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# INFINITE SERIES CONCERNING HARMONIC NUMBERS AND QUINTIC CENTRAL BINOMIAL COEFFICIENTS

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**Abstract**

By examining two hypergeometric series transformations, we establish several remarkable infinite series identities involving harmonic numbers and quintic central binomial coefficients, including five conjectured recently by Z.-W. Sun ['Series with summands involving harmonic numbers', Preprint, 2023, arXiv:2210.07238v7]. This is realised by 'the coefficient extraction method' implemented by *Mathematica* commands.

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## 1. Introduction and outline

There are many infinite series representations for $\pi$ and related mathematical constants (see [2, 3, 7, 9, 10, 13, 17, 22]). Recently, by employing computer algebra, Guillera [14–16] discovered several marvellous evaluations of Ramanujan-like series [19]. Two beautiful examples are

$$\boxed{\text{Equation (2.4)}} \qquad \frac{8}{\pi^2}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\{1+8n+20n^2\},$$

$$\boxed{\text{Equation (2.8)}} \qquad \frac{7}{2}\zeta(3)=\sum_{n=0}^{\infty}\frac{(-2^8)^n}{\binom{2n}{n}^5}\frac{5+14n+10n^2}{(2n+1)^5}.$$

These two identities and further similar ones have been proved uniformly by means of the hypergeometric series approach (see [6, 8, 11]). The same approach has been used to examine analogous series involving harmonic numbers (see [5, 7, 9, 10, 21]). We evaluate several challenging series involving harmonic numbers and quintic central binomial coefficients in closed form, including five experimentally conjectured by

Sun [20] (see (2.5), (2.6), (2.9), (2.10), (3.2)). Five representative examples are

$$\boxed{\text{Equation (3.3)}}\quad \frac{64\zeta(3)}{\pi^2}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\Big\{\frac{8}{1+2n}+(1+8n+20n^2)\mathbf{H}_n^{\langle 3\rangle}\Big\}.$$

$$\boxed{\text{Equation (3.4)}}\quad -\frac{56\pi^2}{45}=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\{(1+8n+20n^2)(\mathbf{H}_n^{\langle 4\rangle}+64\mathbf{O}_n^{\langle 4\rangle})\}.$$

$$\boxed{\text{Equation (2.11)}}\quad -98\zeta^2(3)=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{n^5\binom{2n}{n}^5}\Big\{\frac{1}{n}+4(1-6n+10n^2)\mathbf{O}_n^{\langle 3\rangle}\Big\}.$$

$$\boxed{\text{Equation (2.12)}}\quad -127\zeta(7)=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{n^5\binom{2n}{n}^5}(1-6n+10n^2)(\mathbf{H}_{n-1}^{\langle 4\rangle}+4\mathbf{O}_n^{\langle 4\rangle}).$$

$$\boxed{\text{Equation (3.6)}}\quad 14\zeta(3)\{8-7\zeta(3)\}=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{\binom{2n}{n}^5}\frac{1+4n(1+6n+10n^2)\mathbf{O}_{n+1}^{\langle 3\rangle}}{n^6(2n+1)^4}.$$

To facilitate the subsequent presentation, we briefly review basic facts about harmonic numbers, the Γ-function and the 'coefficient extraction' method.

**1.1. Harmonic numbers.** For $x\in\mathbb{R}$ and $n\in\mathbb{N}_0$, these numbers are defined by

$$\mathbf{H}_n^{\langle\lambda\rangle}(x)=\sum_{k=0}^{n-1}\frac{1}{(x+k)^\lambda},\qquad \bar{\mathbf{H}}_n^{\langle\lambda\rangle}(x)=\sum_{k=0}^{n-1}\frac{(-1)^k}{(x+k)^\lambda};$$
$$\mathbf{O}_n^{\langle\lambda\rangle}(x)=\sum_{k=0}^{n-1}\frac{1}{(x+2k)^\lambda},\qquad \bar{\mathbf{O}}_n^{\langle\lambda\rangle}(x)=\sum_{k=0}^{n-1}\frac{(-1)^k}{(x+2k)^\lambda}.$$

When $\lambda=1$ and/or $x=1$, they will be suppressed from the notation. We record also the following simple, but useful relations:

$$\mathbf{H}_{2n}^{\langle\lambda\rangle}=\mathbf{O}_n^{\langle\lambda\rangle}+2^{-\lambda}\mathbf{H}_n^{\langle\lambda\rangle},\ \mathbf{H}_n^{\langle\lambda\rangle}(\tfrac{1}{2})=2^\lambda\mathbf{O}_n^{\langle\lambda\rangle};\quad \bar{\mathbf{H}}_{2n}^{\langle\lambda\rangle}=\mathbf{O}_n^{\langle\lambda\rangle}-2^{-\lambda}\mathbf{H}_n^{\langle\lambda\rangle},\ \bar{\mathbf{H}}_n^{\langle\lambda\rangle}(\tfrac{1}{2})=2^\lambda\bar{\mathbf{O}}_n^{\langle\lambda\rangle}.$$

**1.2. The Gamma function.** It is defined by the Euler integral (see [18, Section 8]),

$$\Gamma(x)=\int_0^\infty \tau^{x-1}e^{-\tau}\,d\tau\quad\text{for }\mathrm{Re}(x)>0,$$

with its quotient form being abbreviated to

$$\Gamma\begin{bmatrix}\alpha,\beta,\ldots,\gamma\\A,B,\ldots,C\end{bmatrix}=\frac{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\gamma)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}.$$

As usual, denote the Riemann zeta function and the Euler–Mascheroni constant by

$$\zeta(x)=\sum_{n=1}^{\infty}\frac{1}{n^x}\quad\text{and}\quad\gamma=\lim_{n\to\infty}(\mathbf{H}_n-\ln n).$$

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Then we have power series expansions of the Γ-function [5]:

$$\Gamma(1-x) = \exp\Big\{\sum_{k\geq 1}\frac{\sigma_k}{k}x^k\Big\}, \tag{1.1}$$

$$\Gamma\Big(\frac{1}{2}-x\Big) = \sqrt{\pi}\exp\Big\{\sum_{k\geq 1}\frac{\tau_k}{k}x^k\Big\}; \tag{1.2}$$

where $\sigma_k$ and $\tau_k$ are defined respectively by

$$\begin{aligned} &\sigma_1 = \gamma && \text{and} \quad \sigma_m = \zeta(m) && \text{for } m \geq 2;\\ &\tau_1 = \gamma + 2\ln 2 && \text{and} \quad \tau_m = (2^m-1)\zeta(m) && \text{for } m \geq 2. \end{aligned}$$

**1.3. Coefficient extraction.** For an indeterminate $x$, the rising factorials are

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for } n \in \mathbb{N}.$$

Let $[x^m]\varphi(x)$ stand for the coefficient of $x^m$ in the formal power series $\varphi(x)$. By means of the generating function method, it is not difficult to show (see [4, 7, 8]):

$$[x^m]\frac{(\lambda-x)_n}{(\lambda)_n} = \Omega_m(-\text{hh}) \quad \text{and} \quad [x^m]\frac{(\lambda)_n}{(\lambda-x)_n} = \Omega_m(\text{hh}). \tag{1.3}$$

Here ‘$\text{hh}_k$’ stands for the harmonic number $\text{hh}_k := \mathbf{H}_n^{\langle k\rangle}(\lambda)$ of order $k$, and the Bell polynomials [12, Section 3.3] are expressed explicitly as

$$\Omega_m(\pm\text{hh}) = \sum_{\omega(m)}\prod_{k=1}^{m}\frac{\{\pm\mathbf{H}_n^{\langle k\rangle}(\lambda)\}^{\ell_k}}{\ell_k!\ k^{\ell_k}}. \tag{1.4}$$

The multiple sum on the right runs over $\omega(m)$, the set of all $m$-partitions represented by $m$-tuples $(\ell_1, \ell_2, \ldots, \ell_m) \in \mathbb{N}_0^m$ subject to the condition $\sum_{k=1}^m k\ell_k = m$.

