

# Appendix

## A.1 QDN Notation

Symbol	Meaning
$\times$	Cartesian product, direct product
$\otimes$	tensor product
$\Sigma_n$	stage $n$
$\mathcal{A}_n$	apparatus at stage $\Sigma_n$
$r_n$	number of real or virtual detectors in $\mathcal{A}_n =$ the rank of $\mathcal{A}_n$
$i_n$	the $i$ th real or virtual detector in $\mathcal{A}_n$
$Q_n^i$	the qubit representing $i_n, i = 1, 2, \dots, r_n$
$\mathcal{Q}_n \equiv Q_n^1 Q_n^2 \dots Q_n^{r_n}$	the quantum register at stage $\Sigma_n$ : a rank- $r_n$ tensor product
$I_n$	the observer's information about $\mathcal{Q}_n$
$H_n \equiv (\mathcal{Q}_n, I_n)$	Heisenberg net at stage $\Sigma_n$
$2^{r_n}$	dimension of $\mathcal{Q}_n \equiv \dim \mathcal{Q}_n$
$B_n^i$	the preferred basis for $\mathcal{Q}_n^i, i = 1, 2, \dots, r$
$\mathcal{R}_n^r \equiv B_n^1 B_n^2 \dots B_n^{r_n}$	the preferred basis for $\mathcal{Q}_n$ : a Cartesian product
$\mathbf{k}_n$	$k$ th element of computational basis $B_n,$ $k = 0, 1, 2, \dots, 2^{r_n} - 1$
$\overline{\mathbf{k}}_n$	dual of $\mathbf{k}_n$
$\Psi_n$	pure labstate at stage $\Sigma_n$ : an element of $\mathcal{Q}_n$
$\hat{\mathbb{A}}_n^i$	$i$ th signal destruction operator at stage $\Sigma_n$
$\hat{\mathbb{A}}_n^i$	$i$ th signal creation operator at stage $\Sigma_n$
$\mathbb{P}_n^i \equiv \hat{\mathbb{A}}_n^i \hat{\mathbb{A}}_n^i,$	$i$ th no-signal projection operator
$\hat{\mathbb{P}}_n^i \equiv \hat{\mathbb{A}}_n^i \mathbb{A}_n^i$	$i$ th signal projection operator
$S_n^i \equiv \{\mathbb{P}_n^i, \hat{\mathbb{P}}_n^i, \mathbb{A}_n^i, \hat{\mathbb{A}}_n^i\}$	$i$ th signal set
$\mathbb{T}_{mn}^{ij} \equiv \mathbf{i}_m \mathbf{j}_n$	transition operator

## A.2 Lab Time and Frame Fields

In general relativity (GR), spacetime is modeled as a four-dimensional manifold with a Lorentz signature metric. In general, GR spacetimes cannot be covered by a single coordinate patch, particularly if there are closed time-like curves (CTCs) as in the case of the Gödel metric (Gödel, 1949). In spacetimes with CTCs, a global temporal foliation cannot be constructed, which means that a global laboratory perspective cannot be contemplated in such cases.

The principles of quantized detector networks (QDN) are well suited to deal with such issues. First, QDN is an endophysical approach to empirical physics, meaning that it does not attempt a global (exophysical) description of the Universe. In addition, finiteness is the order of the day, which means that actual physics laboratories are regarded as of finite extent and duration, and that no infinities are measurable. QDN does not normally attempt to discuss systems under observation (SUOs) in terms of an infinite number of real or virtual detectors. The QDN discussion of the bosonic and fermionic oscillators in Chapter 24 are given to illustrate the remarkable theoretical properties of infinite-rank quantum registers. In applications to real SUOs, QDN invariably involves finite-rank quantum registers.

In GR, a relatively localized laboratory description typically involves a complex of four *frame fields*,  $\{e_\mu : \mu = 0, 1, 2, 3\}$  constituting the laboratory frame, or coordinate patch adapted to cover a real physical laboratory. These basis vectors are usually chosen to satisfy the orthonormality relations

$$g(e_\mu, e_\nu) = \eta_{\mu\nu}, \quad (\text{A.1})$$

where  $g$  is the metric tensor and the  $\eta_{\mu\nu}$  are the components of a  $4 \times 4$  matrix displaying the Lorentz signature of a Minkowski metric:

$$[\eta_{\mu\nu}] \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (\text{A.2})$$

The significant frame field is  $e_0$ , which is time-like and indicates the temporal foliation that dictates clock time over the laboratory. The three other frame fields are space-like and lie in the space-like hypersurfaces of relative simultaneity that the observer has set up in their laboratory using some chosen synchronization protocol.

