

VARIANCE-OPTIMAL HEDGING IN GENERAL AFFINE STOCHASTIC VOLATILITY MODELS

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Abstract

We consider variance-optimal hedging in general continuous-time affine stochastic volatility models. The optimal hedge and the associated hedging error are determined semiexplicitly in the case that the stock price follows a martingale. The integral representation of the solution opens the door to efficient numerical computation. The setup includes models with jumps in the stock price and in the activity process. It also allows for correlation between volatility and stock price movements. Concrete parametric models will be illustrated in a forthcoming paper.

Keywords: Variance-optimal hedging; stochastic volatility; Galtchouk–Kunita–Watanabe decomposition; affine process; Laplace transform

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1. Introduction

Often observed so-called stylized features of stock returns include semiheavy tails, volatility clustering, and the leverage effect (i.e. negative correlation between changes in volatility and stock prices). Stochastic volatility (SV) models account for these observations. Examples in the literature include the Heston model [14] and the Lévy-driven stochastic volatility models put forward in [2]. Other SV models are based on time-changed Lévy processes as in [7]. All these examples are affine in the sense of [10]. Further instances of such affine stochastic volatility models are discussed in [18] and [33].

Stochastic volatility typically leads to an incomplete market, i.e. perfect hedging strategies do not exist for many contingent claims. As a way out, we may try to minimize the expected squared hedging error

$$E[(v + \vartheta \cdot S_T - H)^2]$$

over all initial endowments $v \in \mathbb{R}$ and all admissible hedging strategies ϑ . Here, H denotes the discounted payoff at time T of a European-style contingent claim and S denotes the discounted price process of the underlying stock. The stochastic integral $\vartheta \cdot S$ stands for the gains from dynamic trading in the stock according to strategy ϑ .

This problem and in particular its general structure have been extensively studied in the literature; cf., e.g. [1], [8], [29], [35], and the references therein. If S is a martingale, the solution is determined in [11] based on the Galtchouk–Kunita–Watanabe decomposition. To be more specific, let $V_t := E[H \mid \mathcal{F}_t]$ denote the martingale generated by H . Then $v^* := V_0$ is

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the optimal initial endowment in the above hedging problem and the optimal hedging strategy can be written as

$$\vartheta_t^* = \frac{d\langle V, S \rangle_t}{d\langle S, S \rangle_t}, \quad (1.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the predictable covariation process. The corresponding hedging error equals

$$E[(v^* + \vartheta^* \cdot S_T - H)^2] = E[\langle V, V \rangle_T - \langle \vartheta^* \cdot S, \vartheta^* \cdot S \rangle_T]. \quad (1.2)$$

Alternative representations of the solution are provided in [5] using the *carré du champ* operator and in [4] based on Malliavin derivatives. All these general characterisations do not immediately lead to numerical results in concrete models. For example, it may not be obvious how to evaluate (1.1) and (1.2) because analytical expressions for V are typically not available.

Numerical approaches are discussed in, e.g. [9], [13], and [16]. Heath *et al.* [13] used partial differential equation methods for specific continuous stochastic volatility models. Cont *et al.* [9] and similarly Hubalek and Sgarra [15] considered an SV model involving jumps. A partial integro-differential equation is solved by finite-difference schemes in order to obtain the process V above. The hedging error is computed by Monte Carlo simulation. This approach is applied to exotic contingent claims and it allows for options as hedging instruments.

In this paper we study variance-optimal hedging in a general affine stochastic volatility model. The objective is to determine semiexplicit expressions for the optimal hedging strategy and the hedging error which can be evaluated without implementing involved numerical schemes or computer-intensive Monte Carlo simulations. They are obtained with the help of integral transform techniques which are used widely in option pricing (cf., e.g. [6] and [31]) and in [16] for the mean-variance hedging problem without stochastic volatility.

We focus on the case that S is a martingale. Firstly, this allows us to cover a broader class of volatility structures than without this restriction, e.g. those involving a leverage term. Secondly, we believe that the excess drift of asset prices is of secondary importance for the hedging problem. Finally, quadratic hedging appears as an auxiliary problem in a first-order approximation to utility-based derivative pricing and hedging; cf. [3], [19], [21], [24], [25], and [26]. Here the variance-optimal hedge must be determined under some equivalent martingale measure, i.e. S is by default a martingale under the relevant measure. For a treatment of the more involved non-martingale case, we refer the reader to [23], which generalizes some of the present results. In such context a measure change to the—generally signed—variance-optimal martingale measure plays a major role implicitly or explicitly. However, the setup in [23] allows for leverage only in very special cases. Moreover, it is written on a less rigorous mathematical level.

The structure of the paper is as follows. In Section 2 we introduce the general affine stochastic volatility model. Subsequently, we discuss an integral representation of the contingent claim that is to be hedged. Section 4 contains the solution to the hedging problem in this setup. For numerical results in concrete parametric models, we refer to the forthcoming paper [20]. Proofs of the main results are to be found in the final section.

Unexplained notation is used as in [17]. Superscripts generally refer to components of a vector or vector-valued process rather than powers. The few exceptions should be obvious from the context. As a key tool, we need the notion of semimartingale characteristics (B, C, ν) . For a summary of important results, we refer the reader to Appendix A. Here $h = (h_1, h_2)$ generally denotes a componentwise truncation function on \mathbb{R}^2 , i.e. $h(x_1, x_2) = (\tilde{h}(x_1), \tilde{h}(x_2))$, where $\tilde{h}: \mathbb{R} \rightarrow \mathbb{R}$ is a one-dimensional truncation function, which can, e.g. be chosen of the

form

$$\tilde{h}(x_k) = \begin{cases} 0 & \text{if } x_k = 0, \\ (1 \wedge |x_k|) \frac{x_k}{|x_k|} & \text{otherwise.} \end{cases} \tag{1.3}$$

All characteristics and Lévy–Khintchine triplets on \mathbb{R} and \mathbb{R}^2 , respectively, are expressed relative to truncation functions \tilde{h} and h , respectively. For ease of exposition, we occasionally use the particular function (1.3) in the proofs, but the results hold for arbitrary \tilde{h} .

2. The general affine stochastic volatility model

We denote the discounted price process of a univariate stock by $S = S_0 \exp(Z)$, i.e. Z stands for logarithmic returns. Moreover, we consider a positive *activity process* y leading to randomly changing volatility in our general setup. We assume that the bivariate process (y, Z) is an affine semimartingale in the sense of [10]. More specifically, we suppose that the characteristics $(B^{y,Z}, C^{y,Z}, \nu^{y,Z})$ of the $\mathbb{R}^+ \times \mathbb{R}$ -valued semimartingale (y, Z) are of the form

$$B_t^{y,Z} = \int_0^t (\beta_{(0)} + \beta_{(1)}y_{s-}) ds, \tag{2.1}$$

$$C_t^{y,Z} = \int_0^t (\gamma_{(0)} + \gamma_{(1)}y_{s-}) ds,$$

$$\nu^{y,Z}([0, t] \times G) = \int_0^t (\varphi_{(0)}(G) + \varphi_{(1)}(G)y_{s-}) ds, \tag{2.2}$$

for all $G \in \mathcal{B}^2$ and $t \in [0, T]$, where $(\beta_{(j)}, \gamma_{(j)}, \varphi_{(j)})$, $j = 0, 1$, are Lévy–Khintchine triplets on \mathbb{R}^2 which are admissible in the sense of [10, Definition 2.6] or [18, Definition 3.1], i.e.

- $\beta_{(j)} \in \mathbb{R}^2$, $\gamma_{(j)}$ is a symmetric, nonnegative matrix in $\mathbb{R}^{2 \times 2}$, and $\varphi_{(j)}$ is a σ -finite measure on $\mathbb{R}^2 \setminus \{0\}$ satisfying $\int (1 \wedge |x|^2)\varphi_{(j)}(dx) < \infty$,
- $\gamma_{(0)}^{1,1} = \gamma_{(0)}^{1,2} = \gamma_{(0)}^{2,1} = 0$,
- $\varphi_{(0)}((\mathbb{R}_+ \times \mathbb{R})^C) = \varphi_{(1)}((\mathbb{R}_+ \times \mathbb{R})^C) = 0$, i.e. $\varphi_{(0)}$ and $\varphi_{(1)}$ are actually Lévy measures on $\mathbb{R}_+ \times \mathbb{R}$,
- $\int h_1(x)\varphi_{(0)}(dx) < \infty$ and $\beta_{(0)}^1 - \int h_1(x)\varphi_{(0)}(dx) \geq 0$,
- $\int x_1\varphi_{(1)}(dx) < \infty$. This additional condition prevents explosion in finite time; cf. [10, Lemma 9.2].

We refer to this setup as a *general affine stochastic volatility model*. We provide a few examples of popular affine SV models in the literature.