By applying the ‘coefficient extraction’ method (see [5, 7, 9, 10, 21]) to the two hypergeometric series transformations due to Chu and Zhang [11], we shall establish several new infinite series identities involving harmonic numbers and quintic central binomial coefficients. In the next section, this approach will be illustrated by deducing identities from one of these transformations. Then in Section 3, more infinite series evaluations will be presented by examining the other transformation.

## 2. Sample series with harmonic numbers of lower orders

For $\{a, b, c, d, e\} \in \mathbb{C}$ subject to the condition $\text{Re}(1+2a-b-c-d-e) > 0$, Chu and Zhang [11, Theorem 10] discovered the transformation formula

$$\begin{aligned}\sum_{k=0}^{\infty}&\frac{(a+2k)\,(b)_k(c)_k(d)_k(e)_k}{(1+a-b)_k(1+a-c)_k(1+a-d)_k(1+a-e)_k}\\ &= \sum_{n=0}^{\infty}(-1)^n\frac{\mathcal{U}_n(a,b,c,d,e)}{(1+2a-b-c-d-e)_{2n+2}}\end{aligned}$$

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$$\times \frac{\left\{ \begin{matrix} (1+a-b-c)_n(1+a-b-d)_n(1+a-b-e)_n \\ (1+a-c-d)_n(1+a-c-e)_n(1+a-d-e)_n \end{matrix} \right\}}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(1+a-e)_n}, \tag{2.1}$$

where the cubic polynomial $\mathcal{U}_n(a,b,c,d,e)$ is given by

$$\begin{aligned}\mathcal{U}_n(a,b,c,d,e) &= (a-e+n)(1+2a-b-c-d+2n)(2+2a-b-c-d-e+2n)\\ &\quad + (1+a-b-c+n)(1+a-b-d+n)(1+a-c-d+n).\end{aligned}$$

By specifying $e = a$ in (2.1) and assuming the condition $\mathrm{Re}(1+a-b-c-d) > 0$, we derive, by making use of Dougall's summation theorem for the very well-poised ${}_5F_4$-series [1, Section 4.4], the summation formula

$$\begin{aligned}&\Gamma\left[\begin{matrix} 1+a-b, 1+a-c, 1+a-d, 1+a-b-c-d \\ a, 1+a-b-c, 1+a-b-d, 1+a-c-d \end{matrix}\right]\\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(1-b)_n(1-c)_n(1-d)_n}{(1+a-b-c-d)_{2n+2}} \mathrm{U}_n(a,b,c,d)\\ &\quad \times \frac{(1+a-b-c)_n(1+a-b-d)_n(1+a-c-d)_n}{n!\,(1+a-b)_n(1+a-c)_n(1+a-d)_n},\end{aligned} \tag{2.2}$$

where the cubic polynomial $\mathrm{U}_n(a,b,c,d)$ is given by

$$\begin{aligned}\mathrm{U}_n(a,b,c,d) &= n(1+2a-b-c-d+2n)(2+a-b-c-d+2n)\\ &\quad + (1+a-b-c+n)(1+a-b-d+n)(1+a-c-d+n).\end{aligned}$$

The parameter restriction $\mathrm{Re}(1+a-b-c-d) > 0$ can be removed by analytic continuation. The formula holds in the whole complex space of dimension 4 except for the hyperplanes determined by $\{b-a \in \mathbb{N}\} \cup \{c-a \in \mathbb{N}\} \cup \{d-a \in \mathbb{N}\} \cup \{b+c+d-a \in \mathbb{N}\}$, which is also covered by the convergence domain of the infinite series.

Now we are going to illustrate how to use (2.2) and then (2.1) to derive infinite series identities involving harmonic numbers and quintic central binomial coefficients.

**2.1. Quintic central binomial coefficients in numerators.** Under the parameter replacements

$$a = \tfrac{1}{2} + ax, \quad b = \tfrac{1}{2} + bx, \quad c = \tfrac{1}{2} + cx, \quad d = \tfrac{1}{2} + dx,$$

the summation formula in (2.2) becomes

$$\begin{aligned}&\Gamma\left[\begin{matrix} 1+ax-bx, 1+ax-cx, 1+ax-dx, 1+ax-bx-cx-dx \\ \frac{1}{2}+ax, \frac{1}{2}+ax-bx-cx, \frac{1}{2}+ax-bx-dx, \frac{1}{2}+ax-cx-dx \end{matrix}\right]\\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{2}-bx)_n(\frac{1}{2}-cx)_n(\frac{1}{2}-dx)_n}{n!\,(1+ax-bx-cx-dx)_{2n+1}} \mathcal{P}_n(a,b,c,d;x)\\ &\quad \times \frac{(\frac{1}{2}+ax-bx-cx)_n(\frac{1}{2}+ax-bx-dx)_n(\frac{1}{2}+ax-cx-dx)_n}{(1+ax-bx)_n(1+ax-cx)_n(1+ax-dx)_n},\end{aligned} \tag{2.3}$$

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where the cubic polynomial $\mathcal{P}_n(a,b,c,d;x)$ is given by

$$\begin{aligned}\mathcal{P}_n(a,b,c,d;x) &= n(1+ax-bx-cx-dx+2n)(\tfrac{1}{2}+2ax-bx-cx-dx+2n)\\ &+(\tfrac{1}{2}+ax-bx-cx+n)(\tfrac{1}{2}+ax-bx-dx+n)(\tfrac{1}{2}+ax-cx-dx+n).\end{aligned}$$

Observe that both sides of equation (2.3) are analytic in $x$ in the neighbourhood of $x=0$. By means of (1.1) and (1.2), the $\Gamma$-function quotient on the left-hand side can be expanded into a Maclaurin series. The same can be done for the right-hand side via (1.3) and (1.4). By comparing the coefficients $\mathrm{A}_m(a,b,c,d)$ of $x^m$ across the resulting series, we can establish the following infinite series identities.