QDN requires such a framework in order for quantum principles to be applicable: space and time have different roles in QM as it is encountered in the laboratory. The concept of *stage* is intimately linked to the existence of such a framework.

## A.3 Lab Time and Stages

Consider an experiment of the Stern–Gerlach type, wherein a beam of particles is passed through a beam splitter feeding onto two detectors denoted  $A$  and  $B$ .

Suppose each run consists of a beam prepared at labtime reset to  $t = 0$  with the observer looking at each detector separately, at labtime  $T_A$  in the case of detector  $A$ , and labtime  $T_B$  in the case of detector  $B$ . We are interested here in the possibility that  $T_B$  is very much greater than  $T_A$ . For instance, suppose  $T_A$  is of the order of a millionth of a second and  $T_B$  is a million years. Then it is an empirical fact that the two times are not simultaneous relative to the laboratory concerned, and then the question arises as to whether the observation at  $A$  could causally impact on the observation at  $B$ .

In the framework of special relativity, the answer is determined by the lightcone structure of Minkowski spacetime. If  $B$  is outside the forward lightcone centered on  $A$ , then Einstein causality tells us that  $A$  could not influence  $B$ . In that situation, an inertial frame could always be found in which these two events were simultaneous, relative to that frame. Since probabilities are related to outcome frequencies, which is a counting process of signals, we would expect the same outcomes in such a frame as in the original frame where  $T_A$  and  $T_B$  were vastly different.

We expect the same results would hold if  $B$  was inside the forward lightcone of  $A$  but adequate shielding was in place. In modern electronics, the problem of undesirable signal interference due to one detector affecting another is known as *crosstalk*. *Shielding* is our term for the elimination of crosstalk. It is a problem but one that in principle can be overcome. Indeed, it could be argued that the very existence of physically distinct persistent SUOs co-existing at the same time is evidence for shielding. The fact that atoms are generally stable and can be regarded as distinct is direct evidence for that.

This line of reasoning leads to the interesting idea that *space* itself is a manifestation of shielding. Consider two hydrogen atoms. According to standard nonrelativistic QM, the combined SUO consists of two identical protons and two identical electrons. If the atoms were separated by, say, 2 angstroms, we might be tempted to describe their combined wave function in terms of a two-proton, two-electron wave function, properly antisymmetrized on account of the indistinguishability and fermionic nature of the constituents. On the other hand, if the two atoms were, say, a light year apart, no one would ever think of these atoms as anything other than two separate SUOs, each described by a one-proton, one-electron wave function (in standard nonrelativistic QM).

This raises the interesting question: which comes first, space or shielding? According to Schwinger, quoted in Chapter 24, the space-time is an idealization of empirical context, so the implication is that the emergent concept of shielding comes before we can define the reductionist concept of space.

The above arguments tell us that labtime simultaneity or strict lightcone causality is not essential in signal detection; the important criterion is that different signal detectors should not interfere causally with, or have the possibility of interfering causally with, other signal detectors in the experiment. It is this condition, referred to as shielding, that defines our concept of *stage*. Signal detectors that are looked at in such a way constitute a single *stage* in an

experiment. A stage is an collection of detectors that are looked at by an observer in such a way that no subset of those detectors can causally influence any other subset.

A stage is therefore a contextual classification of detectors that collectively plays the same role in QDN as events on a space-like hyperplane of “simultaneity” in some time-like foliation in general relativity.

The concept of stage undermines the contextually incomplete Block Universe concept: stages require the existence of observers, and the experimental protocol (context) to be given explicitly, something that the Block Universe has no place for. The “Consistent Histories” approach to QM is an interesting development of QM in that it deals with empirical propositions at multiple different observation times (Griffiths, 1984), and these could be regarded as stages.