Example 2.1. The Heston [14] model can be written as

$$dZ_t = (\mu + \delta y_t) dt + \sqrt{y_t} dW_t^1, \quad dy_t = (\kappa - \lambda y_t) dt + \sigma \sqrt{y_t} dW_t^2,$$

where $\kappa \geq 0$, μ , δ , λ , and σ denote constants, and W^1 and W^2 are Wiener processes with constant correlation ρ . The bivariate process (y, Z) is affine in the sense of (2.1)–(2.2) with

$$\begin{aligned} (\beta_{(0)}, \gamma_{(0)}, \varphi_{(0)}) &= \left(\begin{pmatrix} \kappa \\ \mu \end{pmatrix}, 0, 0 \right), \\ (\beta_{(1)}, \gamma_{(1)}, \varphi_{(1)}) &= \left(\begin{pmatrix} -\lambda \\ \delta \end{pmatrix}, \begin{pmatrix} \sigma^2 & \rho\sigma \\ \rho\sigma & 1 \end{pmatrix}, 0 \right). \end{aligned}$$

Example 2.2. The so-called BNS model of [2] is of the form

$$dZ_t = (\mu + \delta y_{t-}) dt + \sqrt{y_{t-}} dW_t + \varrho dz_t, \quad dy_t = -\lambda y_{t-} dt + dz_t,$$

where $\mu, \delta, \lambda,$ and ϱ are constants and W is a Wiener process. Here z denotes a subordinator, i.e. an increasing Lévy process whose Lévy–Khintchine triplet we write as $(b^z, 0, F^z)$. The process (y, Z) is affine in the sense of (2.1)–(2.2) with

$$\beta^{(0)} = \left(\mu + \varrho b^z + \int_0^\infty (\tilde{h}(\varrho x) - \varrho \tilde{h}(x)) F^z(dx) \right), \quad \gamma^{(0)} = 0,$$

$$\varphi_{(0)}(G) = \int_0^\infty \mathbf{1}_G(x, \varrho x) F^z(dx) \quad \text{for all } G \in \mathcal{B}^2,$$

and

$$(\beta_{(1)}, \gamma_{(1)}, \varphi_{(1)}) = \left(\begin{pmatrix} -\lambda \\ \delta \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right);$$

cf. [18, Section 4.3]. In order to obtain more realistic autocorrelation patterns in volatility, Barndorff-Nielsen and Shephard replaced the Lévy-driven Ornstein–Uhlenbeck process y_t by a linear combination of such processes. This extension has a multivariate affine structure according to [18, Section 4.3]. It can be treated along similar lines as the simpler BNS model above. We stick to the bivariate case in this paper in order not to confuse the reader with heavy notation.

Example 2.3. Carr *et al.* [7] considered stochastic volatility models which are based on time-changed Lévy processes, namely

$$Z_t = \mu t + X_{Y_t} + \varrho z_t, \quad dY_t = y_{t-} dt, \quad dy_t = -\lambda y_{t-} dt + dz_t,$$

where $\mu, \varrho,$ and λ are constants, and z and X denote independent Lévy processes with triplets $(b^z, 0, F^z)$ and (b^X, c^X, F^X) , respectively. Here z is supposed to be increasing. The model is of the form (2.1)–(2.2) with

$$\beta^{(0)} = \left(\mu + \varrho b^z + \int_0^\infty (\tilde{h}(\varrho x) - \varrho \tilde{h}(x)) F^z(dx) \right), \quad \gamma^{(0)} = 0,$$

$$\varphi_{(0)}(G) = \int_0^\infty \mathbf{1}_G(x, \varrho x) F^z(dx) \quad \text{for all } G \in \mathcal{B}^2,$$

$$\beta_{(1)} = \begin{pmatrix} -\lambda \\ b^X \end{pmatrix}, \quad \gamma_{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & c^X \end{pmatrix}, \quad \varphi_{(1)}(G) = \int_{\mathbb{R}} \mathbf{1}_G(0, x) F^X(dx)$$

for all $G \in \mathcal{B}^2$; cf. [18, Section 4.4]. If we choose X to be a Brownian motion with drift, we obtain the dynamics of the BNS model above.

Another possible choice of the time change rate y is a square-root process:

$$Z_t = \mu t + X_{Y_t} + \varrho(y_t - y_0), \quad dY_t = y_t dt, \quad dy_t = (\kappa - \lambda y_t) dt + \sigma \sqrt{y_t} dW_t,$$

where $\kappa \geq 0, \mu, \varrho, \lambda,$ and σ are constants, W denotes a Wiener process, and X is an independent

Lévy process with triplet (b^X, c^X, F^X) . This model is of the form (2.1)–(2.2) with

$$\begin{aligned}
 (\beta_{(0)}, \gamma_{(0)}, \varphi_{(0)}) &= \left(\left(\begin{matrix} \kappa \\ \mu + \varrho\kappa \end{matrix} \right), 0, 0 \right), \\
 \beta_{(1)} &= \begin{pmatrix} -\lambda \\ b^X - \varrho\lambda \end{pmatrix}, \quad \gamma_{(1)} = \begin{pmatrix} \sigma^2 & \varrho\sigma^2 \\ \varrho\sigma^2 & \varrho^2\sigma^2 + c^X \end{pmatrix}, \quad \varphi_{(1)}(G) = \int_{\mathbb{R}} \mathbf{1}_G(0, x) F^X(dx)
 \end{aligned}$$

for all $G \in \mathcal{B}^2$; cf. [18, Section 4.4]. If we choose X to be a Brownian motion with drift, we recover the dynamics of the Heston model—up to a rescaling of the volatility process y .

The Lévy–Khintchine triplets $(\beta_{(j)}, \gamma_{(j)}, \varphi_{(j)})$, $j = 0, 1$, can be associated to corresponding Lévy exponents

$$\psi_j(u) := u^\top \beta_{(j)} + \frac{1}{2} u^\top \gamma_{(j)} u + \int (e^{u^\top x} - 1 - u^\top h(x)) \varphi_{(j)}(dx). \tag{2.3}$$

The functions ψ_j , $j = 0, 1$, are defined on

$$U_j := \left\{ u \in \mathbb{C}^2 : \int_{\{|x| \geq 1\}} \exp(\operatorname{Re}(u^\top x)) \varphi_{(j)}(dx) < \infty \right\}.$$

Assumption 2.1. *In order for S to be a locally square-integrable martingale, we assume that $(0, 2) \in U_0 \cap U_1$ and*

$$\psi_0(0, 1) = \psi_1(0, 1) = 0. \tag{2.4}$$

Moreover, we suppose that

$$\psi_0(0, 2) \neq 0 \quad \text{or} \quad \psi_1(0, 2) \neq 0,$$

in order to avoid the degenerate case, $Z = 0$.

Proposition 2.1. *Assumption 2.1 implies that $S \in \mathcal{H}_{\text{loc}}^G$.*

Proof. The differential characteristics (b^Z, c^Z, F^Z) of Z can be easily calculated from (2.1)–(2.2); cf. Proposition A.1 or A.2. They are of the form

$$\begin{aligned}
 b_t^Z &= \beta_{(0)}^2 + \beta_{(1)}^2 y_{t-}, & c_t^Z &= \gamma_{(0)}^{2,2} + \gamma_{(1)}^{2,2} y_{t-}, \\
 F_t^Z(G) &= \int \mathbf{1}_G(x_2) \varphi_{(0)}(dx) + \int \mathbf{1}_G(x_2) \varphi_{(1)}(dx) y_{t-}
 \end{aligned}$$

for all $G \in \mathcal{B}$. Using Propositions A.1 and A.2, we obtain, for the stochastic logarithm $X := \mathcal{L}(S) = 1/S_- \cdot S$,

$$\begin{aligned}
 b_t^X &= b_t^Z + \frac{1}{2} c_t^Z + \int (\tilde{h}(e^x - 1) - \tilde{h}(x)) F_t^Z(dx), \\
 F_t^X(G) &= \int \mathbf{1}_G(e^x - 1) F_t^Z(dx) \quad \text{for all } G \in \mathcal{B}
 \end{aligned}$$

for the differential characteristics (b^X, c^X, F^X) of X . In particular, we have

$$\int x^2 F_t^X(dx) = \int (e^{x^2} - 1)^2 \varphi_{(0)}(dx) + \left(\int (e^{x^2} - 1)^2 \varphi_{(1)}(dx) \right) y_{t-}.$$

In view of Assumption 2.1 and [17, Proposition II.2.29], X is a locally square-integrable semimartingale. Moreover, (2.4) implies that

$$b_t^X + \int (x - \tilde{h}(x)) F_t^X(dx) = \psi_{(0)}(0, 1) + \psi_{(1)}(0, 1)y_{t-} = 0,$$

which implies that X is actually a local martingale. Since $\mathcal{E}(X)_-$ is locally bounded, $S = S_0\mathcal{E}(X) = S_0(1 + \mathcal{E}(X)_- \bullet X)$ is a locally square-integrable martingale as well.

Owing to the results of [10], the characteristic function of the bivariate affine process (y, Z) is known explicitly.

Proposition 2.2. *The conditional characteristic function of (y, Z) is of the form*

$$E[\exp(u_1 y_{t+s} + u_2 Z_{t+s}) \mid \mathcal{F}_t] = \exp(\Psi_0(s, u_1, u_2) + \Psi_1(s, u_1, u_2)y_t + u_2 Z_t),$$

$u \in \mathbb{C}_- \times i\mathbb{R}$, where $\Psi_1: \mathbb{R}_+ \times \mathbb{C}_- \times i\mathbb{R} \rightarrow \mathbb{C}_-$ solves the initial value problem

$$\frac{\partial}{\partial t} \Psi_1(t, u_1, u_2) = \psi_1(\Psi_1(t, u_1, u_2), u_2), \quad \Psi_1(0, u_1, u_2) = u_1, \tag{2.5}$$

and $\Psi_0: \mathbb{R}_+ \times \mathbb{C}_- \times i\mathbb{R} \rightarrow \mathbb{C}$ is given by

$$\Psi_0(t, u_1, u_2) = \int_0^t \psi_0(\Psi_1(s, u_1, u_2), u_2) ds. \tag{2.6}$$

Here \mathbb{C}_- is defined as $\mathbb{C}_- := \{z \in \mathbb{C} : \text{Re}(z) \leq 0\}$.

Proof. This is a special case of [18, Theorem 3.2].