- $\boxed{\mathrm{A}_0(a,b,c,d)}$ Guillera [16] (see also [8, 11]):

$$\frac{8}{\pi^2}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\{1+8n+20n^2\}. \tag{2.4}$$

- $\boxed{\mathrm{A}_1(1,1,0,0)}$ Conjectured by Sun [20, (4.15)]

$$\frac{16\ln 2}{\pi^2}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\{(1+6n)-(1+8n+20n^2)\mathbf{H}_n\}. \tag{2.5}$$

- $\boxed{\mathrm{A}_1(3,3,2,2)}$

$$\frac{16\ln 2}{\pi^2}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\{(3+10n)+10(1+8n+20n^2)\mathbf{O}_n\}.$$

Linear combinations of the above two series yield two further series below, where the first one was experimentally conjectured by Sun [20, (4.16)]:

$$\frac{32\ln 2}{\pi^2}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\{(1+10n)-5(1+8n+20n^2)\mathbf{H}_{2n}\}, \tag{2.6}$$

$$\frac{48\ln 2}{\pi^2}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\{(4+20n)+5(1+8n+20n^2)\bar{\mathbf{H}}_{2n}\}.$$

- $\boxed{\mathrm{A}_2(1+\mathbf{i},0,1,\mathbf{i})}$

$$\frac{128\ln^2 2}{\pi^2}-\frac{8}{3}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\left\{\begin{matrix}4-8(1+6n)\mathbf{H}_n\\ +(1+8n+20n^2)(\mathbf{H}_n^{\langle 2\rangle}+4\mathbf{H}_n^2)\end{matrix}\right\}.$$

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- $\boxed{\mathrm{A}_2(3,7,\sqrt{19}\mathbf{i},-\sqrt{19}\mathbf{i})}$

$$\frac{16\ln^2 2}{5\pi^2}-\frac{4}{5}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\left\{\begin{matrix}1+2(3+10n)\mathbf{O}_n\\ -(1+8n+20n^2)(\mathbf{O}_n^{\langle 2\rangle}-10\mathbf{O}_n^2)\end{matrix}\right\}.$$

- $\boxed{3\mathrm{A}_3(1,1,\mathbf{i},-\mathbf{i})+2\mathrm{A}_3(0,1,-2,1)}$

$$\frac{28\zeta(3)-64\ln^3 2}{\pi^2}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\left\{\begin{matrix}\frac{2}{2n+1}+6\mathbf{H}_n-3(1+6n)(\mathbf{H}_n^2+2\mathbf{O}_n^{\langle 2\rangle})\\ +(1+8n+20n^2)(\mathbf{H}_n^3+6\mathbf{H}_n\mathbf{O}_n^{\langle 2\rangle})\end{matrix}\right\}.$$

- $\boxed{3\mathrm{A}_3(2,0,1+\mathbf{i},1-\mathbf{i})-2\mathrm{A}_3(0,1,-2,1)}$

$$\frac{56\zeta(3)+256\ln^3 2}{\pi^2}-16\ln 2=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\left\{\begin{matrix}\frac{4}{2n+1}-12\mathbf{H}_n+3(1+6n)(4\mathbf{H}_n^2+\mathbf{H}_n^{\langle 2\rangle})\\ -(1+8n+20n^2)(4\mathbf{H}_n^3+3\mathbf{H}_n\mathbf{H}_n^{\langle 2\rangle})\end{matrix}\right\}.$$

Among these series, the first and also the simplest was discovered by Guillera [16] using computer algebra. For the hypergeometric series approach, more identities of Ramanujan type are systematically reviewed in [8, 11].

**2.2. Quintic central binomial coefficients in denominators.** Alternatively, under the parameter setting

$$a=1+ax,\quad b=\tfrac{1}{2}+bx,\quad c=\tfrac{1}{2}+cx,\quad d=\tfrac{1}{2}+dx,\quad e=\tfrac{1}{2}+ex,$$

the transformation corresponding to (2.1) can be simplified into

$$\begin{aligned}\sum_{k=0}^{\infty}&\frac{(1+ax+2k)\,(\frac{1}{2}+bx)_k(\frac{1}{2}+cx)_k(\frac{1}{2}+dx)_k(\frac{1}{2}+ex)_k}{(\frac{1}{2}+ax-bx)_{k+1}(\frac{1}{2}+ax-cx)_{k+1}(\frac{1}{2}+ax-dx)_{k+1}(\frac{1}{2}+ax-ex)_{k+1}}\\ &=\sum_{n=0}^{\infty}(-1)^n\frac{Q_n(a,b,c,d,e;x)}{(1+2ax-bx-cx-dx-ex)_{2n+2}}\\ &\quad\times\frac{\left\{\begin{matrix}(1+ax-bx-cx)_n(1+ax-bx-dx)_n(1+ax-bx-ex)_n\\ (1+ax-cx-dx)_n(1+ax-cx-ex)_n(1+ax-dx-ex)_n\end{matrix}\right\}}{(\frac{1}{2}+ax-bx)_{n+1}(\frac{1}{2}+ax-cx)_{n+1}(\frac{1}{2}+ax-dx)_{n+1}(\frac{1}{2}+ax-ex)_{n+1}},\end{aligned}\tag{2.7}$$

where the cubic polynomial $Q_n=Q_n(a,b,c,d,e;x)$ is given by

$$\begin{aligned}Q_n&=(1+ax-bx-cx+n)(1+ax-bx-dx+n)(1+ax-cx-dx+n)\\ &\quad+(\tfrac{1}{2}+ax-ex+n)(\tfrac{3}{2}+2ax-bx-cx-dx+2n)(2+2ax-bx-cx-dx-ex+2n).\end{aligned}$$

Since both sides of (2.7) are analytic in $x$ in the neighbourhood of $x=0$, we can expand them into Maclaurin series. Even though the series on the left does not admit a closed form expression, we do succeed, by extracting the coefficients $\mathrm{B}_m(a,b,c,d,e)$ of $x^m$ across the resulting series, in recovering Guillera's elegant formula (2.8), confirming two conjectured identities (2.9) and (2.10) made very recently by

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Sun [20], and finding further elegant and difficult evaluation formulae involving harmonic numbers of higher order. They are represented by the following ten examples.

- $\boxed{B_0(a,b,c,d,e)}$ Guillera [16] (see also [8, 11])

$$-28\zeta(3)=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{n^5\binom{2n}{n}^5}\{1-6n+10n^2\}. \tag{2.8}$$

- $\boxed{B_1(2,1,1,1,1)}$ Conjectured by Sun [20, (4.6)]

$$-\frac{\pi^4}{2}=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{n^5\binom{2n}{n}^5}\{2(1-6n+10n^2)\mathbf{O}_n+(1-3n)\}. \tag{2.9}$$

- $\boxed{B_2(0,0,0,1,-1)}$ Conjectured by Sun [20, (4.8)]

$$62\zeta(5)=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{n^5\binom{2n}{n}^5}\{(1-6n+10n^2)(\mathbf{H}_{n-1}^{\langle 2\rangle}-2\mathbf{O}_n^{\langle 2\rangle})+1\}. \tag{2.10}$$

- $\boxed{B_2(2,1,1,1,1)}$

$$-186\zeta(5)=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{n^5\binom{2n}{n}^5}\begin{Bmatrix}1+8(1-3n)\mathbf{O}_n\\ +2(1-6n+10n^2)(4\mathbf{O}_n^2+\mathbf{O}_n^{\langle 2\rangle})\end{Bmatrix}.$$

- $\boxed{B_3(2,1,1,1,1)}$

$$-\frac{\pi^6}{4}=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{n^5\binom{2n}{n}^5}\begin{Bmatrix}3\mathbf{O}_n+3(1-3n)(4\mathbf{O}_n^2+\mathbf{O}_n^{\langle 2\rangle})\\ +(1-6n+10n^2)(8\mathbf{O}_n^3+6\mathbf{O}_n\mathbf{O}_n^{\langle 2\rangle}+\mathbf{O}_n^{\langle 3\rangle})\end{Bmatrix}.$$

- $\boxed{B_3(0,0,1,1,-2)}$

$$-98\zeta^2(3)=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{n^5\binom{2n}{n}^5}\left\{\frac{1}{n}+4(1-6n+10n^2)\mathbf{O}_n^{\langle 3\rangle}\right\}. \tag{2.11}$$

We remark that the above closed form expression is not obtained directly from (2.7) under the parameter setting $B_3(0,0,1,1,-2)$. In this case, the corresponding left-hand side results in another series, which can be simplified (apart from a constant factor) by symmetry as follows:

$$\begin{aligned}\sum_{k=0}^{\infty}\left\{\frac{1}{(1+2k)^6}+\frac{2\mathbf{O}_k^{\langle 3\rangle}}{(1+2k)^3}\right\}&=\sum_{k=0}^{\infty}\frac{1}{(1+2k)^6}+\sum_{\substack{k,j=0\\k\neq j}}^{\infty}\frac{1}{(1+2k)^3(1+2j)^3}\\&=\left\{\sum_{k=0}^{\infty}\frac{1}{(1+2k)^3}\right\}\times\left\{\sum_{j=0}^{\infty}\frac{1}{(1+2j)^3}\right\}=\frac{49}{64}\zeta^2(3).\end{aligned}$$

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By carrying out analogous operations, we can evaluate four further series in closed form.