#### A.4 Ensembles

The ensemble concept arises in physics for two reasons. First, no experiment normally validates a generalized proposition in a single run, even in classical physics, which deals with generalized propositions with a generalized proposition classification (GPC) of 2. Such propositions are based on the classical assertion that SUOs have precise qualities that can be quantified by exact measurement. Regardless of their view of that assertion, experimentalists will know that, in reality, experimental errors and inaccuracies are always present. To overcome this, multiple runs of the basic experimental protocol are generally performed and then averages and other statistical quantities established from the accumulated data. These multiple runs constitute ensembles of one kind or another.

The second reason for the use of ensembles is that quantum physics asserts that the outcome of any given run in an experiment is a random variable, so that statistical analysis based on the Born rule is required as a matter of principle and policy.

Ensembles come in several varieties, each with its particular spatiotemporal architecture, and each characterized by generally unstated, implicit context. Whichever kind of ensemble is chosen depends on several factors usually outside the experimentalist’s control, such as limited resources and time. The following are two important kinds of ensemble.

##### **Spatial Ensembles**

A spatial ensemble is a physical collection in a given laboratory of multiple, mutually isolated (from each other) copies of a given SUO, such as atoms in a crystal, such that a basic run is performed on each copy once. The statistics for the experiment is then established by collecting the outcome data from each copy and assuming that outcome frequencies can be attributed to probabilities associated with the original SUO.

In experiments based on relatively localized laboratories, wherein physical conditions are relatively homogeneous, this ensemble concept is usually reasonable. However, three problems may arise. First, different copies may actually interact with each other. This is the case in magnetic resonance experiments, for instance, where an interaction between nuclear spins and their neighboring spin environment is all the point. The second problem is that inhomogeneities in the laboratory environment may invalidate the above logic. For instance, a spatial ensemble carrying out an Unruh-type experiment in an accelerating laboratory (in contrast to a freely falling one) will almost certainly display laboratory inhomogeneities in what looks like a local gravitational field. The third problem is one of economics. Some experiments cannot be based on the idealized spatial ensemble concept simply because each individual run may be too costly or too big in spatial terms to duplicate in any laboratory. An example is the Large Hadron Collider: there is only one particle accelerator and it is very big and very expensive.

In the case of the Large Hadron Collider, there is a modification of the spatial ensemble concept that works excellently: multiple copies of the same SUO (protons) are contained in a single circulating beam. Assumptions are then made that during the very brief time of interaction involved, each proton behaves as if it was isolated from the other protons in that beam and would interact with only one other proton in the opposing beam. The beam statistics of the Large Hadron Collider, are impressive: in a given run, each beam consists of 2,808 bunches of protons, and each bunch contains about  $10^{11}$  protons.

### Temporal Ensembles

Some experiments are too costly to perform via spatial ensembles, so the standard alternative is to use a temporal ensemble. In such an architecture, multiple runs of the same basic protocol are implemented in temporal succession using the same apparatus each time. In principle, an ideal temporal ensemble should be equivalent to an ideal spatial ensemble, but that is an assertion that can be challenged on the basis of cosmological evidence. It is now believed that the universe is expanding in an irreversible way, relative to all endophysical observers. Therefore, the environment around any laboratory is not quite the same during any given run of an experiment as any other run. Of course, such discrepancies are minute and could be laughed at as generating a pointless debate, were it not for one glaring fact: the expansion *does* have observable effects, namely, the red shift of light from distant galaxies.

The issue here is the relative scale of times: a comparison of the typical time  $\tau$  to complete a given run, the interval  $T$  to the next run, and the age  $A$  of the Universe estimated from the observed Hubble constant. When  $\tau$  and  $T$  are both negligible compared with  $A$ , as is almost always the case, then temporal ensembles should be as good as spatial ensembles.

The impact of cosmological expansion should not be too lightly dismissed. It has been speculated by scientists such as Dirac that the gravitational constant  $G$  and perhaps even the speed of light  $c$  may change over cosmological time scales (Dirac, 1938a). Therefore, any discussion of observed physical properties such as electron mass and other properties should take such issues into account if the context merits it.

We note that Peres did not consider temporal ensembles to be proper ensembles in QM (Peres, 1995), but the fact is that many experiments are indeed carried out via such ensembles.