3. European options

The hedging problem cannot be solved in closed form even for geometric Lévy processes without stochastic volatility. In order to obtain at least semiexplicit solutions, we consider European-style claims whose discounted payoff at time T is of the form $H = f(S_T)$. More specifically, we assume that the function $f: (0, \infty) \rightarrow \mathbb{R}$ can be written in integral form as

$$f(s) = \int s^z \Pi(dz)$$

with some finite complex measure Π on a strip $\mathcal{S}_f := \{z \in \mathbb{C} : R' \leq \text{Re}(z) \leq R\}$, where $R', R \in \mathbb{R}$. The measure Π is supposed to be symmetric in the sense that $\Pi(\bar{A}) = \overline{\Pi(A)}$ for $A \in \mathcal{B}(\mathbb{C})$ and $\bar{A} := \{\bar{z} \in \mathbb{C} : z \in A\}$. In most cases we can choose $R' = R$ and the measure Π has a density; cf. [16]. For example, we have

$$(s - K)^+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(1-z)} dz$$

with $R > 1$ for a European call with strike K , which means that

$$\Pi(dz) = \frac{1}{2\pi i} \frac{K^{1-z}}{z(1-z)} dz$$

on $\mathcal{S}_f := R + i\mathbb{R}$. The put $f(s) = (K - s)^+$ corresponds to the same formula but with $R < 0$. The integral representation of many other payoffs can be found in [16].

For the derivation of formulae in the next section, some moment conditions are needed. We phrase them here in terms of analytical properties of the characteristic exponents in Proposition 2.2.

Assumption 3.1. *We assume that the following conditions hold.*

- (i) *For some $\varepsilon > 0$, the mappings $(u_1, u_2) \mapsto \Psi_0(t, u_1, u_2), \Psi_1(t, u_1, u_2)$ have an analytic extension on*

$$\mathcal{S} := \{u \in \mathbb{C}^2 : (\operatorname{Re}(u_1), \operatorname{Re}(u_2)) \in \mathcal{V}_\varepsilon(0)\}$$

for all $t \in [0, T]$, where

$$M_0 := \sup\{2\Psi_1(t, 0, r) : r \in [R' \wedge 0, R \vee 0], t \in [0, T]\},$$

and

$$\mathcal{V}_\varepsilon(a) := (-\infty, (M_0 \vee 0) + \varepsilon) \times ((2R' \wedge 0) - \varepsilon, (2R \vee a) + \varepsilon)$$

for $a \in \mathbb{R}_+$. These extensions are again denoted by Ψ_0 and Ψ_1 , respectively.

- (ii) *The mappings $t \mapsto \Psi_0(t, u_1, u_2)$ and $t \mapsto \Psi_1(t, u_1, u_2)$ are continuous on $[0, T]$ for any $(u_1, u_2) \in \mathcal{S}$.*
- (iii) $\mathcal{V}_\varepsilon(2) \subset U_0 \cap U_1$. *This is satisfied if the mappings $i\mathbb{R} \rightarrow \mathbb{C}, u \mapsto \psi_j(u)$ for $j = 0, 1$ have analytic extensions to $\mathcal{V}_\varepsilon(2)$, in which case representation (2.3) holds for this extension. Note that this implies the integrability condition $(0, 2) \in U_0 \cap U_1$ in Assumption 2.1.*

Remark 3.1. (i) From [36, Theorem III.13.XI], it follows that the analytic extensions Ψ_0, Ψ_1 solve (2.5) and (2.6) as well. Moreover, $\Psi_0, \Psi_1, D_2\Psi_0$, and $D_2\Psi_1$ are continuous on $[0, T] \times \mathcal{S}$. For later use, we also note that the mapping $t \mapsto \Psi_1(t, 0, z)$ is twice continuously differentiable on $[0, T]$ for all $z \in \mathcal{S}_f$.

- (ii) From [10, Theorem 2.16(ii)], it follows with Assumption 3.1(i) that

$$E[\exp(u_1 y_{t+s} + u_2 Z_{t+s}) \mid \mathcal{F}_t] = \exp(\Psi_0(s, u_1, u_2) + \Psi_1(s, u_1, u_2)y_t + u_2 Z_t) \tag{3.1}$$

for all $u \in \mathcal{S}$. It is easy to see that $\Psi_j(t, u_1, u_2)$ is real-valued for $u \in \mathbb{R}^2 \cap \mathcal{S}, j = 0, 1$.

- (iii) By the first remark we have $\Psi_0(t, 0, z) = \int_0^t \psi_0(\Psi_1(s, 0, z), z) ds$ for $z \in \mathcal{S}$. Equation (3.1) and Jensen’s inequality yield $\operatorname{Re}(\Psi_1(t, 0, z)) \leq \Psi_1(t, 0, \operatorname{Re}(z))$ and, hence, $(\Psi_1(t, 0, z), z) \in \mathcal{S}$ for all $t \in [0, T]$ and all $z \in \mathcal{S}_f$.

Lemma 3.1. *Under Assumption 3.1(i), S_T^z is a square-integrable random variable for any $z \in \mathcal{S}_f$. The same is true for H .*

Proof. In view of Remark 3.1(ii) we have

$$E[|S_T^z|^2] \leq \exp(2(|R'| \vee |R|) |\log S_0|) E[e^{2R'Z_T} + e^{2RZ_T}] < \infty.$$

Applying Hölder’s inequality we have

$$\begin{aligned} \mathbb{E}[|H|^2] &\leq \mathbb{E}\left[\left(\int |S_T^z| |\Pi|(dz)\right)^2\right] \\ &\leq |\Pi|(\mathcal{S}_f) \mathbb{E}\left[\int |S_T^z|^2 |\Pi|(dz)\right] \\ &\leq |\Pi|(\mathcal{S}_f)^2 \exp(2(|R'| \vee |R|) |\log S_0|) \mathbb{E}[e^{2R'Z_T} + e^{2RZ_T}] \\ &< \infty. \end{aligned}$$

Here $|\Pi|$ indicates the total variation measure of Π in the sense of [32, Section 6.1].

A key role in the hedging problem is played by the square-integrable martingale V generated by H , i.e.

$$V_t := \mathbb{E}[H \mid \mathcal{F}_t], \quad t \in [0, T].$$

We call it an *option price process* because it could be used as such without introducing arbitrage to the market. But note that we do not assume H to be traded, let alone with price process V .

Proposition 3.1. *Under Assumption 3.1(i), we have*

$$V_t = \int V(z)_t \Pi(dz), \quad t \in [0, T], \tag{3.2}$$

where the square-integrable martingale $V(z)$ for $z \in \mathcal{S}_f$ is defined as $V(z)_t := \mathbb{E}[S_T^z \mid \mathcal{F}_t]$ for $t \in [0, T]$.

Proof. For all $t \in [0, T]$ and $z \in \mathcal{S}_f$, we have the estimate

$$\begin{aligned} \int \mathbb{E}[|V(z)_t|] |\Pi|(dz) &\leq \int \mathbb{E}[|V(z)_T|] |\Pi|(dz) \\ &\leq \int \mathbb{E}[e^{\operatorname{Re}(z) \log S_T}] |\Pi|(dz) \\ &\leq \mathbb{E}[e^{R' \log S_T} + e^{R \log S_T}] |\Pi|(\mathcal{S}_f) \\ &< \infty \end{aligned}$$

from Remark 3.1(ii) and the finiteness of Π . An application of Fubini’s theorem yields

$$\begin{aligned} \mathbb{E}\left[\int V(z)_t \Pi(dz) \mathbf{1}_C\right] &= \mathbb{E}\left[\int \mathbb{E}[S_T^z \mid \mathcal{F}_t] \mathbf{1}_C \Pi(dz)\right] \\ &= \mathbb{E}\left[\int S_T^z \Pi(dz) \mathbf{1}_C\right] \\ &= \mathbb{E}[V_t \mathbf{1}_C] \end{aligned}$$

for all $C \in \mathcal{F}_t$. This implies the assertion.

The process $V(z)$ will be determined in Theorem 4.1, below, as a by-product. Via (3.2) this leads to an integral representation of the option price process which is of interest in itself. It extends similar formulae in [7] and [33] to the more general class of processes considered in this paper.

4. Variance-optimal hedging

We turn now to the hedging problem itself. We generally assume that Assumptions 2.1 and 3.1 hold. We define the set of admissible trading strategies as

$$\Theta := \{\vartheta \text{ predictable process: } E[|\vartheta|^2 \cdot \langle S, S \rangle_T] < \infty\}.$$

We call an initial capital $v^* \in \mathbb{R}$ and an admissible strategy ϑ^* *variance-optimal (hedge)* if they minimize

$$E[(v + \vartheta \cdot S_T - H)^2]$$

over all such pairs $(v, \vartheta) \in \mathbb{R} \times \Theta$. The residue

$$J_0 := E[(v^* + \vartheta^* \cdot S_T - H)^2]$$

is referred to as the *minimal hedging error*. The following characterization of the variance-optimal hedge constitutes the first main result of this paper.

Theorem 4.1. *The variance-optimal initial capital v^* and the variance-optimal hedging strategy ϑ^* are given by*

$$\begin{aligned} v^* &= \int V(z)_0 \Pi(dz), \\ \vartheta_t^* &= \int \frac{V(z)_{t-} \kappa_0(t, z) + \kappa_1(t, z) y_{t-}}{S_{t-} \delta_0 + \delta_1 y_{t-}} \Pi(dz), \end{aligned} \tag{4.1}$$

where the process $V(z)$ satisfies

$$V(z)_t = S_t^z \exp(\Psi_0(T - t, 0, z) + \Psi_1(T - t, 0, z) y_t), \quad z \in \mathcal{S}_f, \tag{4.2}$$

$\delta_0, \delta_1 \in \mathbb{R}$, and the functions $\kappa_0, \kappa_1 : [0, T] \times \mathcal{S}_f \rightarrow \mathbb{C}$ are defined as

$$\kappa_j(t, z) := \psi_j(\Psi_1(T - t, 0, z), z + 1) - \psi_j(\Psi_1(T - t, 0, z), z), \tag{4.3}$$

$$\delta_j := \psi_j(0, 2), \quad j = 0, 1. \tag{4.4}$$

As is well known, the variance-optimal hedging strategy ϑ^* given by (4.1) also yields the solution to

$$\min_{\vartheta \in \Theta} E[(\tilde{v} + \vartheta \cdot S_T - H)^2], \tag{4.5}$$

where $\tilde{v} \in \mathbb{R}$ denotes a given initial capital instead of the optimizer from the previous theorem. Our second main result concerns the hedging error. This quantity gives an idea of the remaining risk. The seller of the option may take it into account in order to decide what risk premium to charge for the claim.