- $\boxed{\mathrm{B}_4(0,1,-1,-\mathbf{i},\mathbf{i})}$

$$-127\zeta(7)=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{n^5\binom{2n}{n}^5}(1-6n+10n^2)(\mathbf{H}_{n-1}^{\langle 4\rangle}+4\mathbf{O}_n^{\langle 4\rangle}). \tag{2.12}$$

- $\boxed{\mathrm{B}_5(0,0,1,1,-2)}$

$$\frac{217\zeta(3)\zeta(5)}{2}=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{n^5\binom{2n}{n}^5}\left\{\begin{matrix}\frac{\mathbf{H}_{n-1}^{\langle 2\rangle}-2\mathbf{O}_n^{\langle 2\rangle}}{4n}+\mathbf{O}_n^{\langle 3\rangle}+(1-6n+10n^2)\\ \times(\mathbf{H}_{n-1}^{\langle 2\rangle}\mathbf{O}_n^{\langle 3\rangle}-2\mathbf{O}_n^{\langle 2\rangle}\mathbf{O}_n^{\langle 3\rangle}-2\mathbf{O}_n^{\langle 5\rangle}\end{matrix}\right\}.$$

- $\boxed{\mathrm{B}_6(0,1-\sqrt{2}\mathbf{i},1+\sqrt{2}\mathbf{i},-1,-1)}$

$$\frac{2744\zeta^3(3)+2044\zeta(9)}{-3}=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{n^5\binom{2n}{n}^5}\left\{\begin{matrix}\frac{16\mathbf{O}_n^{\langle 3\rangle}}{n}+(1-6n+10n^2)\\ \times[32(\mathbf{O}_n^{\langle 3\rangle})^2+32\mathbf{O}_n^{\langle 6\rangle}-\mathbf{H}_{n-1}^{\langle 6\rangle}]\end{matrix}\right\}.$$

- $\boxed{\mathrm{B}_7(0,1-\sqrt{2}\mathbf{i},1+\sqrt{2}\mathbf{i},-1,-1)}$

$$\frac{889\zeta(3)\zeta(7)}{-4}=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{n^5\binom{2n}{n}^5}\left\{\begin{matrix}\frac{\mathbf{H}_{n-1}^{\langle 4\rangle}+4\mathbf{O}_n^{\langle 4\rangle}}{4n}+(1-6n+10n^2)\\ \times(\mathbf{H}_{n-1}^{\langle 4\rangle}\mathbf{O}_n^{\langle 3\rangle}+4\mathbf{O}_n^{\langle 4\rangle}\mathbf{O}_n^{\langle 3\rangle}+4\mathbf{O}_n^{\langle 7\rangle})\end{matrix}\right\}.$$

## 3. Further series with harmonic numbers of higher orders

Chu and Zhang [11, Theorem 9] give another transformation analogous to (2.1). For $\{a,b,c,d,e\}\in\mathbb{C}$ subject to $\mathrm{Re}(1+2a-b-c-d-e)>0$,

$$\sum_{k=0}^{\infty}\frac{(a+2k)\,(b)_k(c)_k(d)_k(e)_k}{(1+a-b)_k(1+a-c)_k(1+a-d)_k(1+a-e)_k}$$
$$=\sum_{n=0}^{\infty}(-1)^n\frac{\mathcal{V}_n(a,b,c,d,e)}{(1+a-e)_{2n+1}}$$
$$\times\frac{(b)_n(c)_n(d)_n(1+a-b-e)_n(1+a-c-e)_n(1+a-d-e)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(1+2a-b-c-d-e)_{n+1}},$$

where the cubic polynomial $\mathcal{V}_n(a,b,c,d,e)$ is given by

$$\mathcal{V}_n(a,b,c,d,e)=(a-b+n)(1+a-e+2n)(1+2a-c-d-e+2n)$$
$$+(b+n)(1+a-c-e+n)(1+a-d-e+n).$$

Letting $e=a$ in this equality and assuming the condition $\mathrm{Re}(1+a-b-c-d)>0$, we derive, again by making use of Dougall's summation theorem for the well-poised

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${}_5F_4$-series [1, Section 4.4], the following summation formula:

$$\Gamma\left[\begin{matrix}1+a-b, 1+a-c, 1+a-d, 1+a-b-c-d\\ a, 1+a-b-c, 1+a-b-d, 1+a-c-d\end{matrix}\right]$$
$$= \sum_{n=0}^{\infty}(-1)^n \frac{(b)_n(c)_n(d)_n \mathrm{V}_n(a,b,c,d)}{(2n+1)!\,(1+a-b-c-d)_{n+1}}$$
$$\times \frac{(1-b)_n(1-c)_n(1-d)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n}, \tag{3.1}$$

where the cubic polynomial $\mathrm{V}_n(a,b,c,d)$ is given by

$$\mathrm{V}_n(a,b,c,d) = (1+2n)(a-b+n)(1+a-c-d+2n) + (b+n)(1-c+n)(1-d+n).$$

As with (2.2), the parameter restriction $\mathrm{Re}(1+a-b-c-d) > 0$ can be removed by analytic continuation. The formula is valid in the whole complex space of dimension 4 except for the hyperplanes determined by

$$\{b-a\in\mathbb{N}\}\cup\{c-a\in\mathbb{N}\}\cup\{d-a\in\mathbb{N}\}\cup\{b+c+d-a\in\mathbb{N}\}.$$

**3.1. Quintic central binomial coefficients in numerators.** Performing the parameter replacements

$$a=\tfrac{1}{2}+ax,\quad b=\tfrac{1}{2}+bx,\quad c=\tfrac{1}{2}+cx,\quad d=\tfrac{1}{2}+dx,$$

the summation formula in (3.1) becomes

$$\Gamma\left[\begin{matrix}1+ax-bx, 1+ax-cx, 1+ax-dx, 1+ax-bx-cx-dx\\ \frac{1}{2}+ax, \frac{1}{2}+ax-bx-cx, \frac{1}{2}+ax-bx-dx, \frac{1}{2}+ax-cx-dx\end{matrix}\right]$$
$$= \sum_{n=0}^{\infty}(-1)^n \frac{(\frac{1}{2}+bx)_n(\frac{1}{2}+cx)_n(\frac{1}{2}+dx)_n}{(2n+1)!\,(1+ax-bx-cx-dx)_n}\mathrm{P}_n(a,b,c,d;x)$$
$$\times \frac{(\frac{1}{2}-bx)_n(\frac{1}{2}-cx)_n(\frac{1}{2}-dx)_n}{(1+ax-bx)_n(1+ax-cx)_n(1+ax-dx)_n},$$

where the cubic polynomial $\mathrm{P}_n(a,b,c,d;x)$ is given by

$$\mathrm{P}_n(a,b,c,d;x) = (1+2n)(ax-bx+n)(\tfrac{1}{2}+ax-cx-dx+2n) + (\tfrac{1}{2}+bx+n)(\tfrac{1}{2}-cx+n)(\tfrac{1}{2}-dx+n).$$

By expanding both sides of the above equation into Maclaurin series and then comparing the coefficients $\mathrm{C}_m(a,b,c,d)$ of $x^m$, we can deduce several infinite series identities. Ten representative examples are recorded as follows.