### A.5 Vector Spaces

A **vector space**  $(V, \mathbb{F})$  over a field  $\mathbb{F}$  such as the real numbers  $\mathbb{R}$  or complex numbers  $\mathbb{C}$  is a set  $V$  of elements  $V \equiv \{\mathbf{a}, \mathbf{b}, \dots\}$ , known as vectors, with the following properties:

- (i) There is a binary map  $\dot{+} : V \times V \rightarrow V$  such that, for any elements  $\mathbf{a}, \mathbf{b} \in V$ , the object  $\mathbf{a} \dot{+} \mathbf{b} \in V$ . This is called *addition of vectors*, or just *vector addition*. The elements of  $V$  are called *vectors*.

Vector addition is commutative, i.e.,

$$\mathbf{a} \dot{+} \mathbf{b} = \mathbf{b} \dot{+} \mathbf{a}, \quad \forall \mathbf{a}, \mathbf{b} \in V. \quad (\text{A.3})$$

Vector addition is associative, i.e.,

$$\mathbf{a} \dot{+} (\mathbf{b} \dot{+} \mathbf{c}) = (\mathbf{a} \dot{+} \mathbf{b}) \dot{+} \mathbf{c}, \quad \forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in V. \quad (\text{A.4})$$

- (ii) There is a unique element in  $V$ , known as the *zero vector*, denoted by  $\mathbf{0}_V$ , such that

$$\mathbf{a} \dot{+} \mathbf{0}_V = \mathbf{a}, \quad \forall \mathbf{a} \in V. \quad (\text{A.5})$$

- (iii) For every vector  $\mathbf{a}$ , there exists an *additive inverse*, denoted by  $-\mathbf{a}$ , such that

$$\mathbf{a} \dot{+} (-\mathbf{a}) = \mathbf{0}. \quad (\text{A.6})$$

These properties mean that  $V$  is an abelian group under vector addition.

- (iv) For any  $\mathbf{a} \in V$ ,  $\lambda \in \mathbb{F}$ , then the object  $\lambda \mathbf{a} \in V$ . This is known as *multiplication by a scalar*, or just *scalar multiplication*. In this context, the elements of  $\mathbb{F}$  are called *scalars*.

Scalar multiplication satisfies the property

$$\lambda(\mu \mathbf{a}) = (\lambda\mu) \mathbf{a}, \quad \lambda, \mu \in \mathbb{F}, \mathbf{a} \in V. \quad (\text{A.7})$$

- (v) Scalar multiplication is *distributive*, i.e.,

$$\begin{aligned} \lambda(\mathbf{a} \dot{+} \mathbf{b}) &= (\lambda \mathbf{a}) \dot{+} (\lambda \mathbf{b}), \\ (\lambda + \mu) \mathbf{a} &= (\lambda \mathbf{a}) \dot{+} (\mu \mathbf{a}) \end{aligned} \quad (\text{A.8})$$

It is standard practice to use  $V$  to mean  $(V, \mathbb{F})$ . The *ground field*  $\mathbb{F}$  is generally understood, but it is important to know whether it is  $\mathbb{R}$  or  $\mathbb{C}$ . In the former case we say  $V$  is a *real vector space*, while in the latter case we say  $V$  is a *complex vector space*. The space of *three-vectors* used to represent position in physical space is a real vector space, while the Hilbert space of quantum state vectors is a complex vector space.

It is easy to show that,  $\forall \lambda \in \mathbb{F}, \lambda \mathbf{0}_V = \mathbf{0}_V$ . Likewise, if  $0_{\mathbb{F}}$  is the zero element in  $\mathbb{F}$ , then  $0_{\mathbb{F}} \mathbf{a} = \mathbf{0}_V, \forall \mathbf{a} \in V$ . It is important not to confuse the scalar zero  $0_{\mathbb{F}}$  and the vector zero  $\mathbf{0}_V$ .

In practice, we do not bother to use a different symbol for vector addition,  $\dot{+}$ , in order to distinguish addition in the field,  $+$ . Henceforth, the same symbol,  $+$ , will be used for both.

