], we have the following.

Proposition 5.2. *Let* $\Phi \in$ Out(F_3). *Then any relative train track representative of* Φ *has at most one exponential stratum.*

Proof. Suppose that $f : \Gamma \to \Gamma$ is a relative train track representative of Φ. Suppose that H_r is an exponential stratum and that G_r is the corresponding f-invariant subgraph (the union of all the strata, up to and including H_r).

Similarly, let G_{r-1} be the f-invariant subgraph consisting of the union of the strata up to, but not including, H_r.

Let C_r be a connected component of H_r and let C_{r-1} be a connected component of $C_r \cap G_{r-1}$ (which we allow to be empty). Since the transition matrix of H_r is irreducible, some power of f must leave both C_r and C_{r-1} invariant. However, since H_r is an exponential stratum, the rank of $\pi_1(C_r)$ is at least 2 more than the rank of $\pi_1(C_{r-1})$. (For instance, take an edge of H_r incident to C_{r-1} and take a power, k, of f such that $f^k(e)$ crosses e at least three times). This immediately gives the result: either the rank of $\pi_1(C_{r-1})$ is zero and the rank of $\pi_1(C_r)$ is at least two, in which case there can only be zero or polynomial strata above H_r, or the rank of $\pi_1(C_{r-1})$ is one, and there can be no strata above H_r.   □

Corollary 5.3. *Let* $\Phi \in$ Out(F_3) *have exponential growth. Then* Φ *has exactly one attracting lamination. This lamination is carried by a* Φ-*invariant conjugacy class of a free factor, K of F_3.*

The rank of K is either 2 or 3. If the rank of K is 2, then it is unique in the following sense: if K_1 is another free factor of rank 2 which is Φ-invariant (up to conjugacy), then K_1 is conjugate to K.

Proof. Some positive power of Φ has an improved relative train track representative by [BFH00, Theorem 5.1.5].

By the proof of Lemma 3.1.13 of [BFH00], there is a one-to-one correspondence between exponential strata of an eg-aperiodic relative train track map (such as an improved relative train track map) and the attracting laminations of Φ.

However, by Proposition 5.2, our improved relative train track for Φ only admits one exponential stratum, and therefore, Φ only has one attracting lamination which is therefore fixed by Φ (and not just periodic for Φ).

As in [BFH00, Corollary 2.6.5 and Definition 3.2.3], there is a unique free factor whose conjugacy class carries the lamination of Φ. Since the lamination is fixed by Φ, this free factor is Φ-invariant up to conjugacy.

For the final part, we note that an exponential stratum cannot have a component which has rank 1 as a graph (is homotopic to a circle), so K must have rank 2 or 3.

Furthermore, if K has rank 2, and if K_1 is a Φ-invariant free factor (up to conjugacy) also of rank 2, then we can form – via [BFH00, Theorem 5.1.5] – an improved relative train track representative for some power of Φ, where K_1 is the fundamental group of some invariant subgraph, G_r. If the restriction of the relative train track map to G_r is polynomial, then the whole automorphism will have polynomial growth; this is excluded by hypothesis. Hence, by [BFH00, Lemma 3.1.9], there is a lamination carried by (the conjugacy class of) K_1. But there is only one lamination for Φ; hence, K and K_1 are conjugate. □

Conjugacy problem for exponential growth with rank 2 *lamination*

Consider Φ and Ψ two outer automorphisms of F_3 that are of exponential growth, and whose attracting lamination is carried by a free factor of rank 2, respectively K_Φ and K_Ψ.

By Proposition 2.9 (and the unicity of Corollary 5.3), the groups K_Φ and K_Ψ can be computed.

Since both K_Φ and K_Ψ are free factors of same rank, there exists (and one can compute) an automorphism χ of F_3 sending K_Ψ to K_Φ, and after conjugating Ψ by the outer-class X of χ, we may assume that $K_\Phi = K_\Psi$, and we denote it K.