- $\boxed{\mathrm{C}_2(0,1,-1,0)}$ Conjectured by Sun [20, (4.19)]

$$\frac{8}{3} = \sum_{n=0}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\Big\{4-(1+8n+20n^2)(\mathbf{H}_n^{\langle 2\rangle}-8\mathbf{O}_n^{\langle 2\rangle})\Big\}. \tag{3.2}$$

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- $\boxed{\mathrm{C}_3(2,2,1,1)}$

$$\frac{64\zeta(3)}{\pi^2}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\left\{\frac{8}{1+2n}+(1+8n+20n^2)\mathbf{H}_n^{\langle 3\rangle}\right\}. \tag{3.3}$$

- $\boxed{\mathrm{C}_4(1+\mathbf{i},2,0,2\mathbf{i})}$

$$-\frac{56\pi^2}{45}=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\left\{(1+8n+20n^2)(\mathbf{H}_n^{\langle 4\rangle}+64\mathbf{O}_n^{\langle 4\rangle})\right\}. \tag{3.4}$$

- $\boxed{\mathrm{C}_5(2,2,1,1)}$

$$\frac{256\zeta(5)}{\pi^2}-\frac{64\zeta(3)}{3}=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\left\{\begin{matrix}8(\mathbf{H}_n^{\langle 2\rangle}-8\mathbf{O}_n^{\langle 2\rangle})/(1+2n)-4\mathbf{H}_n^{\langle 3\rangle}\\ +(1+8n+20n^2)(\mathbf{H}_n^{\langle 5\rangle}+\mathbf{H}_n^{\langle 2\rangle}\mathbf{H}_n^{\langle 3\rangle}-8\mathbf{H}_n^{\langle 3\rangle}\mathbf{O}_n^{\langle 2\rangle})\end{matrix}\right\}.$$

- $\boxed{\mathrm{C}_6(1,4,-1+3\mathbf{i},-1-3\mathbf{i})}$

$$\frac{512\zeta^2(3)}{\pi^2}-\frac{496\pi^4}{945}=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\left\{\frac{16\mathbf{H}_n^{\langle 3\rangle}}{1+2n}+(1+8n+20n^2)[\mathbf{H}_n^{\langle 6\rangle}+(\mathbf{H}_n^{\langle 3\rangle})^2-128\mathbf{O}_n^{\langle 6\rangle}]\right\}.$$

- $\boxed{\mathrm{C}_7(1,4,-1+3\mathbf{i},-1-3\mathbf{i})}$

$$\begin{aligned}&\frac{1024\zeta(7)}{\pi^2}-\frac{448\pi^2\zeta(3)}{45}\\ &\quad=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\left\{\frac{8(\mathbf{H}_n^{\langle 4\rangle}+64\mathbf{O}_n^{\langle 4\rangle})}{1+2n}+(1+8n+20n^2)[\mathbf{H}_n^{\langle 3\rangle}(\mathbf{H}_n^{\langle 4\rangle}+64\mathbf{O}_n^{\langle 4\rangle})+\mathbf{H}_n^{\langle 7\rangle}]\right\}.\end{aligned}$$

- $\boxed{\mathrm{C}_8(1+\mathbf{i},2,0,2\mathbf{i})}$

$$-\frac{304\pi^6}{14175}=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\left\{(1+8n+20n^2)[(\mathbf{H}_n^{\langle 4\rangle}+64\mathbf{O}_n^{\langle 4\rangle})^2+\mathbf{H}_n^{\langle 8\rangle}-4096\mathbf{O}_n^{\langle 8\rangle}]\right\}.$$

- $\boxed{\mathrm{C}_9(1,4,-1+3\mathbf{i},-1-3\mathbf{i})}$

$$\begin{aligned}\frac{8192\zeta(9)}{\pi^2}+\frac{4096\zeta^3(3)}{\pi^2}-\frac{3968\pi^4\zeta(3)}{315}&=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^5}{(-2^{12})^n}\left\{\frac{24[\mathbf{H}_n^{\langle 6\rangle}+(\mathbf{H}_n^{\langle 3\rangle})^2-128\mathbf{O}_n^{\langle 6\rangle}]}{1+2n}\right.\\ &\left.+(1+8n+20n^2)[2\mathbf{H}_n^{\langle 9\rangle}+(\mathbf{H}_n^{\langle 3\rangle})^3+3\mathbf{H}_n^{\langle 3\rangle}\mathbf{H}_n^{\langle 6\rangle}-384\mathbf{H}_n^{\langle 3\rangle}\mathbf{O}_n^{\langle 6\rangle}]\right\}.\end{aligned}$$

- $\boxed{\mathrm{C}_{12}(1+\mathbf{i},2,0,2\mathbf{i})}$

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$$-\frac{46496\pi^{10}}{70945875}=\sum_{n=1}^{\infty}\binom{2n}{n}^5\frac{1+8n+20n^2}{(-2^{12})^n}\left\{\begin{matrix}(\mathbf{H}_n^{\langle 4\rangle}+64\mathbf{O}_n^{\langle 4\rangle})^3+2(\mathbf{H}_n^{\langle 12\rangle}+262144\mathbf{O}_n^{\langle 12\rangle})\\ +3(\mathbf{H}_n^{\langle 4\rangle}+64\mathbf{O}_n^{\langle 4\rangle})(\mathbf{H}_n^{\langle 8\rangle}-4096\mathbf{O}_n^{\langle 8\rangle})\end{matrix}\right\}.$$

- $\boxed{\mathrm{C}_{20}(1+\mathbf{i},2,0,2\mathbf{i})}$

$$-\frac{986692736\pi^{18}}{714620417135625}=\sum_{n=1}^{\infty}\binom{2n}{n}^5\frac{1+8n+20n^2}{(-2^{12})^n}\mathrm{W}_{20}(n),$$

where

$$\begin{aligned}\mathrm{W}_{20}(n)=&(\mathbf{H}_n^{\langle 4\rangle}+64\,\mathbf{O}_n^{\langle 4\rangle})^5+10\,(\mathbf{H}_n^{\langle 4\rangle}+64\,\mathbf{O}_n^{\langle 4\rangle})^3(\mathbf{H}_n^{\langle 8\rangle}-4096\,\mathbf{O}_n^{\langle 8\rangle})\\ &+24\,(\mathbf{H}_n^{\langle 20\rangle}+1073741824\,\mathbf{O}_n^{\langle 20\rangle})+15\,(\mathbf{H}_n^{\langle 4\rangle}+64\mathbf{O}_n^{\langle 4\rangle})\,(\mathbf{H}_n^{\langle 8\rangle}-4096\mathbf{O}_n^{\langle 8\rangle})^2\\ &+30\,(\mathbf{H}_n^{\langle 4\rangle}+64\,\mathbf{O}_n^{\langle 4\rangle})\,(\mathbf{H}_n^{\langle 16\rangle}-16777216\mathbf{O}_n^{\langle 16\rangle})+20(\mathbf{H}_n^{\langle 8\rangle}-4096\mathbf{O}_n^{\langle 8\rangle})\\ &\times(\mathbf{H}_n^{\langle 12\rangle}+262144\mathbf{O}_n^{\langle 12\rangle})+20(\mathbf{H}_n^{\langle 4\rangle}+64\mathbf{O}_n^{\langle 4\rangle})^2(\mathbf{H}_n^{\langle 12\rangle}+262144\mathbf{O}_n^{\langle 12\rangle}).\end{aligned}$$