Theorem 4.2. *The minimal hedging error is given by*

$$J_0 = \begin{cases} \iiint_0^T J_1(t, z_1, z_2) dt \Pi(dz_1) \Pi(dz_2) & \text{if } \delta_0 \neq 0, \delta_1 \neq 0, \\ \iiint_0^T J_2(t, z_1, z_2) dt \Pi(dz_1) \Pi(dz_2) & \text{if } \delta_0 = 0, \\ \iiint_0^T J_3(t, z_1, z_2) dt \Pi(dz_1) \Pi(dz_2) & \text{if } \delta_1 = 0. \end{cases}$$

The integrals over \mathcal{I}_f have to be understood in the sense of the Cauchy principal value (cf. the following remark). The integrands $J_k: [0, T] \times \mathcal{I}_f^2 \rightarrow \mathbb{C}$, $k = 1, 2, 3$, in these expressions are defined as

$$\begin{aligned}
 J_1(t, z_1, z_2) &= S_0^{z_1+z_2} e^{\xi_0} \left(\exp(\Psi_0(t, \xi_1, z_1 + z_2) + \Psi_1(t, \xi_1, z_1 + z_2) y_0) \right. \\
 &\quad \times \left(\frac{\eta_2}{\delta_1} (D_2 \Psi_0(t, \xi_1, z_1 + z_2) + D_2 \Psi_1(t, \xi_1, z_1 + z_2) y_0) + \frac{\eta_1 \delta_1 - \eta_2 \delta_0}{\delta_1^2} \right) \\
 &\quad + \frac{\eta_0 \delta_1^2 - \eta_1 \delta_0 \delta_1 + \eta_2 \delta_0^2}{\delta_1^3} e^{-\delta_0 \xi_1 / \delta_1} \\
 &\quad \times \int_0^1 \left(\frac{\delta_1}{\delta_0} + \xi_1 s \right) \exp \left(\frac{\delta_0}{\delta_1} \xi_1 s + \Psi_0 \left(t, \frac{\delta_1}{\delta_0} \log(s) + \xi_1 s, z_1 + z_2 \right) \right. \\
 &\quad \left. \left. + \Psi_1 \left(t, \frac{\delta_1}{\delta_0} \log(s) + \xi_1 s, z_1 + z_2 \right) y_0 \right) ds \right), \\
 J_2(t, z_1, z_2) &= \frac{S_0^{z_1+z_2} e^{\xi_0}}{\delta_1} \exp(\Psi_0(t, \xi_1, z_1 + z_2) + \Psi_1(t, \xi_1, z_1 + z_2) y_0) \\
 &\quad \times (\eta_1 + (D_2 \Psi_0(t, \xi_1, z_1 + z_2) + D_2 \Psi_1(t, \xi_1, z_1 + z_2) y_0) \eta_2), \\
 J_3(t, z_1, z_2) &= \frac{S_0^{z_1+z_2} \eta_0}{\delta_0} \exp((\psi_0(0, z_1) + \psi_0(0, z_2))(T - t) + \psi_0(0, z_1 + z_2)t).
 \end{aligned}$$

The constants δ_0 and δ_1 are defined as in (4.4). The remaining variables are specified as follows:

$$\begin{aligned}
 \alpha_j &= \alpha_j(t, z_1, z_2) \\
 &:= \psi_j(\xi_1(t, z_1, z_2), z_1 + z_2) - \psi_j(\Psi_1(T - t, 0, z_1), z_1) - \psi_j(\Psi_1(T - t, 0, z_2), z_2), \\
 \eta_0 &= \eta_0(t, z_1, z_2) := \delta_0 \alpha_0(t, z_1, z_2) - \kappa_0(t, z_1) \kappa_0(t, z_2), \\
 \eta_1 &= \eta_1(t, z_1, z_2) \\
 &:= \delta_0 \alpha_1(t, z_1, z_2) + \delta_1 \alpha_0(t, z_1, z_2) - \kappa_1(t, z_1) \kappa_0(t, z_2) - \kappa_1(t, z_2) \kappa_0(t, z_1), \\
 \eta_2 &= \eta_2(t, z_1, z_2) := \delta_1 \alpha_1(t, z_1, z_2) - \kappa_1(t, z_1) \kappa_1(t, z_2), \\
 \xi_j &= \xi_j(t, z_1, z_2) := \Psi_j(T - t, 0, z_1) + \Psi_j(T - t, 0, z_2), \quad j = 0, 1,
 \end{aligned}$$

with κ_0 and κ_1 from (4.3). For ease of notation, we dropped the arguments of some functions in the formulae above. The mappings $\eta_0, \eta_1, \eta_2, \alpha_0, \alpha_1, \xi_0$, and ξ_1 are defined on $[0, T] \times \mathcal{I}_f^2$.

Remark 4.1. (i) The integrals in the previous theorem are to be understood in the sense that

$$J_0 = \lim_{c \uparrow \infty} \int_{\mathcal{I}_f^c} \int_{\mathcal{I}_f^c} \int_0^T J_k(t, z_1, z_2) dt \Pi(dz_1) \Pi(dz_2),$$

where

$$\mathcal{I}_f^c := \{z \in \mathbb{C}: R' \leq \text{Re}(z) \leq R, |\text{Im}(z)| \leq c\}.$$

It is not obvious whether integrability holds on all of \mathcal{I}_f .

(ii) Condition $\delta_1 = 0$ actually implies that Z is a Lévy process. The integral of J_3 relative to t can easily be evaluated in closed form, which is done in [16, Theorem 3.2].

(iii) The minimal hedging error in (4.5) for fixed initial endowment \tilde{v} instead of the optimal v equals

$$J_0 + (\tilde{v} - v)^2.$$

For concrete models and numerical results, we refer the reader to the companion paper [20].

5. Proofs of the main results

This section is devoted to the proofs of Theorems 4.1 and 4.2. We start by analyzing the martingales $V(z)$ in Proposition 3.1.

Lemma 5.1. *$V(z)$ in Proposition 3.1 is of the form (4.2).*

Proof. The proof follows immediately from (3.1).

The Galtchouk–Kunita–Watanabe (GKW) decomposition of $V(z)$ is determined in the following lemma.

Lemma 5.2. *Fix $z \in \mathcal{B}_f$. If we set*

$$\begin{aligned} \vartheta(z)_t &:= \frac{V(z)_{t-} \kappa_0(t, z) + \kappa_1(t, z) y_{t-}}{S_{t-} \delta_0 + \delta_1 y_{t-}}, \\ L(z) &:= V(z) - V(z)_0 - \vartheta(z) \bullet S, \end{aligned} \tag{5.1}$$

then $\vartheta(z) \in \Theta$ (here liberally extended to complex-valued processes), $L(z)$ is a square-integrable martingale, and LS is a local martingale. Here $\kappa_j(t, z)$ and δ_j , $j = 0, 1$, are defined as in (4.3) and (4.4), respectively.

Proof. The denominator in (5.1) is positive because the constants $\delta_0, \delta_1 \in \mathbb{R}_+$ do not both vanish (cf. Assumption 2.1). The differential characteristics (b^S, c^S, F^S) of S can be obtained from (2.1)–(2.2) and Proposition A.2. We have

$$\begin{aligned} c_t^S &= S_{t-}^2 (\gamma_{(0)}^{2,2} + \gamma_{(1)}^{2,2} y_{t-}) \\ \text{and } F_t^S(G) &= \int \mathbf{1}_G(S_{t-}(e^{x_2} - 1)) \varphi_{(0)}(dx) + \int \mathbf{1}_G(S_{t-}(e^{x_2} - 1)) \varphi_{(1)}(dx) y_{t-} \end{aligned}$$

for all $G \in \mathcal{B}$. By [17, Proposition II.2.29b], this implies that

$$\langle S, S \rangle_t = \int_0^t \left(c_s^S + \int_{\mathbb{R}} x^2 F_s^S(dx) \right) ds = \int_0^t S_{s-}^2 (\delta_0 + \delta_1 y_{s-}) ds, \tag{5.2}$$

where the second equality follows from

$$\gamma_{(j)}^{2,2} + \int (e^{x_2} - 1)^2 \varphi_{(j)}(dx) = \psi_j(0, 2) - 2\psi_j(0, 1) = \psi_j(0, 2) = \delta_j.$$

Another application of Proposition A.2 allows us to compute the differential characteristics $(b^{V(z), S}, c^{V(z), S}, F^{V(z), S})$ of $(V(z), S) = g(I, y, Z)$, where $I_t = t$ denotes the identity process and

$$g(t, x_1, x_2) := \begin{pmatrix} S_0^z \exp(\Psi_0(T - t, 0, z) + \Psi_1(T - t, 0, z)x_1 + zx_2) \\ S_0 \exp(x_2) \end{pmatrix}.$$

Again, using [17, Proposition II.2.29b], this leads to

$$\begin{aligned} \langle V(z), S \rangle_t &= \int_0^t \left((c^{V(z), S})_s^{2,1} + \int_{\mathbb{R}^2} x_1 x_2 F_s^{V(z), S}(d(x_1, x_2)) \right) ds \\ &= \int_0^t V(z)_{s-} S_{s-} (\kappa_0(s, z) + \kappa_1(s, z) y_{s-}) ds. \end{aligned} \tag{5.3}$$

We conclude that

$$\vartheta(z) \bullet \langle S, S \rangle = \langle V(z), S \rangle. \tag{5.4}$$

Hence, $\langle L, S \rangle = 0$, which implies that LS is a local martingale. Equation (5.4) also yields

$$\langle L(z), \overline{L(z)} \rangle + |\vartheta(z)|^2 \bullet \langle S, S \rangle = \langle L(z), \overline{L(z)} \rangle + (\vartheta(z) \overline{\vartheta(z)}) \bullet \langle S, S \rangle = \langle V(z), \overline{V(z)} \rangle.$$

Since $V(z)$ is a square-integrable martingale, we have $E[\langle V(z), \overline{V(z)} \rangle_T] < \infty$, which implies that $\vartheta(z) \in \Theta$ and $L(z) \in \mathcal{H}^2$.