We often modify the notation involving the additive inverse  $-\mathbf{b}$ , writing

$$\mathbf{a} + (-\mathbf{b}) = \mathbf{a} - \mathbf{b}, \quad (\text{A.9})$$

thereby suggesting a new binary process known as *subtraction*. This is not necessary, but it is useful and should always be interpreted in terms of the addition of vectors.

What is immensely astounding is that the above theory, which may appear no more than a mathematician's game, seems necessary to describe empirically validated quantum physics, with the additional surprise that the field  $\mathbb{F}$  is required to be  $\mathbb{C}$  and not  $\mathbb{R}$ .

### Subspaces of a Vector Space

Suppose  $U$  is a subset of a vector space  $V$  over some field  $\mathbb{F}$ . If  $U$  is a vector space over  $\mathbb{F}$ , using the same rules for vector addition and scalar multiplication as for  $V$ , then we say  $U$  is a *subspace* of  $V$ . Every subspace of  $V$  necessarily contains the zero vector  $\mathbf{0}_V$  of  $V$ .

Given two subspaces  $U_1, U_2$  of  $V$ , then the *intersection*  $U_1 \cap U_2$  is the set of elements common to  $U_1$  and  $U_2$ , and is also a subspace of  $V$ .

### Spanning Sets

Suppose  $S \equiv \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$  is a set of vectors in some vector space  $V$  with ground field  $\mathbb{F}$ . Let  $M$  be the set of all vectors of the form  $x^1 \mathbf{v}_1 + x^2 \mathbf{v}_2 + \dots + x^k \mathbf{v}_k$ , where  $x^1, x^2, \dots, x^k \in \mathbb{F}$ . Then  $M$  is a subspace of  $V$ , *spanned* by  $S$ . We write

$$M \equiv [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]. \quad (\text{A.10})$$

$S$  is called a *spanning set* for  $M$ .

### Linear Independence

An expression of the form  $x^1 \mathbf{v}_1 + x^2 \mathbf{v}_2 + \dots + x^k \mathbf{v}_k$ , where  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$  and  $x^1, x^2, \dots, x^k \in \mathbb{F}$  is called a *linear combination* of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ .

If the  $x^i$  are not all zero, then it is called a *nontrivial* linear combination. Otherwise it is called *trivial*.

A set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$  is *linearly dependent* if there exists a nontrivial linear combination equal to the zero vector  $\mathbf{0}_V$ . In other words,  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$  is linearly dependent if the equation

$$x^1\mathbf{v}_1 + x^2\mathbf{v}_2 + \dots + x^k\mathbf{v}_k = \mathbf{0}_V \tag{A.11}$$

has a solution for which at least one of the  $x^i$  is nonzero.

A set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$  is *linearly independent* if the only solution to Eq. (A.11) is  $x^1 = x^2 = \dots = x^k = 0_{\mathbb{F}}$ .

**Theorem A.1** *The nonzero vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$  are linearly dependent if and only if one of the vectors  $\mathbf{v}_k$  is a linear combination of the preceding ones  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k-1}$ .*

A single nonzero vector  $\mathbf{v}$  is necessarily independent, since the  $x\mathbf{v} = \mathbf{0}_V$  if and only if  $x = 0_{\mathbb{F}}$ .

A linearly independent spanning set is called a *basis*.

(A.12)

**Theorem A.2** *Any vector space that has a finite spanning set contains a basis.*

A vector space is *finite dimensional* if it has a finite basis (i.e., one consisting of a finite number of vectors). Hence, every vector space spanned by a finite spanning set is finite dimensional.

**Theorem A.3** *If  $V$  is a finite-dimensional vector space with basis  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$  then every vector  $\mathbf{v}$  in  $V$  can be expressed in one and only one way as a linear combination*

$$\mathbf{v} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + \dots + x^n\mathbf{e}_n = x^i\mathbf{e}_i \tag{A.13}$$

using the summation convention.

All bases of a finite-dimensional vector space have the same number of elements. The *dimension*  $\dim V$  of a finite-dimensional vector space  $V$  is the number of elements of a basis.