Since by Corollary 5.3, this invariant free factor is unique, the two automorphisms Φ and Ψ are conjugated in $\text{Out}(F_3)$ if and only if they are conjugated in $\text{Out}(F_3, K)$. This can be decided by Theorem 3.1.

6. Exponential growth: mapping tori

In this section, we take the point of view of analysing the semi-direct products of F_3 that are associated to automorphisms (see also [Sel95, Dah16, Dah17, DT21]). Although this point of view allows to treat the conjugacy problem for all the exponentially growing outer automorphisms of F_3, we restrict our presentation to the remaining case in Flowchart 1 – namely, the case of automorphisms whose attracting lamination carrier is the entire group F_3.

Given $\phi \in \text{Aut}(F_n)$, the associated mapping torus is $F_n \rtimes_\phi \langle t \rangle$. The normal subgroup $F_n < F_n \rtimes_\phi \langle t \rangle$ is called a *fiber*. If $\phi = \psi \, \text{Ad}(g)$, then we have $F_n \rtimes_\psi \langle t \rangle = F_n \rtimes_\phi \langle tg \rangle$; in particular, $\Phi = [\phi] \in \text{Out}(F_n)$ has a well-defined mapping torus. The following proposition describes how it relates to conjugacy in $\text{Out}(F_n)$. We refer the reader to [DT21] for precise definititions.

Proposition 6.1 (Standard mapping torus criterion for conjugacy). *Let G be a group, $\Phi, \Psi \in \text{Out}(G)$ and $\phi \in \Phi, \psi \in \Psi$. Then Φ is conjugate to Ψ in $\text{Out}(G)$ if and only if there is an isomorphism*

$$f : G \rtimes_\phi \langle t \rangle \to G \rtimes_\psi \langle t \rangle$$

such that $f(G) = G$ and $f(t) = tw$ for some $w \in G$.

The following theorem relates dynamical characteristics of outer automorphisms to the structure of their mapping tori.

Theorem 6.2 (Relative hyperbolicity in exponential growth). *For any* $\Phi \in \mathrm{Out}(F_n)$, *its mapping torus admits a properly relatively hyperbolic (possibly word-hyperbolic) metric if, and only if, Φ has exponential growth. Peripheral subgroups of the relatively hyperbolic structure can be taken to be the mapping tori of Φ restricted to maximal polynomially growing subgroups.*

Proof. The converse implication was obtained in [Gho23, Theorem 3.1] and [DL20, Theorem 4]. The direct implication is found in [Mac02], [Dah17, Prop 1.3] see also [Hag19], [Bri00, Theorem 1.2]. □

Using this strategy, in [DT21], the conjugacy problem in $\mathrm{Out}(F_n)$ is completely reduced to specific algorithmic problems in the peripheral subgroups. These problems are the algorithmic tractability for their subgroups (effective coherence, conjugacy problem, generation problem), the Minkowski property for certain subgroups, the mixed Whitehead problem, and the conjugacy problem for the induced automorphisms on maximal polynomially growing subgroups. In this reduction, the exponential growth part of the outer automorphisms is completely evacuated from the discussion (it is treated during the reduction).

In the case of exponentially growing automorphisms of F_3, the polynomially growing subgroups are sufficiently small that we can complete a solution of the conjugacy problem in their case. We will explain this in the remaining case of Flowchart 1, in which the polynomially growing subgroups are even simpler.

This will require two steps. The first step is giving the solution to the criterion of Proposition 6.1 in the case the mapping tori of the given autmorphisms are so-called almost toral. The second step is proving that if the mapping torus of an automorphism is not almost toral, then there is a rank 2 free factor carrying the attracting lamination. This allows to conclude since this later case was already treated. Treating it alternatively through the criterion of Proposition 6.1 is still possible and involves cases covered by the larger study [DT23], but it is not done here.