**3.2. Quintic central binomial coefficients in denominators.** Alternatively, under the parameter setting

$$a=\tfrac{3}{2}+ax,\quad b=1+bx,\quad c=1+cx,\quad d=1+dx,$$

we derive from (3.1) the transformation formula

$$\begin{aligned}&\Gamma\left[\begin{matrix}\frac{1}{2}+ax-bx,\frac{1}{2}+ax-cx,\frac{1}{2}+ax-dx,\frac{1}{2}+ax-bx-cx-dx\\ \frac{3}{2}+ax,\frac{1}{2}+ax-bx-cx,\frac{1}{2}+ax-bx-dx,\frac{1}{2}+ax-cx-dx\end{matrix}\right]\\ &=\sum_{n=0}^{\infty}(-1)^n\frac{\mathrm{Q}_n(a,b,c,d;x)\times(-bx)_n(-cx)_n(-dx)_n}{(2n+1)!\,(\frac{1}{2}+ax-bx-cx-dx)_n}\\ &\quad\times\frac{(1+bx)_n(1+cx)_n(1+dx)_n}{(\frac{1}{2}+ax-bx)_{n+1}(\frac{1}{2}+ax-cx)_{n+1}(\frac{1}{2}+ax-dx)_{n+1}},\end{aligned}\tag{3.5}$$

where the cubic polynomial $\mathrm{Q}_n(a,b,c,d;x)$ is given by

$$\begin{aligned}\mathrm{Q}_n(a,b,c,d;x)=&(1+2n)(\tfrac{1}{2}+ax-bx+n)(\tfrac{1}{2}+ax-cx-dx+2n)\\ &+(1+bx+n)(n-cx)(n-dx).\end{aligned}$$

According to this transformation formula, numerous infinite series identities can be obtained by extracting the coefficients $\mathrm{D}_m(a,b,c,d)$ of $x^m$ across the equation. We are going to highlight seventeen remarkable ones below. Among them, the first four identities are derived from (3.5) after multiplying across by $\frac{1}{2}+ax$. The remaining ones are directly obtained from (3.5) by comparing coefficients.

- $\boxed{\mathrm{D}_4(\mathbf{i},2,-2,2\mathbf{i})}$

$$-8=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{\binom{2n}{n}^5}\left\{\frac{1+10n+40n^2+80n^3+88n^4+40n^5}{n^4(2n+1)^5}\right\}.$$

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- $\boxed{\mathrm{D}_5(2,2,1+\sqrt{3}\mathbf{i},1-\sqrt{3}\mathbf{i})}$

$$-8=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{\binom{2n}{n}^5}\Big\{\frac{1+10n+40n^2+80n^3+80n^4+40n^5}{n^5(2n+1)^5}\Big\}.$$

- $\boxed{\mathrm{D}_4(1,1,1,-2)}$

$$\frac{\pi^4}{8}-8=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{\binom{2n}{n}^5}\Big\{\frac{8+10n}{(2n+1)^5}-\frac{(1+8n+24n^2+20n^3)\mathbf{O}_n}{n^3(2n+1)^4}\Big\}.$$

- $\boxed{\mathrm{D}_5(\mathbf{i},2,-2,2\mathbf{i})}$

$$31\zeta(5)-28=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{\binom{2n}{n}^5}\Big\{\frac{4(7+10n)}{(2n+1)^6}+\frac{(1+8n+24n^2+20n^3)(\mathbf{H}_n^{\langle 2\rangle}-2\mathbf{O}_n^{\langle 2\rangle})}{n^3(2n+1)^4}\Big\}.$$

---

- $\boxed{\mathrm{D}_8(2,2,1+\sqrt{3}\mathbf{i},1-\sqrt{3}\mathbf{i})}$

$$14\zeta(3)\{8-7\zeta(3)\}=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{\binom{2n}{n}^5}\frac{1+4n(1+6n+10n^2)\mathbf{O}_{n+1}^{\langle 3\rangle}}{n^6(2n+1)^4}. \tag{3.6}$$

- $\boxed{\mathrm{D}_3(1+\mathbf{i},2,0,2\mathbf{i})}$

$$12-14\zeta(3)=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{\binom{2n}{n}^5}\frac{1+8n+24n^2+20n^3}{n^3(2n+1)^4}. \tag{3.7}$$

- $\boxed{\mathrm{D}_4(1+\mathbf{i},2,0,2\mathbf{i})}$

$$14\zeta(3)-20=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{\binom{2n}{n}^5}\frac{1+9n+30n^2+40n^3+20n^4}{n^4(2n+1)^5}. \tag{3.8}$$

- $\boxed{\mathrm{D}_5(1+\mathbf{i},2,0,2\mathbf{i})=\mathrm{D}_6(1+\mathbf{i},2,0,2\mathbf{i})}$

$$16-14\zeta(3)=\sum_{n=1}^{\infty}\frac{(-2^8)^n}{\binom{2n}{n}^5}\frac{1+6n+10n^2}{2n^5(2n+1)^4}. \tag{3.9}$$

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- $\boxed{\mathrm{D}_7(1, 1+\mathbf{i}, 0, 1-\mathbf{i})}$

$$127\zeta(7) + 112\zeta(3) - 224 = 2\sum_{n=1}^{\infty} \frac{(-2^8)^n}{\binom{2n}{n}^5} \left\{ \frac{1+6n+10n^2}{n^7(2n+1)^3} - \frac{(1+8n+24n^2+20n^3)(\mathbf{H}_n^{\langle 4\rangle} + 4\mathbf{O}_{n+1}^{\langle 4\rangle})}{n^3(2n+1)^4} \right\}. \tag{3.10}$$

- $\boxed{\mathrm{D}_8(1, 1+\mathbf{i}, 0, 1-\mathbf{i})}$

$$288 - 112\zeta(3) - 127\zeta(7) = 2\sum_{n=1}^{\infty} \frac{(-2^8)^n}{\binom{2n}{n}^5} \left\{ \frac{(1+n)(1+6n+10n^2)}{n^8(2n+1)^4} - \frac{(1+9n+30n^2+40n^3+20n^4)(\mathbf{H}_n^{\langle 4\rangle} + 4\mathbf{O}_{n+1}^{\langle 4\rangle})}{n^4(2n+1)^5} \right\}. \tag{3.11}$$