### Linear Transformations

Let  $U$  and  $V$  be two vector spaces, not necessarily of the same dimension, over the same field  $\mathbb{F}$ . A *linear transformation* (or *linear mapping*)  $T$  of  $U$  into  $V$  is a mapping that assigns to every  $\mathbf{u} \in U$  a unique vector  $T(\mathbf{u}) \in V$ , such that

$$\begin{aligned} T(\mathbf{u}_1 + \mathbf{u}_2) &= T(\mathbf{u}_1) + T(\mathbf{u}_2), & \forall \mathbf{u}_1, \mathbf{u}_2 \in U, \\ T(\lambda\mathbf{u}) &= \lambda T(\mathbf{u}), & \forall \mathbf{u} \in U, \quad \forall \lambda \in \mathbb{F}. \end{aligned} \tag{A.14}$$

$T(\mathbf{u})$  is the *image* of  $\mathbf{u}$  under  $T$ .

The set of all linear mappings of  $U$  into  $V$  is denoted  $L(U, V)$ .

Given a linear transformation  $T(U, V)$ , the set of all vectors  $\mathbf{u} \in U$  such that

$$T\mathbf{u} = \mathbf{0}_V \tag{A.15}$$

is called the *kernel* of  $T$  and written  $\ker T$ .

The set of all vectors  $T(\mathbf{u})$ ,  $\mathbf{u} \in U$  is called the *image* of  $U$  under  $T$ , and is denoted by  $T(U)$ .

The following is a critical theorem in QDN.

**Theorem A.4** (Tropper, 1969) *If  $T \in L(U, V)$ , then  $\ker T$  is a subspace of  $U$  and  $T(U)$  is a subspace of  $V$ , such that*

$$\dim \ker T + \dim T(U) = \dim U. \tag{A.16}$$

If  $\dim \ker T$  is called the *nullity* of  $T$  and  $\dim T(U)$  is called the *rank* of  $T$ , then the above theorem can be stated as

$$\text{nullity of } T + \text{rank of } T = \dim U. \tag{A.17}$$

**Linear Functionals**

A *linear functional*  $\tilde{f}$  is a linear mapping of a vector space  $V$  into its ground field  $\mathbb{F}$ , such that

$$\tilde{f}(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\tilde{f}(\mathbf{u}) + \beta\tilde{f}(\mathbf{v}) \in \mathbb{F}, \quad \forall \alpha, \beta \in \mathbb{F}, \quad \forall \mathbf{u}, \mathbf{v} \in V. \tag{A.18}$$

Note that summation on the left-hand side is in  $V$ , while summation on the right-hand side is in  $\mathbb{F}$ . Denote the set of all linear functionals over  $V$  by  $L(V, \mathbb{F})$ .

When  $\mathbb{F} = \mathbb{R}$ , then  $\tilde{f}$  is a *real-valued linear functional*, whereas if  $\mathbb{F} = \mathbb{C}$  then  $\tilde{f}$  is a *complex-valued linear functional*.

**One-Forms**

Given any two linear functionals  $\tilde{f}, \tilde{g} \in L(V, \mathbb{F})$ , define the linear combination  $\alpha\tilde{f} + \beta\tilde{g}$ , where  $\alpha, \beta \in \mathbb{F}$  by the rule

$$\left(\alpha\tilde{f} + \beta\tilde{g}\right)(\mathbf{v}) \equiv \alpha\tilde{f}(\mathbf{v}) + \beta\tilde{g}(\mathbf{v}), \quad \alpha, \beta \in \mathbb{F}, \quad \mathbf{v} \in V. \tag{A.19}$$

With this rule,  $L(V, \mathbb{F})$  is itself a vector space, known as the *dual vector space* and denoted by  $V^*$ . Elements of this vector space will be called *one-forms*.

The convention we shall follow as much as possible is that vectors will be represented by symbols in bold, such as  $\mathbf{v}$ , while one-forms will be denoted by symbols with a tilde, or a bar, such as  $\tilde{\omega}$  or  $\bar{\omega}$ .

The one-form/vector relation is employed in QDN in our representation of questions and answers, as discussed in Chapter 2.

An important fact is that when  $V$  is finite dimensional, then  $V^*$  has the same dimension, namely,  $\dim V^* = \dim V$ .