For later use, we need a technical result.

Lemma 5.3. *Let $\tau_n := \inf\{t > 0 : y_t \notin [1/n, n] \text{ or } S_t \notin [1/n, n]\} \wedge T$. For any $n \in \mathbb{N}$, there exists a constant $c(n) < \infty$ such that*

$$\int_0^T E[\mathbf{1}_{[0, \tau_n]} |V(z)_t|^2 (\alpha_0(t, z, \bar{z}) + \alpha_1(t, z, \bar{z}) y_t)] dt \leq c(n) \tag{5.5}$$

holds for all $z \in \mathcal{J}_f$. The functions α_0 and α_1 are defined as in Theorem 4.2.

Proof. Fix $n \in \mathbb{N}$. In this proof c denotes a generic constant that does not depend on $t \in [0, T]$ or $z \in \mathcal{J}_f$. It may vary from line to line. Setting $\varrho := \max\{|R'|, |R|\}$ implies that $\mathbf{1}_{[0, \tau_n]} e^{\text{Re}(z) \log(S_t)} \leq n^\varrho \leq c$. Since $\psi_j(\bar{z}_1, \bar{z}_2) = \overline{\psi_j(z_1, z_2)}$, we have $\Psi_1(t, \bar{z}_1, \bar{z}_2) = \overline{\Psi_1(t, z_1, z_2)}$ for $z_1, z_2 \in i\mathbb{R}$ and, hence, for $z_1, z_2 \in \mathcal{J}_f$ by uniqueness of analytic continuations. We obtain

$$\begin{aligned} \alpha_j(t, z, \bar{z}) &= \psi_j(2\text{Re}(\Psi_1(T-t, 0, z)), 2\text{Re}(z)) - 2\text{Re}(\psi_j(\Psi_1(T-t, 0, z), z)) \\ &\geq (\text{Re}(\Psi_1(T-t, 0, z)), \text{Re}(z)) \gamma_{(j)} (\text{Re}(\Psi_1(T-t, 0, z)), \text{Re}(z))^\top \\ &\quad + (\text{Im}(\Psi_1(T-t, 0, z)), \text{Im}(z)) \gamma_{(j)} (\text{Im}(\Psi_1(T-t, 0, z)), \text{Im}(z))^\top \\ &\quad + \int (\exp(\text{Re}(\Psi_1(T-t, 0, z)) x_1 + \text{Re}(z) x_2) - 1)^2 \varphi_{(j)}(dx) \\ &\geq 0 \end{aligned}$$

for $j = 0, 1$ because $\gamma_{(j)}$ is positive semi-definite. Hence,

$$\begin{aligned} K_1(z) &:= \int_0^T E[\mathbf{1}_{[0, \tau_n]} \mathbf{1}_{\{\text{Re}(\Psi_1(T-t, 0, z)) \leq 0\}} \\ &\quad \times \exp(2\text{Re}(\Psi_0(T-t, 0, z)) + 2\text{Re}(\Psi_1(T-t, 0, z)) y_t) S_t^{2\text{Re}(z)} \\ &\quad \times (\alpha_0(t, z, \bar{z}) + \alpha_1(t, z, \bar{z}) y_t)] dt \\ &\leq \int_0^T \exp\left(2\text{Re}(\Psi_0(T-t, 0, z)) + \frac{2}{n} \text{Re}(\Psi_1(T-t, 0, z))\right) c \\ &\quad \times (\psi_0(2\text{Re}(\Psi_1(T-t, 0, z)), 2\text{Re}(z)) - 2\text{Re}(\psi_0(\Psi_1(T-t, 0, z), z)) \\ &\quad + n(\psi_1(2\text{Re}(\Psi_1(T-t, 0, z)), 2\text{Re}(z)) \\ &\quad - 2\text{Re}(\psi_1(\Psi_1(T-t, 0, z), z)))) dt. \end{aligned}$$

Using (2.5) and (2.6), we obtain

$$\begin{aligned}
 K_1(z) &\leq c \int_0^T \left| \exp\left(2\text{Re}(\Psi_0(T-t, 0, z)) + \frac{2}{n}\text{Re}(\Psi_1(T-t, 0, z))\right) \right. \\
 &\quad \left. \times \psi_0(2\text{Re}(\Psi_1(T-t, 0, z)), 2\text{Re}(z)) \right| dt \\
 &\quad + c \int_0^T \left| \exp\left(2\text{Re}(\Psi_0(T-t, 0, z)) + \frac{2}{n}\text{Re}(\Psi_1(T-t, 0, z))\right) \right. \\
 &\quad \left. \times \psi_1(2\text{Re}(\Psi_1(T-t, 0, z)), 2\text{Re}(z)) \right| dt \\
 &\quad + c \left| \int_0^T \frac{\partial}{\partial t} \left(2\text{Re}(\Psi_0(T-t, 0, z)) + \frac{2}{n}\text{Re}(\Psi_1(T-t, 0, z)) \right) \right. \\
 &\quad \left. \times \exp\left(2\text{Re}(\Psi_0(T-t, 0, z)) + \frac{2}{n}\text{Re}(\Psi_1(T-t, 0, z))\right) dt \right| \\
 &=: c(K_2(z) + K_3(z) + |K_4(z)|).
 \end{aligned}$$

Equation (3.1) and Jensen’s inequality yield

$$2\text{Re}(\Psi_j(t, 0, z)) \leq \Psi_j(t, 0, 2\text{Re}(z)) \quad \text{for all } (t, z) \in [0, T] \times \mathcal{F}_f.$$

By continuity of the mappings Ψ_0 and Ψ_1 (cf. Assumption 3.1(ii)) we have

$$\begin{aligned}
 |K_4(z)| &= \left| \exp\left(\frac{2}{n}\text{Re}(z)\right) - \exp\left(2\text{Re}(\Psi_0(T, 0, z)) + \frac{2}{n}\text{Re}(\Psi_1(T, 0, z))\right) \right| \\
 &\leq \exp\left(\frac{2}{n}\text{Re}(z)\right) + \exp\left(\Psi_0(T, 0, 2\text{Re}(z)) + \frac{1}{n}\Psi_1(T, 0, 2\text{Re}(z))\right) \\
 &\leq c.
 \end{aligned}$$

Similarly, we obtain estimates

$$\begin{aligned}
 g(t, z) &:= 2\text{Re}(\Psi_1(T-t, 0, z)) \leq \Psi_1(T-t, 0, 2\text{Re}(z)) \leq c, \\
 2\text{Re}(\Psi_0(T-t, 0, z)) &\leq \Psi_0(T-t, 0, 2\text{Re}(z)) \leq c.
 \end{aligned}$$

For $j = 0, 1$, we have

$$\begin{aligned}
 &\left| \frac{1}{n} e^{g(t,z)/n} \psi_j(g(t, z), 2\text{Re}(z)) \right| \\
 &\leq \left[\frac{2}{n} (|\beta_{(j)}^2| + \varrho |\gamma_{(j)}^{2,2}|) \varrho + (|\beta_{(j)}^1| + 2\varrho |\gamma_{(j)}^{1,2}|) |g_n(t, z)| + \frac{n}{2} |\gamma_{(j)}^{1,1}| |g_n(t, z)|^2 \right] e^{g_n(t,z)} \\
 &\quad + \frac{1}{n} e^{g_n(t,z)} \int |e^{g(t,z)x_1 + 2\text{Re}(z)x_2} - 1 - g(t, z)\tilde{h}(x_1) - 2\text{Re}(z)\tilde{h}(x_2)| \varphi_{(j)}(dx),
 \end{aligned}$$

where $g_n(t, z) := g(t, z)/n$. Boundedness of g implies that we have

$$e^{g_n(t,z)} |g_n(t, z)|^k \leq c$$

for $k = 0, 1, 2$ and some constant c . Furthermore, we have

$$\begin{aligned} & \frac{1}{n} e^{g_n(t,z)} \int |e^{g(t,z)x_1 + 2\text{Re}(z)x_2} - 1 - g(t,z)\tilde{h}(x_1) - 2\text{Re}(z)\tilde{h}(x_2)|\varphi_{(j)}(dx) \\ & \leq \frac{1}{n} e^{g_n(t,z)} \left(\int_{\{|x|>1\}} e^{M_0x_1 + 2Rx_2} \varphi_{(j)}(dx) + \int_{\{|x|>1\}} e^{M_0x_1 + 2R'x_2} \varphi_{(j)}(dx) \right. \\ & \quad \left. + (1 + 2\varrho + |g(t,z)|)\varphi_{(j)}(\{|x| > 1\}) \right) \\ & \quad + \frac{1}{n} e^{g_n(t,z)} \int_{\{|x|\leq 1\}} |e^{g(t,z)x_1 + 2\text{Re}(z)x_2} - 1 - g(t,z)x_1 - 2\text{Re}(z)x_2|\varphi_{(j)}(dx). \end{aligned}$$

Since $\varphi_{(j)}$ is a Lévy measure, $\varphi_{(j)}(\{|x| > 1\})$ is finite. The first two integrals on the right-hand side are bounded in view of Assumption 3.1(iii). Note that

$$|e^{g(t,z)x_1 + 2\text{Re}(z)x_2} - 1 - g(t,z)x_1 - 2\text{Re}(z)x_2| \leq c(x_1^2 + x_2^2)$$

for all $|x| \leq 1$. Since $\int_{\{|x|\leq 1\}} (x_1^2 + x_2^2)\varphi_{(j)}(dx) < \infty$, we obtain

$$\frac{1}{n} e^{g_n(t,z)} \int |e^{g(t,z)x_1 + 2\text{Re}(z)x_2} - 1 - g(t,z)\tilde{h}(x_1) - 2\text{Re}(z)\tilde{h}(x_2)|\varphi_{(j)}(dx) \leq c.$$