The almost toral case

We say $\Phi \in \mathrm{Out}(F_n)$ or its mapping torus $F_n \rtimes_\phi \mathbb{Z}$, $\phi \in \Phi$ is *almost toral relatively hyperbolic* if $F_n \rtimes_\phi \mathbb{Z}$ is hyperbolic relative to a collection of subgroups isomorphic to $\mathbb{Z} \times \mathbb{Z}$ or to $\mathbb{Z} \rtimes \mathbb{Z}$, the fundamental group of a Klein bottle. By extension, in that case, we say that the automorphism is almost toral relatively hyperbolic.

In the relatively hyperbolic structure of Theorem 6.2, the peripheral subgroups are semi-direct products of free groups; hence, the groups $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z} \rtimes \mathbb{Z}$ are the only possible virtually abelian peripheral subgroups. Actually, the following is immediate from Theorem 6.2 and Nielsen-Schreier theorem.

Proposition 6.3. *The mapping torus of an outer automorphism of F_n is almost toral if and only if its maximal polynomially growing subgroups have rank one.*

It is furthermore decidable whether the mapping torus of a given automorphism of F_n is almost toral, by [DG13] (this is decidable in the following sense: there is an algorithm that will terminate if the automorphism is exponentially growing, and provide presentations for each conjugacy representative of peripheral subgroup and indicate whether or not they are abelian or isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}$).

Theorem 6.4. *The conjugacy problem in $\mathrm{Out}(F_n)$ is decidable among the almost toral relatively hyperbolic automorphisms.*

Proof. We are given ϕ, ψ two automorphisms of F_n, such that $F_n \rtimes_\phi \langle t \rangle$ and $F_n \rtimes_\phi \langle t \rangle$ are relatively hyperbolic groups with peripheral subgroups either $\mathbb{Z}^2$ or $K = \mathbb{Z} \rtimes \mathbb{Z}$.

By [DG13], we may compute the peripheral subgroups of both groups. More specifically, we may get a list of conjugacy representatives of these subgroups given by generating pairs.

If the numbers of conjugacy classes of peripheral subgroups in $F_n \rtimes_\phi \langle t \rangle$ and in $F_n \rtimes_\phi \langle t \rangle$ are different, then the two groups cannot be isomorphic, and ϕ and ψ cannot be conjugate in $\mathrm{Out}(F_n)$. We thus assume these numbers are equal.

If there are no peripheral subgroups in $F_n \rtimes_\phi \langle t \rangle$ and in $F_n \rtimes_\phi \langle t \rangle$, these groups are hyperbolic, and we may use [Dah16] to determine whether ϕ and ψ are conjugate in $\mathrm{Out}\, F_n$. We now assume that there are peripheral subgroups in both mapping tori.

Using [DT21], in order to determine whether ϕ and ψ are conjugate in $\mathrm{Out}(F_n)$, it suffices to establish that these possible peripheral subgroups (i.e., $\mathbb{Z}^2$ and $K = \mathbb{Z} \rtimes \mathbb{Z}$) and their subgroups form an algorithmically tractable class of groups, with the (effective) Minkowski property, and with solvable fiber-and-orientation preserving mixed Whitehead problem.

First, notice that all the subgroups of $\mathbb{Z}^2$ and K are themselves isomorphic to either $\mathbb{Z}^2, K, \mathbb{Z},$ or $\{1\}$, so we only discuss the properties for $\mathbb{Z}^2$ and K. Algorithmic tractability is actually immediate (despite its definition).

We now address the Minkowski property for $\mathbb{Z}^2$ and K. Recall that it means finding a characteristic finite quotient of them such that every finite order outer automorphism of $\mathbb{Z}^2$ or K induces a non-inner automorphism in it.

For $\mathbb{Z}^2$, the quotient $\mathbb{Z}^2 \to (\mathbb{Z}/3)^2$ is sufficient because (and this is the classic observation of Minkowski giving the name of the property) $GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}/3)$ has a torsion-free kernel.