### Dual Basis

Suppose  $V$  is an  $n$ -dimensional vector space with basis  $B(V) \equiv \{\mathbf{e}^a: a = 1, 2, \dots, n\}$ . Then an arbitrary vector  $\mathbf{v} \in V$  can be written in the form

$$\mathbf{v} = \sum_{a=1}^n v^a \mathbf{e}^a, \quad (\text{A.20})$$

where the  $\{v^i\}$  are known as the *components* of  $\mathbf{v}$  relative to the basis  $B(V)$ .

Given  $B(V)$ , we can always find a basis  $B(V^*) \equiv \{\tilde{\mathbf{e}}^a: a = 1, 2, \dots, n\}$  for the dual space  $V^*$  such that  $\tilde{\mathbf{e}}^a(\mathbf{e}^b) = \delta^{ab}$ . We call  $B^*(V)$  the *conjugate basis*. This greatly simplifies calculations.

### Bracket Notation

Given a vector  $\mathbf{v} \in V$  and one-form  $\tilde{\omega} \in V^*$ , the bracket notation  $\langle \tilde{\omega}, \mathbf{v} \rangle \equiv \tilde{\omega}(\mathbf{v})$  is often used. In quantum mechanics, vectors are often written as *kets*, such as  $|\psi\rangle$ , and dual vectors as *bra-vectors*, such as  $\langle \phi|$ , a notation used extensively by Dirac (Dirac, 1958). Then the “inner product”  $\langle \phi|\psi\rangle$  is called the *bracket* of  $|\psi\rangle$  and  $\langle \phi|$ .

Quantum mechanics vector spaces use a complex-valued field and the following rule is imposed:  $\langle \phi|\psi\rangle^* = \langle \psi|\phi\rangle$ .

### Tensor Product Spaces

Suppose  $U$  and  $V$  are vector spaces over the same field  $\mathbb{F}$ . It is possible for  $U$  and  $V$  to be copies of the same vector space, but not necessarily so. If they were, we would simply label them  $U^1$  and  $U^2$ , respectively. Significantly,  $U$  and  $V$  need not have the same dimension.

If  $\mathbf{u} \in U$  and  $\mathbf{v} \in V$ , then the *direct product*  $\mathbf{u} \otimes \mathbf{v}$  is identified with  $(\mathbf{u}, \mathbf{v})$ , an element of the Cartesian product space  $U \times V$ .

**Example A.5** Consider the vector space of all real  $2 \times 2$  matrices  $M(2, \mathbb{R})$  and the vector space of all real  $3 \times 3$  matrices  $M(3, \mathbb{R})$ . Then elements of the Cartesian product  $M(2, \mathbb{R}) \times M(3, \mathbb{R})$  are given by *Kronecker products* of matrices. For example, if  $A \equiv [A_{ab}] \in M(2, \mathbb{R})$  and  $B \equiv [B_{ij}] \in M(3, \mathbb{R})$ , then the Kronecker product  $A \otimes B \equiv [C_{ai,bj}]$  is an array with double matrix indices, such that

$$C_{ai,bj} \equiv A_{ab}B_{ij}, \quad a, b = 1, 2, \quad i, j = 1, 2, 3. \quad (\text{A.21})$$

Unfortunately, direct product vector spaces are **not** vector spaces themselves, which can be readily seen by considering linear combinations of arbitrary elements.

Because we need vector addition to represent superposition in QM, and we find ourselves dealing with tensor products, we overcome this problem by extending  $U \times V$  to a larger space, the *tensor product* of  $U$  and  $V$ , denoted by  $U \otimes V$ . This tensor product space is defined to contain **all** linear combinations of elements of the Cartesian product space  $U \times V$  and satisfies all the axioms of a vector space

over the common ground field  $\mathbb{F}$ . Elements of  $U \otimes V$  are either of the form  $\mathbf{u} \otimes \mathbf{v}$  (which is in  $U \times V$ ) or linear combinations of such direct products, which may or may not be in  $U \times V$ .

Elements of  $U \otimes V$  that are of the form  $\mathbf{u} \otimes \mathbf{v}$  for  $\mathbf{u} \in U, \mathbf{v} \in V$  are called *separable*. Elements of  $U \otimes V$  that are not separable are called *entangled*.

Entanglement is of great significance in quantum mechanics. The physically observable properties of entangled quantum states lie at the heart of the problems with the interpretation of quantum mechanics.