Altogether, we conclude that

$$\left| \frac{1}{n} e^{g(t,z)/n} \psi_j(g(t,z), 2\text{Re}(z)) \right| \leq c < \infty$$

uniformly in $(t, z) \in [0, T] \times \mathcal{I}_f$, $j = 0, 1$. This in turn yields $K_2(z) \leq c$, $K_3(z) \leq c$, and, hence,

$$0 \leq K_1(z) \leq c < \infty \tag{5.6}$$

for all $z \in \mathcal{I}_f$. Analogously, we show that

$$\begin{aligned} K_5(z) & := \int_0^T \mathbb{E}[\mathbf{1}_{[0, \tau_n]} \mathbf{1}_{\{\text{Re}(\Psi_1(T-t, 0, z)) > 0\}} \\ & \quad \times \exp(2\text{Re}(\Psi_0(T-t, 0, z)) + 2\text{Re}(\Psi_1(T-t, 0, z))y_t -) S_t^{2\text{Re}(z)} \\ & \quad \times (\alpha_0(t, z, \bar{z}) + \alpha_1(t, z, \bar{z})y_t)] dt \\ & \leq c. \end{aligned} \tag{5.7}$$

Assertion (5.5) follows now from (5.6) and (5.7) because

$$|V(z)_t|^2 = S_t^{2\text{Re}(z)} \exp(2\text{Re}(\Psi_0(T-t, 0, z)) + 2\text{Re}(\Psi_1(T-t, 0, z))y_t)$$

due to Lemma 5.1.

Corollary 5.1. *Using the notation of Lemma 5.3, we have*

$$\mathbb{E} \left[\left(\int |\vartheta(z)|^2 |\Pi|(dz) \right) \cdot \langle S, S \rangle_{\tau_n} \right] < \infty, \tag{5.8}$$

$$\int \mathbb{E}[\langle L(z), \overline{L(\bar{z})} \rangle_{\tau_n}] |\Pi|(dz) < \infty. \tag{5.9}$$

Proof. By (5.1) and (5.2) we have

$$\begin{aligned}
 K &:= \mathbb{E} \left[\left(\int | \vartheta(z) |^2 | \Pi | (dz) \right) \cdot \langle S, S \rangle_{\tau_n} \right] \\
 &= \mathbb{E} \left[\int_0^{\tau_n} \int |V(z)_t|^2 \frac{|\kappa_0(t, z) + \kappa_1(t, z)y_t|^2}{\delta_0 + \delta_1 y_t} | \Pi | (dz) dt \right].
 \end{aligned}$$

Similar to the derivation of (5.3) we derive

$$\langle V(z_1), V(z_2) \rangle_t = \int_0^t V(z_1)_s V(z_2)_s (\alpha_0(s, z_1, z_2) + \alpha_1(s, z_1, z_2)y_s) ds$$

for $z_1, z_2 \in \mathcal{S}_f$. Equations (5.2) and (5.3) imply that

$$\begin{aligned}
 \vartheta(z_2) \cdot \langle V(z_1), S \rangle_t &= \vartheta(z_1) \cdot \langle S, V(z_2) \rangle_t \\
 &= (\vartheta(z_1)\vartheta(z_2)) \cdot \langle S, S \rangle_t \\
 &= \int_0^t V(z_1)_s V(z_2)_s \frac{(\kappa_0(s, z_1) + \kappa_1(s, z_1)y_s)(\kappa_0(s, z_2) + \kappa_1(s, z_2)y_s)}{\delta_0 + \delta_1 y_s} ds.
 \end{aligned}$$

This leads to

$$\begin{aligned}
 \langle L(z_1), L(z_2) \rangle_t &= \int_0^t V(z_1)_s V(z_2)_s \\
 &\quad \times \left(\alpha_0(s, z_1, z_2) + \alpha_1(s, z_1, z_2)y_s \right. \\
 &\quad \left. - \frac{(\kappa_0(s, z_1) + \kappa_1(s, z_1)y_s)(\kappa_0(s, z_2) + \kappa_1(s, z_2)y_s)}{\delta_0 + \delta_1 y_s} \right) ds \tag{5.10}
 \end{aligned}$$

for $z_1, z_2 \in \mathcal{S}_f$. The angle bracket $\langle L(z), \overline{L(z)} \rangle = \langle L(z), L(\overline{z}) \rangle$ is a nonnegative increasing process. Hence,

$$0 \leq \frac{|\kappa_0(t, z) + \kappa_1(t, z)y_t|^2}{\delta_0 + \delta_1 y_t} \leq \alpha_0(t, z, \overline{z}) + \alpha_1(t, z, \overline{z})y_t \tag{5.11}$$

for all $t \in [0, T]$ because $\kappa_j(t, \overline{z}) = \overline{\kappa_j(t, z)}$ for $j = 0, 1$. With the help of Fubini’s theorem we deduce that

$$K \leq \iint_0^T \mathbb{E}[\mathbf{1}_{[0, \tau_n]} |V(z)_t|^2 (\alpha_0(t, z, \overline{z}) + \alpha_1(t, z, \overline{z})y_t)] dt | \Pi | (dz).$$

Since Π is a finite measure on \mathcal{S}_f , estimate (5.8) follows from Lemma 5.3. In view of (5.10) and (5.11) we have

$$\langle L(z), \overline{L(z)} \rangle_{\tau_n} \leq \int_0^{\tau_n} |V(z)_t|^2 (\alpha_0(t, z, \overline{z}) + \alpha_1(t, z, \overline{z})y_t) dt. \tag{5.12}$$

Hence, (5.9) follows from Lemma 5.3 as well.

We can now determine the GKW decomposition of V .

Lemma 5.4. *A real-valued admissible trading strategy is defined by*

$$\vartheta^* := \int \vartheta(z) \Pi(dz).$$

Moreover,

$$L := \int L(z) \Pi(dz)$$

is a real-valued square-integrable martingale orthogonal to S (i.e. such that LS is a local martingale). Finally, we have

$$V = V_0 + \vartheta^* \cdot S + L.$$

Proof. Using Hölder’s inequality and (5.8), we obtain

$$\begin{aligned} \mathbb{E} \left[\int_0^{\tau_n} |\vartheta_t^*|^2 d\langle S, S \rangle_t \right] &\leq \mathbb{E} \left[\int_0^{\tau_n} \left(\int |\vartheta(z)_t| |\Pi|(dz) \right)^2 d\langle S, S \rangle_t \right] \\ &\leq |\Pi|(\mathcal{S}_f) \mathbb{E} \left[\left(\int |\vartheta(z)|^2 |\Pi|(dz) \right) \cdot \langle S, S \rangle_{\tau_n} \right] \\ &< \infty \end{aligned}$$

for the stopping times τ_n from Lemma 5.3. Hence, $\vartheta^* \in L_{loc}^2(S)$. Lemma 5.2 implies that $SL(z) \in \mathcal{M}_{loc}$ for all $z \in \mathcal{S}_f$. Let $(\sigma_n)_{n \geq 1}$ denote a localizing sequence for $S^{\sigma_n} \in \mathcal{H}_{loc}^G$. Since $L(z) \in \mathcal{H}^2$, it follows that $(SL(z))^{\sigma_n} \in \mathcal{M}$; cf. [17, Theorem I.4.2]. Now we set $\tilde{\tau}_n := \tau_n \wedge \sigma_n$. Since $\langle L(z), L(\bar{z}) \rangle$ is an increasing process, Jensen’s inequality yields

$$\mathbb{E}[\langle L(z)_\tau \rangle]^2 \leq \mathbb{E}[\langle L(z)_\tau \rangle^2] \leq \mathbb{E}[\langle L(z), \overline{L(\bar{z})} \rangle_\tau] \tag{5.13}$$

for any stopping time τ . In Lemma 5.3 we have shown that

$$\int_0^T \mathbb{E}[\mathbf{1}_{[0, \tilde{\tau}_n]} |V(z)_t|^2 (\alpha_0(t, z, \bar{z}) + \alpha_1(t, z, \bar{z}) y_t)] dt$$

is uniformly bounded in $z \in \mathcal{S}_f$. Owing to $S^{\tilde{\tau}_n} \in \mathcal{H}^2$, we have

$$K := \sup\{\mathbb{E}[(S_t^{\tilde{\tau}_n})^2] : t \in [0, T]\} < \infty.$$

In view of the Cauchy–Schwarz inequality, (5.12), and (5.13), we obtain

$$\begin{aligned} &\int \mathbb{E}[|S_t^{\tilde{\tau}_n} L(z)_t^{\tilde{\tau}_n}|] |\Pi|(dz) \\ &\leq \sqrt{K} \int \left(\int_0^T \mathbb{E}[\mathbf{1}_{[0, \tilde{\tau}_n]} |V(z)_t|^2 (\alpha_0(t, z, \bar{z}) + \alpha_1(t, z, \bar{z}) y_t)] dt \right)^{1/2} |\Pi|(dz) \\ &< \infty \end{aligned}$$

and, similarly,

$$\mathbb{E} \left[\int |L(z)_{\tau_n \wedge t}| |\Pi|(dz) \right] = \int \mathbb{E}[|L(z)_t^{\tau_n}|] |\Pi|(dz) < \infty$$

for $t \in [0, T]$. In particular, the integral in the definition of L is finite. An application of