- $\boxed{\mathrm{D}_9(1, 1+\mathbf{i}, 0, 1-\mathbf{i}) = \mathrm{D}_{10}(1, 1+\mathbf{i}, 0, 1-\mathbf{i})}$

$$127\zeta(7) + 112\zeta(3) - 256 = \sum_{n=1}^{\infty} \frac{(-2^8)^n}{\binom{2n}{n}^5} \left\{ \frac{(1+6n+10n^2)(1 - n^4\mathbf{H}_n^{\langle 4\rangle} - 4n^4\mathbf{O}_{n+1}^{\langle 4\rangle})}{n^9(2n+1)^4} \right\}. \tag{3.12}$$

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- $\boxed{\mathrm{D}_{11}(1+\mathbf{i}, 2, 0, 2\mathbf{i})}$

$$5632 - 1792\zeta(3) - 2032\zeta(7) - 2047\zeta(11) = 4\sum_{n=1}^{\infty} \frac{(-2^8)^n}{\binom{2n}{n}^5} \left\{ \begin{array}{l} \frac{2(1+6n+10n^2)}{n^{11}(2n+1)^3}[1 - n^4(\mathbf{H}_n^{\langle 4\rangle} + 4\mathbf{O}_{n+1}^{\langle 4\rangle})] \\ + \frac{1+8n+24n^2+20n^3}{n^3(2n+1)^4} \left[ \begin{array}{l} (\mathbf{H}_n^{\langle 4\rangle} + 4\mathbf{O}_{n+1}^{\langle 4\rangle})^2 \\ -\mathbf{H}_n^{\langle 8\rangle} + 16\mathbf{O}_{n+1}^{\langle 8\rangle} \end{array} \right] \end{array} \right\}. \tag{3.13}$$

- $\boxed{\mathrm{D}_{12}(1+\mathbf{i}, 2, 0, 2\mathbf{i})}$

$$1792\zeta(3) + 2032\zeta(7) + 2047\zeta(11) - 6656 = 4\sum_{n=1}^{\infty} \frac{(-2^8)^n}{\binom{2n}{n}^5} \left\{ \begin{array}{l} \frac{2(1+n)(1+6n+10n^2)}{n^{12}(2n+1)^4}[1 - n^4(\mathbf{H}_n^{\langle 4\rangle} + 4\mathbf{O}_{n+1}^{\langle 4\rangle})] \\ + \frac{1+9n+30n^2+40n^3+20n^4}{n^4(2n+1)^5} \left[ \begin{array}{l} (\mathbf{H}_n^{\langle 4\rangle} + 4\mathbf{O}_{n+1}^{\langle 4\rangle})^2 \\ -\mathbf{H}_n^{\langle 8\rangle} + 16\mathbf{O}_{n+1}^{\langle 8\rangle} \end{array} \right] \end{array} \right\}. \tag{3.14}$$

- $\boxed{\mathrm{D}_{13}(1+\mathbf{i}, 2, 0, 2\mathbf{i}) = \mathrm{D}_{14}(1+\mathbf{i}, 2, 0, 2\mathbf{i})}$

$$6144 - 1792\zeta(3) - 2032\zeta(7) - 2047\zeta(11) = 2\sum_{n=1}^{\infty} \frac{(-2^8)^n}{\binom{2n}{n}^5} \frac{1+6n+10n^2}{n^{13}(2n+1)^4} \left\{ \begin{array}{l} (1 - n^4\mathbf{H}_n^{\langle 4\rangle} - 4n^4\mathbf{O}_{n+1}^{\langle 4\rangle})^2 \\ +1 - n^8\mathbf{H}_n^{\langle 8\rangle} + 16n^8\mathbf{O}_{n+1}^{\langle 8\rangle} \end{array} \right\}. \tag{3.15}$$

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- $\boxed{\mathrm{D}_{15}(1, 1+\mathbf{i}, 0, 1-\mathbf{i})}$

$$\begin{aligned}&28672\zeta(3)+32512\zeta(7)+32752\zeta(11)+32767\zeta(15)-122880\\&=\frac{16}{3}\sum_{n=1}^{\infty}\frac{(-2^8)^n}{\binom{2n}{n}^5}\frac{1+6n+10n^2}{n^{15}(2n+1)^3}\left\{\begin{bmatrix}6(1-n^4\mathbf{H}_n^{\langle 4\rangle}-4n^4\mathbf{O}_{n+1}^{\langle 4\rangle})-3n^8\mathbf{H}_n^{\langle 8\rangle}\\+3n^8(\mathbf{H}_n^{\langle 4\rangle}+4\mathbf{O}_{n+1}^{\langle 4\rangle})^2+48n^8\mathbf{O}_{n+1}^{\langle 8\rangle}\end{bmatrix}\right.\\&\quad\left.-\frac{n^{12}(1+8n+24n^2+20n^3)}{(1+2n)(1+6n+10n^2)}\begin{bmatrix}(\mathbf{H}_n^{\langle 4\rangle}+4\mathbf{O}_{n+1}^{\langle 4\rangle})^3+2\mathbf{H}_n^{\langle 12\rangle}+128\mathbf{O}_{n+1}^{\langle 12\rangle}\\-3(\mathbf{H}_n^{\langle 4\rangle}+4\mathbf{O}_{n+1}^{\langle 4\rangle})(\mathbf{H}_n^{\langle 8\rangle}-16\mathbf{O}_{n+1}^{\langle 8\rangle})\end{bmatrix}\right\}.\end{aligned}\tag{3.16}$$

- $\boxed{\mathrm{D}_{16}(1, 1+\mathbf{i}, 0, 1-\mathbf{i})}$

$$\begin{aligned}&139264-28672\zeta(3)-32512\zeta(7)-32752\zeta(11)-32767\zeta(15)\\&=\frac{16}{3}\sum_{n=1}^{\infty}\frac{(-2^8)^n}{\binom{2n}{n}^5}\left\{\frac{(1+n)(1+6n+10n^2)}{n^{16}(2n+1)^4}\begin{bmatrix}6(1-n^4\mathbf{H}_n^{\langle 4\rangle}-4n^4\mathbf{O}_{n+1}^{\langle 4\rangle})-3n^8\mathbf{H}_n^{\langle 8\rangle}\\+3n^8(\mathbf{H}_n^{\langle 4\rangle}+4\mathbf{O}_{n+1}^{\langle 4\rangle})^2+48n^8\mathbf{O}_{n+1}^{\langle 8\rangle}\end{bmatrix}\right.\\&\quad\left.-\frac{(1+9n+30n^2+40n^3+20n^4)}{n^4(1+2n)^5}\begin{bmatrix}(\mathbf{H}_n^{\langle 4\rangle}+4\mathbf{O}_{n+1}^{\langle 4\rangle})^3+2\mathbf{H}_n^{\langle 12\rangle}+128\mathbf{O}_{n+1}^{\langle 12\rangle}\\-3(\mathbf{H}_n^{\langle 4\rangle}+4\mathbf{O}_{n+1}^{\langle 4\rangle})(\mathbf{H}_n^{\langle 8\rangle}-16\mathbf{O}_{n+1}^{\langle 8\rangle})\end{bmatrix}\right\}.\end{aligned}\tag{3.17}$$

