Appendix A

An Aside on $O(4)$

$O(4)$ is the group defined by the multiplication properties of the set of orthogonal matrices which keep the quadratic form

$$\sum_{i=1}^{4} x_i^2 \tag{A.1}$$

invariant. If

$$\vec{x}' = \mathcal{O}\vec{x} \tag{A.2}$$

then

$$\begin{aligned} \vec{x}' \cdot \vec{x}' &= \vec{x} \cdot \mathcal{O}^{\mathrm{T}} \mathcal{O} \cdot \vec{x} = \vec{x} \cdot \vec{x} \\ &\Rightarrow \mathcal{O}^{\mathrm{T}} \mathcal{O} = 1 \end{aligned} \tag{A.3}$$

where $\vec{x}$ is a four-dimensional vector. Looking in the neighbourhood of the identity, we find, with $\mathcal{O} = 1 + \delta$, then

$$\begin{aligned} \mathcal{O}^{\mathrm{T}} \mathcal{O} &= \left(1 + \delta^{\mathrm{T}}\right)(1 + \delta) = 1 + \delta + \delta^{\mathrm{T}} + o(\delta^2) = 1 \\ &\Rightarrow \delta + \delta^{\mathrm{T}} = 0. \end{aligned} \tag{A.4}$$

This means that δ must be an anti-symmetric, 4×4 matrix. This defines the Lie algebra of $O(4)$. The complete set of anti-symmetric 4×4 matrices is given by

$$\begin{aligned} (M_{\mu\nu})_{\sigma\tau} &= \delta_{\mu\sigma}\delta_{\nu\tau} - \delta_{\mu\tau}\delta_{\nu\sigma} \\ &= \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\epsilon_{\lambda\rho\sigma\tau}. \end{aligned} \tag{A.5}$$

It is easy to calculate

$$[M_{\mu\nu}, M_{\sigma\tau}] = \frac{1}{4}\left(\epsilon_{\mu\nu\lambda\rho}\epsilon_{\lambda\rho\gamma\delta}\epsilon_{\sigma\tau\delta\beta}\epsilon_{\sigma\tau\alpha\beta} - \epsilon_{\alpha\beta\sigma\tau}\epsilon_{\sigma\tau\gamma\delta}\epsilon_{\mu\nu\lambda\rho}\epsilon_{\lambda\rho\delta\omega}\right). \tag{A.6}$$

We can expand this further; it is easy to do some of the sums over dummy indices, but it is more illuminating to define

$$J_i = \frac{1}{2}\epsilon_{ijk}\left(M_{jk}\right)_{lm} = \frac{1}{2}\epsilon_{ijk}\left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right) = \epsilon_{ilm} \tag{A.7}$$

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and

$$K_i = (M_{0i})_{lm}\,. \tag{A.8}$$

Then the commutators

$$\begin{aligned}[J_i, J_j] &= \epsilon_{ijk} J_k \\ [J_i, K_j] &= \epsilon_{ijk} K_k\end{aligned} \tag{A.9}$$

follow directly, with $J_1 = M_{23}$, $J_2 = M_{31}$ and $J_3 = M_{12}$. To calculate $[K_i, K_j]$ we consider the generators

$$\tilde{J}_1 = M_{12}, \quad \tilde{J}_2 = M_{20}, \quad \tilde{J}_3 = M_{01}, \tag{A.10}$$

which generate the subgroup that leaves the form $x_0^2 + x_1^2 + x_2^2$ invariant. Then because of rotational symmetry we must have

$$\left[\tilde{J}_i, \tilde{J}_j\right] = \epsilon_{ijk} \tilde{J}_k. \tag{A.11}$$

(We can check this, for example, with $\left[\tilde{J}_1, \tilde{J}_2\right] = [M_{12}, M_{20}] = [M_{12}, M_{20}] = [J_3, -K_2] = -\epsilon_{321} K_1 = K_1 = M_{01} = \tilde{J}_3$.) Thus

$$\left[\tilde{J}_2, \tilde{J}_3\right] = [M_{20}, M_{01}] = [-K_2, K_1] = \tilde{J}_1 = M_{12} = J_3 \tag{A.12}$$

thus

$$[K_1, K_2] = J_3 \tag{A.13}$$

hence rotational covariance dictates the general relation

$$[K_i, K_j] = \epsilon_{ijk} J_k. \tag{A.14}$$

The combinations

$$M_i^{\pm} = \frac{1}{2}\left(J_i \pm K_i\right) \tag{A.15}$$

satisfy the commutators

$$\left[M_i^{\pm}, M_j^{\pm}\right] = \epsilon_{ijk} M_k^{\pm} \tag{A.16}$$

while

$$\left[M_i^{+}, M_j^{-}\right] = 0. \tag{A.17}$$

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