Fubini’s theorem yields

$$E[(S_t^{\tilde{\tau}_n} L_t^{\tilde{\tau}_n} - S_s^{\tilde{\tau}_n} L_s^{\tilde{\tau}_n}) \mathbf{1}_A] = \int E[(S_t^{\tilde{\tau}_n} L(z)_t^{\tilde{\tau}_n} - S_s^{\tilde{\tau}_n} L(z)_s^{\tilde{\tau}_n}) \mathbf{1}_A] \Pi(dz) = 0$$

for $A \in \mathcal{F}_s$ and $s \leq t$. Hence, $SL \in \mathcal{M}_{loc}$ and, therefore, $\langle S, L \rangle = 0$. A similar Fubini-type argument shows that $L^{\tau_n} \in \mathcal{M}$ because $L(z) \in \mathcal{H}^2$ for all $z \in \mathcal{S}_f$ and \mathcal{H}^2 is stable under stopping. Furthermore, (5.9) leads to

$$\begin{aligned} \sup_{t \in [0, T]} E[|L_t^{\tau_n}|^2] &= E[|L_{\tau_n}|^2] \\ &\leq \iint E[|L(z_1)_{\tau_n} L(z_2)_{\tau_n}|] |\Pi|(dz_1) |\Pi|(dz_2) \\ &\leq |\Pi|(\mathcal{S}_f) \int E[|L(z)_{\tau_n}|^2] |\Pi|(dz) \\ &\leq |\Pi|(\mathcal{S}_f) \int E[\langle L(z), \overline{L(z)} \rangle_{\tau_n}] |\Pi|(dz) \\ &< \infty \end{aligned}$$

and, hence, $L \in \mathcal{H}_{loc}^G$. Obviously, L starts in zero. From Proposition 3.1 and Lemma 5.2, we know that

$$V = V_0 + \int (\vartheta(z) \cdot S) \Pi(dz) + L.$$

In view of (5.8), Fubini’s theorem for stochastic integrals [30, Theorem IV.65] yields

$$\int (\vartheta(z) \cdot S) \Pi(dz) = \left(\int \vartheta(z) \Pi(dz) \right) \cdot S = \vartheta^* \cdot S.$$

From

$$(\vartheta^* - \overline{\vartheta^*}) \cdot \langle S, S \rangle = \langle V, S \rangle - \langle \overline{V}, S \rangle = 0,$$

it follows that ϑ^* and hence also L is real valued. The admissibility of ϑ^* and the square integrability of L follow as in Lemma 5.2 from $V \in \mathcal{H}^2$.

The GKW decomposition of V leads to the variance-optimal hedge.

Proof of Theorem 4.1. The assertion follows from Lemma 5.4 combined with [34, Theorem 3 and Corollary 10].

Finally, we turn to the proof of the formula for the hedging error. Observe that the superscript c in the following does *not* refer to the continuous martingale part of processes.

5.1. Proof of Theorem 4.2

Let us consider the truncated payoff function

$$f^c(s) := \int s^z \Pi^c(dz),$$

where $c \in \mathbb{R}_+$, $\Pi^c(A) := \Pi(\mathcal{S}_f \cap A)$ for all $A \in \mathcal{B}(\mathcal{S}_f)$, and

$$\mathcal{S}_f^c := \{z \in \mathbb{C} : R' \leq \text{Re}(z) \leq R, |\text{Im}(z)| \leq c\}.$$

Observe that f^c is real valued because Π is symmetric. We define $H^c := f^c(S_T)$ by the corresponding contingent claim. Its price process V^c is defined by $V_t^c := E[H^c \mid \mathcal{F}_t]$ as in Section 3. By Proposition 3.1 we have

$$V_t^c = \int V(z)_t \Pi^c(dz) = \int_{\mathcal{S}_f^c} V(z)_t \Pi(dz).$$

Hölder’s inequality yields

$$\begin{aligned} |V_T^c - V_T|^2 &\leq \left(\int_{\mathcal{S}_f^c} |V(z)_T| \Pi(dz) \right)^2 \\ &\leq |\Pi|(\mathcal{S}_f) \int |V(z)_T|^2 \Pi(dz) \\ &\leq (|\Pi|(\mathcal{S}_f))^2 (e^{2R \log S_T} + e^{2R' \log S_T}). \end{aligned}$$

Since the expression on the right-hand side is square integrable, we have

$$H^c = V_T^c \xrightarrow{L^2} V_T = H \quad \text{as } c \rightarrow \infty$$

by dominated convergence. If we set

$$(\vartheta^c)^* := \int \vartheta(z) \Pi^c(dz) \quad \text{and} \quad L^c := \int L(z) \Pi^c(dz),$$

then

$$V^c = V^c(0) + \vartheta^* \bullet S + L$$

is the GKW decomposition of V^c in the sense of Lemma 5.4. Theorem 3.8 of [27] shows that L_T^c converges to L_T in L^2 . Hence, we have

$$J_0 = E[L_T^2] = \lim_{c \rightarrow \infty} E[(L_T^c)^2] = \lim_{c \rightarrow \infty} E[\langle L^c, L^c \rangle_T]. \tag{5.14}$$

In order to finish the proof, we need the following Fubini-type theorem for the predictable covariation.

Lemma 5.5. *We have*

$$\langle L^c, L^c \rangle = \iint \langle L(z_1), L(z_2) \rangle \Pi^c(dz_1) \Pi^c(dz_2),$$

where $\langle L(z_1), L(z_2) \rangle$ is given by (5.10).

Proof. This is shown similarly as in the proof of [16, Theorem 3.2]; cf. [28, Lemma 3.26] for details.

We can now complete the proof of Theorem 4.2. From (5.14), it follows with Lemma 5.5 that

$$E[L_T^2] = \lim_{c \rightarrow \infty} E \left[\int_{\mathcal{S}_f^c} \int_{\mathcal{S}_f^c} \langle L(z_1), L(z_2) \rangle_T \Pi(dz_1) \Pi(dz_2) \right]. \tag{5.15}$$

We focus on the case in which $\delta_0, \delta_1 \neq 0$. The others follow along the same lines. Note that $\langle X, \bar{X} \rangle$ is an increasing and integrable process for all \mathbb{C} -valued square-integrable martingales X . From (5.10), it follows that

$$\begin{aligned} \mathbb{E}[\langle L(z_1), L(z_2) \rangle_T] &= \int_0^T \left(\frac{\eta_2}{\delta_1} \mathbb{E}[V(z_1)_t V(z_2)_t y_t] + \frac{\eta_1 \delta_1 - \eta_2 \delta_0}{\delta_1^2} \mathbb{E}[V(z_1)_t V(z_2)_t] \right. \\ &\quad \left. + \frac{\eta_0 \delta_1^2 - \eta_1 \delta_0 \delta_1 + \eta_2 \delta_0^2}{\delta_1^2} \mathbb{E} \left[\frac{V(z_1)_t V(z_2)_t}{\delta_0 + \delta_1 y_t} \right] \right) dt, \end{aligned} \tag{5.16}$$

where η_0, η_1 , and η_2 defined as in the assertion depend on t, z_1 , and z_2 . In view of Assumption 3.1(i) and Remark 3.1(ii), we obtain

$$\mathbb{E}[V(z_1)_t V(z_2)_t] = e^{\xi_0} S_0^{z_1+z_2} \exp(\Psi_0(t, \xi_1, z_1 + z_2) + \Psi_1(t, \xi_1, z_1 + z_2) y_0), \tag{5.17}$$

where ξ_0 and ξ_1 defined as in the assertion depend on t, z_1 , and z_2 . For $(x_1, x_2) \in \mathcal{A}$, we set $g_0(x_1, x_2) := \exp(x_1 y_t + x_2 Z_t)$ and $g_1(x_1, x_2) := \mathbb{E}[g_0(x_1, x_2)]$. Moreover, let $B(\xi_1, \tilde{\varepsilon}) := \{z \in \mathbb{C} : |z - \xi_1| < \tilde{\varepsilon}\}$ denote the ball around ξ_1 with radius $\tilde{\varepsilon}$. Choose $\tilde{\varepsilon} := \varepsilon/2$ for ε as in Assumption 3.1(i). We have

$$\tilde{\varepsilon} \sup\{|D_1 g_0(\xi, z_1 + z_2)| : \xi \in B(\xi_1, \tilde{\varepsilon})\} \leq e^{(M_0+2\tilde{\varepsilon})y_t} (e^{2RZ_t} + e^{2R'Z_t}).$$

The right-hand side of this inequality has finite expectation by Assumption 3.1(i) and Remark 3.1(ii). Since

$$\left| \frac{g_0(\xi_1 + \eta, z_1 + z_2) - g_0(\xi_1, z_1 + z_2)}{\eta} \right| \leq \tilde{\varepsilon} + \sup\{|D_1 g_0(\xi, z_1 + z_2)| : \xi \in B(\xi_1, \tilde{\varepsilon})\}$$

for sufficiently small $|\eta|$, dominated convergence yields

$$\begin{aligned} &\mathbb{E}[V(z_1)_t V(z_2)_t y_t] \\ &= e^{\xi_0} S_0^{z_1+z_2} \mathbb{E}[D_1 g_0(\xi_1, z_1 + z_2)] \\ &= e^{\xi_0} S_0^{z_1+z_2} D_1 g_1(\xi_1, z_1 + z_2) \\ &= \mathbb{E}[V(z_1)_t V(z_2)_t] (D_2 \Psi_0(t, \xi_1, z_1 + z_2) + D_2 \Psi_1(t, \xi_1, z_1 + z_2) y_0). \end{aligned} \tag{5.18}$$