As with ordinary arithmetic, the tensor product operation  $\otimes$  takes precedence over the vector summation operation  $+$  in  $U \otimes V$ , so we may leave out the brackets and just write

$$(\mathbf{u}_1 \otimes \mathbf{v}_1) + (\mathbf{u}_2 \otimes \mathbf{v}_1) \equiv \mathbf{u}_1 \otimes \mathbf{v}_1 + \mathbf{u}_2 \otimes \mathbf{v}_2 \in U \otimes V. \tag{A.22}$$

Note that  $U \otimes V$  is a vector space over the field  $\mathbb{F}$  common to  $U$  and  $V$ . Multiplication by a scalar can be considered in several ways:

$$\lambda \{\mathbf{u} \otimes \mathbf{v}\} = (\lambda \mathbf{u}) \otimes \mathbf{v} = \mathbf{u} \otimes (\lambda \mathbf{v}) = \lambda \mathbf{u} \otimes \mathbf{v}. \tag{A.23}$$

Similarly,

$$\lambda \{\mathbf{u}_1 \otimes \mathbf{v}_1 + \mathbf{u}_2 \otimes \mathbf{v}_2\} = \lambda \mathbf{u}_1 \otimes \mathbf{v}_1 + \lambda \mathbf{u}_2 \otimes \mathbf{v}_2. \tag{A.24}$$

Denote the zero vectors in  $U, V$ , and  $U \otimes V$  by  $\mathbf{0}_U, \mathbf{0}_V$ , and  $\mathbf{0}_{U \otimes V}$ , respectively. Then for any  $\mathbf{u} \in U, \mathbf{v} \in V$ , we have

$$\mathbf{0}_U \otimes \mathbf{v} = \mathbf{u} \otimes \mathbf{0}_V = \mathbf{0}_{U \otimes V}. \tag{A.25}$$

Likewise, if  $0_{\mathbb{F}}$  is the zero element of the field  $\mathbb{F}$ , then

$$0_{\mathbb{F}}(\mathbf{u} \otimes \mathbf{v}) = \mathbf{0}_{U \otimes V}. \tag{A.26}$$

**Rank**

The *rank* of a tensor product space is the number of vector spaces in the product. For example,  $U \otimes V$  is a *rank-two* tensor product space;  $U^1 \otimes U^2 \otimes \dots \otimes U^n$  is of rank  $n$ .

Elements of a given tensor product have the rank of that tensor product space. Hence scalars have rank zero, while vectors and one-forms have rank one.

Given a number of vectors spaces  $V^1, V^2, \dots, V^r$ , then the rank- $r$  tensor product space  $V^1 \otimes V^2 \otimes \dots \otimes V^r$  has dimension equal to the product of all the individual dimensions, that is,

$$\dim \{V^1 \otimes V^2 \otimes \dots \otimes V^r\} = \{\dim V^1\} \{\dim V^2\} \dots \{\dim V^r\}. \tag{A.27}$$

Likewise, if  $V^{1*}, V^{2*}, \dots, V^{s*}$  are dual vector spaces, we define the rank- $s$  tensor product space  $V^{1*} \otimes V^{2*} \otimes \dots \otimes V^{s*}$  in an analogous way, and similarly for mixed tensor product spaces such as  $V^1 \otimes V^{2*} \otimes \dots$ .

**Separable Bases**

If  $\{\mathbf{u}^a : a = 1, 2, \dots, \dim U\}$  is a basis for  $U$  and  $\{\mathbf{v}^b : b = 1, 2, \dots, \dim V\}$  is a basis for  $V$ , then a frequently useful basis for  $U \otimes V$  is given by  $\{\mathbf{u}^a \otimes \mathbf{v}^b : a = 1, 2, \dots, \dim U, b = 1, 2, \dots, \dim V\}$ . Every element of this basis is a separable element of the tensor product space  $U \otimes V$ , so we call this a *separable basis*. From this we immediately conclude that

$$\dim \{U \otimes V\} = \{\dim U\} \cdot \{\dim V\}. \quad (\text{A.28})$$

This result generalizes to higher rank tensor product spaces.

**Hilbert spaces**

Hilbert spaces are finite or infinite dimensional vector spaces with a complete inner product (Streater and Wightman, 1964).