For K, let us write it as $K = \langle a \rangle \rtimes \langle t \rangle$. Recall that the derived subgroup is $\langle a^2 \rangle$, and its abelianisation is $K/\langle a^2 \rangle \simeq (\mathbb{Z}/2) \times \mathbb{Z}$. It follows that $K \to (\mathbb{Z}/2) \times (\mathbb{Z}/3)$ (sending a to $(1,0)$ and t to $(0,1)$) is a characteristic quotient. We will show that it is a suitable quotient for the Minkowski property.

Any automorphisms of K preserve $\langle t^2 \rangle$ (the center) and $\langle a \rangle$ (the preimage of the torsion of the abelianisation). Since the set of square roots of t^2 is $\{a^k t, k \in \mathbb{Z}\}$, the automorphisms of K are of the form $(a \mapsto a^{\epsilon_1}, t \mapsto a^k t^{\epsilon_2})$ for $\epsilon_1; \epsilon_2 \in \{-1, 1\}$ and $k \in \mathbb{Z}$. Since the automorphisms $(a \mapsto a^{-1}, t \mapsto t)$ and $(a \mapsto a, t \mapsto a^2 t)$ are both inner, it follows that

$$\mathrm{Out}\, K \simeq (\mathbb{Z}/2) \times (\mathbb{Z}/2),$$

with representatives being $(a \mapsto a, t \mapsto a^u t^\epsilon)$, for $u \in \{0, 1\}$ and $\epsilon \in \{-1, 1\}$. This shows that the proposed quotient is suitable for establishing the Minkowski property of K.

It remains to solve the fiber-and-orientation preserving mixed Whitehead problem for $\mathbb{Z}^2 = \langle a \rangle \times \langle t \rangle$, and for $K = \langle a \rangle \rtimes \langle t \rangle$ (the fiber being $\langle a \rangle$). It is the problem of determining whether two tuples of conjugacy classes of tuples are in the same orbit for the group of automorphisms preserving $\langle a \rangle$ and $t \langle a \rangle$.

In the case of $\mathbb{Z}^2 = \langle a \rangle \times \langle t \rangle$, this subgroup of automorphisms is $\{(a \mapsto a^\epsilon, t \mapsto a^k t), \epsilon \in \{-1, 1\}, k \in \mathbb{Z}\}$. The tuples of conjugacy classes of tuples are just tuples of elements in an abelianisation group. This immediately translates as a trivial divisibility problem for integers.

In the final case of $K = \langle a \rangle \rtimes \langle t \rangle$, the subgroup of fiber and orientation preserving automorphisms consists of only two outer classes (since $\mathrm{Out}\, K \simeq (\mathbb{Z}/2) \times (\mathbb{Z}/2)$). The orbit problem is then easily solved by applying the two automorphisms and using a solution to the conjugacy problem. □

The polynomially growing subgroups in F_3

By Levitt's theorem [Lev09, Theorem 4.1], given an outer automorphism of F_n, its maximal polynomially growing subgroups have rank $\leq n$, they have rank $< n$ if the automorphism has an exponentially growing conjugacy class, and in the later case, if one such group has rank $n - 1$, it is unique up to conjugacy.

In the case of $n = 3$, the possible polynomially growing subgroups for an exponentially growing outer automorphism of F_3 are then either trivial, or cyclic, or of rank 2. Given the previous discussion, we consider the case of a single conjugacy class of polynomially growing subgroup of rank 2.

It turns out, as we will show, that in this case, the polynomially growing subgroup must be placed very specifically in the group F_3, revealing a lamination carrier of rank 2. The next proposition shows in particular that all such automorphisms must be conjugated to automorphisms given in Example 1.4.

Although this particular detail is not used in the solution to the conjugacy problem, it gives insights into how to represent the conjugacy classes of outer automorphisms that fall into this case.

Recall that if D is a Φ-invariant subgroup, we can find $\phi \in \Phi$ such that $\phi(D) = D$.