- $\boxed{\mathrm{D}_{17}(1, 1+\mathbf{i}, 0, 1-\mathbf{i})=\mathrm{D}_{18}(1, 1+\mathbf{i}, 0, 1-\mathbf{i})}$

$$\begin{aligned}&28672\zeta(3)+32512\zeta(7)+32752\zeta(11)+32767\zeta(15)-131072\\&=\frac{8}{3}\sum_{n=1}^{\infty}\frac{(-2^8)^n}{\binom{2n}{n}^5}\frac{1+6n+10n^2}{n^{17}(2n+1)^4}\left\{\begin{bmatrix}6(1-n^4\mathbf{H}_n^{\langle 4\rangle}-4n^4\mathbf{O}_{n+1}^{\langle 4\rangle})-3n^8\mathbf{H}_n^{\langle 8\rangle}\\+3n^8(\mathbf{H}_n^{\langle 4\rangle}+4\mathbf{O}_{n+1}^{\langle 4\rangle})^2+48n^8\mathbf{O}_{n+1}^{\langle 8\rangle}\end{bmatrix}\right.\\&\quad\left.-n^{12}\begin{bmatrix}(\mathbf{H}_n^{\langle 4\rangle}+4\mathbf{O}_{n+1}^{\langle 4\rangle})^3+2\mathbf{H}_n^{\langle 12\rangle}+128\mathbf{O}_{n+1}^{\langle 12\rangle}\\-3(\mathbf{H}_n^{\langle 4\rangle}+4\mathbf{O}_{n+1}^{\langle 4\rangle})(\mathbf{H}_n^{\langle 8\rangle}-16\mathbf{O}_{n+1}^{\langle 8\rangle})\end{bmatrix}\right\}.\end{aligned}\tag{3.18}$$

Observing these series carefully, we find a curious fact. Among the four groups

$$(3.7)\approx(3.8)\approx(3.9),\qquad(3.10)\approx(3.11)\approx(3.12),$$
$$(3.13)\approx(3.14)\approx(3.15),\qquad(3.16)\approx(3.17)\approx(3.18),$$

the three series in each group have almost the same value apart from integer differences.

We uncovered an explanation of this phenomenon by examining (3.5). Switching the initial term with $n=0$ to the left-hand side, dividing across by '$-c$' and then letting

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$c \to 0$, we can write, after some simplifications, the resulting transformation as

$$\frac{8bdx^3(3+2ax)}{(1+2ax)^2(1+2ax-2bx)(1+2ax-2dx)} - \frac{2x}{1+2ax}\Psi(x)$$
$$= bdx^3 \sum_{n=1}^{\infty} \frac{(-1)^n \mathrm{W}_n(a,b,d;x) \times n!\,(n-1)!\,(1+bx)_n(1-bx)_{n-1}(1+dx)_n(1-dx)_{n-1}}{(2n+1)!\,(\frac{1}{2}+ax-bx-dx)_n(\frac{1}{2}+ax)_{n+1}(\frac{1}{2}+ax-bx)_{n+1}(\frac{1}{2}+ax-dx)_{n+1}}, \tag{3.19}$$

where

$$\Psi(x) = \psi(\tfrac{1}{2}+ax-bx) + \psi(\tfrac{1}{2}+ax-dx) - \psi(\tfrac{1}{2}+ax) - \psi(\tfrac{1}{2}+ax-bx-dx),$$

the reduced cubic polynomial $\mathrm{W}_n(a,b,d;x)$ is

$$\mathrm{W}_n(a,b,d;x) = n(1+bx+n)(n-dx) + (1+2n)(\tfrac{1}{2}+ax-bx+n)(\tfrac{1}{2}+ax-dx+2n)$$

and the digamma function [18, Section 9] is defined by

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \sum_{k=0}^{\infty}\frac{x-1}{(k+1)(k+x)}.$$

Now it is trivial to check that

$$[x^0]\{\psi(\tfrac{1}{2}+x\lambda) - \psi(\tfrac{1}{2}+x\mu)\} = 0.$$

For $m > 0$, we can determine the coefficient

$$[x^m]\{\psi(\tfrac{1}{2}+x\lambda) - \psi(\tfrac{1}{2}+x\mu)\} = [x^m]\sum_{k=0}^{\infty}\Big\{\frac{1}{\frac{1}{2}+k+x\mu} - \frac{1}{\frac{1}{2}+k+x\lambda}\Big\}$$
$$= \sum_{k=0}^{\infty}\frac{(-2)^{m+1}(\lambda^m-\mu^m)}{(1+2k)^{m+1}} = (-1)^m(\lambda^m-\mu^m)(1-2^{m+1})\zeta(m+1).$$

Then, apart from a rational constant, the coefficient of $[x^m]$ in (3.19) is

$$\Lambda_m(a,b,d) := [x^m]\Big\{\frac{-2x}{1+2ax}\Psi(x)\Big\} = \sum_{k=0}^{m-1}\frac{(-2a)^{m-k}}{a}[x^k]\Psi(x).$$

This leads to the explicit formula

$$\Lambda_m(a,b,d) = \frac{(-1)^m}{a}\sum_{k=1}^{m-1}(2a)^{m-k}(2^{k+1}-1)\Delta_k(a,b,d)\zeta(k+1),$$

where for the sake of brevity, we introduce the symbol

$$\Delta_k(a,b,d) := a^k - (a-b)^k - (a-d)^k + (a-b-d)^k.$$

In particular, it is routine to check that

$$\Delta_k(1, 1+\mathbf{i}, 1-\mathbf{i}) = 1^k + (-1)^k - \mathbf{i}^k - (-\mathbf{i})^k$$
$$= 2\chi(k-\text{even}) - 2\mathbf{i}^k\chi(k-\text{even}) = 4\chi(k \equiv_4 2),$$

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where $\chi$ stands for the logical function with $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$, and $i \equiv_4 j$ means '$i$ is congruent to $j$ modulo 4' for $i, j \in \mathbb{Z}$.

Then for $m \in \mathbb{N}$ and $\ell = 0, \pm 1, 2$, we can write explicitly

$$\Lambda_{4m+\ell}(1, 1+\mathbf{i}, 1-\mathbf{i}) = (-1)^{\ell} \sum_{k=1}^{4m+\ell-1} 2^{4m-k+\ell}(2^{k+1}-1)\Delta_k(1, 1+\mathbf{i}, 1-\mathbf{i})\zeta(k+1).$$

Making the replacement $k \to 4k-2$, we derive the simplified formula

$$\Lambda_{4m+\ell}(1, 1+\mathbf{i}, 1-\mathbf{i}) = 2^{4+4m+\ell}(-1)^{\ell} \sum_{k=1}^{m} \frac{2^{4k-1}-1}{2^{4k}}\zeta(4k-1).$$

Since the above sum is independent of $\ell$, we conclude that for any fixed $m \in \mathbb{N}$, the four linear sums (corresponding to $\ell = 0, \pm 1, 2$) of zeta values for $\mathrm{D}_{4m+\ell}(1, 1+\mathbf{i}, 0, 1-\mathbf{i})$ are proportional, despite complexities of the sums on another side involving combined products of Bell polynomials of harmonic numbers. Analogously, we assert that the same statement is also true for $\mathrm{D}_{4m+\ell}(1+\mathbf{i}, 2, 0, 2\mathbf{i})$, which can be justified by the following particular values:

$$\begin{aligned}\Delta_k(1+\mathbf{i}, 2, 2\mathbf{i}) &= (1+\mathbf{i})^k + (-1-\mathbf{i})^k - (\mathbf{i}-1)^k - (1-\mathbf{i})^k \\ &= 2(1+\mathbf{i})^k\chi(k-\text{even}) - 2(1-\mathbf{i})^k\chi(k-\text{even}) = 4(2\mathbf{i})^{\frac{k}{2}}\chi(k \equiv_4 2).\end{aligned}$$

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