Moreover,

$$\frac{e^{\xi_1(y_t + \delta_0/\delta_1)}}{y_t + \delta_0/\delta_1} = \int_{\Gamma} e^{z(y_t + \delta_0/\delta_1)} dz = \int_0^1 \left(\frac{\delta_1}{\delta_0} + \xi_1 s \right) \exp\left(\left(\frac{\delta_1}{\delta_0} \log(s) + \xi_1 s \right) y_t + \frac{\delta_0}{\delta_1} \xi_1 s \right) ds$$

with $\Gamma : [0, 1] \rightarrow \mathbb{C}, s \mapsto (\delta_1/\delta_0) \log(s) + \xi_1 s$. It follows that

$$\begin{aligned} &\mathbb{E} \left[\frac{V(z_1)_t V(z_2)_t}{\delta_0 + \delta_1 y_t} \right] \\ &= e^{\xi_0} S_0^{z_1+z_2} \frac{e^{-\delta_0 \xi_1/\delta_1}}{\delta_1} \\ &\quad \times \left(\frac{\delta_1}{\delta_0} \mathbb{E} \left[\int_0^1 \exp\left(\left(\frac{\delta_1}{\delta_0} \log(s) + \xi_1 s \right) y_t + \frac{\delta_0}{\delta_1} \xi_1 s + (z_1 + z_2) Z_t \right) ds \right] \right. \\ &\quad \left. + \xi_1 \mathbb{E} \left[\int_0^1 s \exp\left(\left(\frac{\delta_1}{\delta_0} \log(s) + \xi_1 s \right) y_t + \frac{\delta_0}{\delta_1} \xi_1 s + (z_1 + z_2) Z_t \right) ds \right] \right). \end{aligned}$$

By $\delta_0, \delta_1 > 0$ we have

$$\left| \exp\left(\left(\frac{\delta_1}{\delta_0} \log(s) + \xi_1 s\right)y_t + \frac{\delta_0}{\delta_1} \xi_1 s + (z_1 + z_2)Z_t\right) \right| \leq e^{\delta_0(M_0 \vee 0)/\delta_1} (e^{(M_0 \vee 0)y_t + 2RZ_t} + e^{(M_0 \vee 0)y_t + 2R'Z_t}) \tag{5.19}$$

for all $s \in (0, 1]$. Since the right-hand side of this inequality has finite expectation (cf. Assumption 3.1(i) and Remark 3.1(ii)), Fubini’s theorem and, once more, Remark 3.1(ii) yield

$$\begin{aligned} \mathbb{E}\left[\frac{V(z_1)_t V(z_2)_t}{\delta_0 + \delta_1 y_t}\right] &= e^{\xi_0} S_0^{z_1+z_2} \frac{e^{-\delta_0 \xi_1/\delta_1}}{\delta_1} \\ &\quad \times \int_0^1 \left(\frac{\delta_1}{\delta_0} + \xi_1 s\right) e^{\delta_0 \xi_1 s/\delta_1} \\ &\quad \times \exp\left(\Psi_0\left(t, \frac{\delta_1}{\delta_0} \log(s) + \xi_1 s, z_1 + z_2\right) \right. \\ &\quad \left. + \Psi_1\left(t, \frac{\delta_1}{\delta_0} \log(s) + \xi_1 s, z_1 + z_2\right) y_0\right) ds. \end{aligned} \tag{5.20}$$

From (5.17)–(5.19) and from the continuity of $\Psi_0, \Psi_1, D_2\Psi_0$, and $D_2\Psi_1$ on $[0, T] \times \mathcal{X}$, we obtain

$$\begin{aligned} \mathbb{E}[|V(z)_t|^2] &= \mathbb{E}[V(z)_t V(\bar{z})_t] \leq k_1(c), \\ \mathbb{E}[|V(z)_t|^2 y_t] &= \mathbb{E}[V(z)_t V(\bar{z})_t y_t] \leq k_2(c), \\ \mathbb{E}\left[\frac{|V(z)_t|^2}{\delta_0 + \delta_1 y_t}\right] &= \mathbb{E}\left[\frac{V(z)_t V(\bar{z})_t}{\delta_0 + \delta_1 y_t}\right] \leq k_3(c), \end{aligned}$$

for all $(t, z) \in [0, T] \times \mathcal{X}_f^c$ and some finite numbers $k_1(c), k_2(c)$, and $k_3(c)$ that may depend on c , but not on t or z . Consequently, (5.16) yields

$$\begin{aligned} \mathbb{E}[\langle L(z), \overline{L(\bar{z})} \rangle_T] &= \mathbb{E}[\langle L(z), L(\bar{z}) \rangle_T] \\ &\leq \int_0^T \left(|\eta_2(t, z, \bar{z})| \frac{k_2(c)}{\delta_1} + \left(\delta_1 |\eta_1(t, z, \bar{z})| + \delta_0 |\eta_2(t, z, \bar{z})| \right) \frac{k_1(c)}{\delta_1^2} \right. \\ &\quad \left. + (\delta_1^2 |\eta_0(t, z, \bar{z})| + \delta_0 \delta_1 |\eta_1(t, z, \bar{z})| + \delta_0^2 |\eta_2(t, z, \bar{z})|) \frac{k_3(c)}{\delta_1^2} \right) dt \\ &\leq k_4(c) \end{aligned}$$

for all $z \in \mathcal{X}_f^c$ and a positive constant $k_4(c)$ depending only on c because the mappings

$$(t, z) \mapsto \eta_0(t, z, \bar{z}), \eta_1(t, z, \bar{z}), \eta_2(t, z, \bar{z})$$

are continuous on $[0, T] \times \mathcal{X}_f^c$ by Assumption 3.1(ii). Hence, Fubini’s theorem yields

$$\mathbb{E}[L_T^2] = \lim_{c \rightarrow \infty} \int_{\mathcal{X}_f^c} \int_{\mathcal{X}_f^c} \mathbb{E}[\langle L(z_1), L(z_2) \rangle_T] \Pi(dz_1) \Pi(dz_2)$$

in view of (5.15). The assertion follows now from (5.16)–(5.18) and (5.20).

Appendix A

The dynamic behaviour of a semimartingale can be described in terms of predictable characteristics (B, C, ν) (cf. [17] for a definition and further background). For most continuous-time processes in applications, the characteristics factorize as follows.

Definition A.1. Let X be an \mathbb{R}^d -valued semimartingale with characteristics (B, C, ν) relative to some truncation function $h: \mathbb{R}^d \rightarrow \mathbb{R}^d$. If there are some \mathbb{R}^d -valued predictable process b , some predictable $\mathbb{R}^{d \times d}$ -valued process c whose values are nonnegative, symmetric matrices, and some transition kernel F from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}^d, \mathcal{B}^d)$ such that

$$B_t = \int_0^t b_s \, ds, \quad C_t = \int_0^t c_s \, ds,$$

$$\nu([0, t] \times G) = \int_0^t F_s(G) \, ds \quad \text{for } t \in [0, T], G \in \mathcal{B}^d,$$

we call (b, c, F) differential characteristics of X .

We can interpret b_t or rather $b_t + \int (x - h(x)) F_t(dx)$ as a drift rate, c_t as a diffusion coefficient, and F_t as a local jump measure. For a Lévy process, the triplet (b, c, F) is deterministic and does not depend on t . In this case it coincides with the Lévy–Khintchine triplet appearing in the Lévy–Khintchine formula. In a sense the differential characteristics generalize this triplet for more general semimartingales. They are typically derived from other ‘local’ representations of the process, e.g. in terms of a stochastic differential equation.

Proposition A.1. Let X be an \mathbb{R}^d -valued semimartingale with differential characteristics (b, c, F) , and let ξ be an $\mathbb{R}^{n \times d}$ -valued predictable process whose components ξ^j , $j = 1, \dots, n$, are integrable with respect to X . Then the \mathbb{R}^n -valued integral process $\xi \bullet X := (\xi^j \bullet X)_{j=1, \dots, n}$ has differential characteristics $(\tilde{b}, \tilde{c}, \tilde{F})$ of the form

$$\tilde{b}_t = \xi_t b_t + \int (\tilde{h}(\xi_t x) - \xi_t h(x)) F_t(dx),$$

$$\tilde{c}_t = \xi_t c_t \xi_t^\top,$$

$$\tilde{F}_t(G) = \int \mathbf{1}_G(\xi_t x) F_t(dx) \quad \text{for all } G \in \mathcal{B}^n \text{ with } 0 \notin G.$$

Here, h and \tilde{h} denote truncation functions on \mathbb{R}^d and \mathbb{R}^n , respectively.

Proof. Cf. [17, IX.5.3] and [22, Lemma 3].

Itô’s formula can be expressed in terms of characteristics as follows.

Proposition A.2. Let X denote an \mathbb{R}^d -valued semimartingale with differential characteristics (b, c, F) . Suppose that $g: U \rightarrow \mathbb{R}^n$ is twice continuously differentiable on some open subset $U \subset \mathbb{R}^d$ such that X and X_- are U -valued. Then the \mathbb{R}^n -valued semimartingale $g(X)$ has differential characteristics $(\tilde{b}, \tilde{c}, \tilde{F})$ of the form

$$\tilde{b}_t^i = \sum_{k=1}^d D_k g^i(X_{t-}) b_t^k + \frac{1}{2} \sum_{k,l=1}^d D_{k,l}^2 g^i(X_{t-}) c_t^{k,l}$$

$$+ \int \left(\tilde{h}^i(g(X_{t-} + x) - g(X_{t-})) - \sum_{k=1}^d D_k g^i(X_{t-}) h^k(x) \right) F_t(dx),$$

$$\tilde{c}_t^{i,j} = \sum_{k,l=1}^d D_k g^i(X_{t-}) c_t^{k,l} D_l g^j(X_{t-}),$$

$$\tilde{F}_t(G) = \int \mathbf{1}_G(g(X_{t-} + x) - g(X_{t-})) F_t(dx) \quad \text{for all } G \in \mathcal{B}^n \text{ with } 0 \notin G,$$

$i, j = 1, \dots, n$. Here, D_k etc. denote partial derivatives and h and \tilde{h} truncation functions on \mathbb{R}^d and \mathbb{R}^n , respectively.

Proof. Cf. [12, Corollary A.6].

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