Proposition 6.5. *If $\phi \in \mathrm{Aut}(F_3)$ is of exponential growth and has an invariant subgroup D of rank at least 2 on which it induces a polynomially growing automorphism, then it preserves the conjugacy class of a rank 2 free factor Q on which it induces an exponentially growing outer automorphism.*

Moreover, the conjugacy class of such a free factor is unique, and there exists a conjugate D' of D, containing a subgroup C' of $Q \cap D'$, such that C' is generated by the commutator of a basis in Q, and $F_3 = Q *_{C'} D'$.

Proof. We assume without loss of generality that D is a maximal ϕ-invariant subgroup on which $\phi|_D$ is polynomially growing. By maximality, we cannot have $D < H < F_3$ with H of rank 2 and ϕ-invariant. Indeed, if this were the case, either $\phi|_H$ is either polynomially growing, which contradicts maximality, or exponentially growing with a polynomially growing invariant subgroup of rank 2, which is impossible in $\mathrm{Out}(F_2)$. Furthermore, D cannot be a free factor of F_3, as the expression (1) from Lemma 3.2 would imply that ϕ is also polynomially growing. It follows that (F_3, D) is relatively one-ended.

By [GJLL98, Theorem II.2], F_3 acts faithfully on an $\mathbb{R}$-tree T_∞ with trivial arc stabilizers, and there is a homothety $H : T_\infty \to T_\infty$ such that for all $x \in T_\infty$ and $f \in F_3$, $f \cdot H(x) = H((f\phi)x)$ (here, a homothety is a map satisfying $d(H(x), H(y)) = \lambda d(x, y)$ for some fixed stretch factor $\lambda \geq 1$). Note that from [GJLL98, §B - §E], the action of F_3 on T_∞ is obtained as a limit of rescaled actions on a fixed free cocompact action of F_3 on a simplicial metric tree τ. Exponential growth of ϕ implies that D fixes a point in T_∞.

Trivial arc stabilizers and the nontrivial subgroup D (which cannot be in a proper free factor) that fixes a point (so the action is not free) imply that we may apply [Hor14, Lemma 4.6] to the action of F_3 on T_∞ with the empty free factor system. This lemma gives three possibilities. Two of these imply that point stabilizers must all either be cyclic or contained in proper free factors of F_3, which is impossible because of the properties of D. The remaining possibility is that the action of F_3 on T_∞ has a so-called dynamical proper free factor. This implies that T_∞ is not a simplicial $\mathbb{R}$-tree, since by definition, a dynamical proper free factor must act on its minimal invariant tree with dense orbits.

We now apply [Gui08, Theorem 5.1]. The triviality of arc stabilizers, one-endeness of F_3 relative to D, and the faithfullness of the action of F_3 on T_∞ imply that the action of F_3 on T_∞ decomposes into a graph of actions where each vertex action is either simpicial, Seifert type or axial. Free groups cannot admit faithful axial actions, and since T_∞ is not simplicial, the graph of actions must contain a Seifert type vertex. Orbifolds with free fundamental groups are surfaces with boundary. Therefore, F_3 decomposes as a graph of groups Y, and one of the vertex groups Y_q must be isomorphic to the fundamental group of a surface Σ with boundary, equipped with a measured foliation $\mathcal{F}$ with dense leaves, for which the action of Y_q on the minimal invariant subtree $(T_\infty)_{Y_q}$ is equivariantly isomorphic to the action of $\pi_1(\Sigma)$ on the $\mathbb{R}$-tree dual to lifted measured foliation on the universal cover $(\tilde{\Sigma}, \tilde{\mathcal{F}})$. The subgroup D, elliptic in T_∞, must also lie in some vertex group Y_D. Since D is a free group of rank 2 that fixes a point, it cannot be conjugate into the subgroup Y_q, so $Y_q \neq Y_D$.

By the definition of a graph of actions (see [Gui08, §1.3]), the edge group incident to Y_q must be a point stabilizer, which in turn must either be trivial or conjugate to the π_1-image of $\partial\Sigma$. The former case gives a free decomposition of F_3 relative to D which contradicts earlier considerations. The latter case implies that all the edges adjacent to q have cyclic edge groups. It also follows that if we collapse all edges of Y that have noncyclic edge group to get a new graph of groups $\bar{Y}$, Y_q is still a vertex group of this collapsed splitting and $\bar{Y}$ has another vertex group containing D. It follows that F_3 admits a nontrivial cyclic splitting relative to D containing Y_q as a *quadratically hanging (QH) subgroup* – that is to say, it is isomorphic to $\pi_1(\Sigma)$, where Σ is a compact surface with boundary and the (conjugacy classes) of the incident edge groups coincide with the π_1-image of $\partial\Sigma$.

Because F_3 is one ended relative to D, by [GL17, Theorem 9.5], F_3 admits a canonical cyclic JSJ decomposition J relative to D. By our description of $\bar{Y}$ above, we know that J has at least two vertex

groups and that one of them is a maximal QH subgroup. Arc stabilizers of the Bass-Serre tree dual to J are cyclic, and there are at least two vertex orbits so we can apply – for example, [GL95] – to conclude that J has exactly two noncyclic vertex groups and that these vertex groups have rank exactly 2. One of these vertex groups contains (a conjugate of) Y_q and is a maximal QH subgroup. The other vertex group contains D. The automorphism invariance of JSJ decompositions, [GL17, Corollary 7.4], implies that ϕ must preserve J (i.e., ϕ maps vertex groups and edge groups to conjugates of vertex and edge groups (respectively)). Since Q is the unique flexible subgroup of J, we must have that $\phi(Q)$ is mapped to a conjugate of Q.

Now the vertex group Y_q from the graph of actions Y sits inside the QH subgroup Q as the π_1-image of some subsurface since both groups have the same rank, $Y_q = Q$ (up to conjugacy). By [CV91], the only Seifert-type action of F_2 is dual to an irrational foliation on a once punctured torus. It follows that the QH strucutre on Q is that of an orientable surface of genus 1 with one boundary component. Therefore, there is exactly one edge group incident to Q.

Let $\bar{J}$ be the graph of groups obtained by contracting all edges except the unique edge adjacent to the unique QH vertex group Q. Then $\bar{J}$ is the amalgamated product $F_3 = D' *_C Q$. D' must contain a conjugate of D, and by construction, this splitting is ϕ-invariant, so $D' = D$. Since C is conjugate to the π_1-image of $\partial\Sigma$ in $\pi_1(\Sigma) = Q$, we have that Q is one-ended relative to C. Since F_3 is many-ended, by [She55, Swa86, Tou15], D is forced to admit a free decomposition $D = \langle d \rangle * \langle c \rangle$ with $C \leq \langle c \rangle$. This gives

$$F_3 = \underbrace{\langle d \rangle * (\langle c \rangle}_{D} *_C Q).$$

Since Q must be the fundamental group of a torus Σ with a boundary component, we have that C, the π_1-image of $\partial\Sigma$, must be a commutator and therefore cannot be a proper power; thus, $C = \langle c \rangle$, so Q is a free factor of F_3, as required. Its uniqueness follows from the canonicity of the JSJ decomposition.

By [GJLL98, Theorem II.1], if the stretch factor of the homothety H is $\lambda = 1$, then T_∞ is simplicial. Since that Q acts on $(T_\infty)_Q$ with dense orbits, we have $\lambda > 1$. Since up to composition with an inner automorphism we have $\phi(Q) = Q$, we have that ϕ induces an exponentially growing automorphism on Q. □

In particular, we get the following.

Corollary 6.6. *Let $\Phi \in \mathrm{Out}(F_3)$ be exponentially growing, not fully irreducible, and such that no power has an invariant proper free factor of rank 2. Then the mapping torus $F_3 \rtimes_\phi \mathbb{Z}$ is almost toral relatively hyperbolic.*

This concludes all cases of Flowchart 1, and thus proves Theorem 1.1.

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