THE VANISHING PROBLEM OF THE STRING CLASS WITH DEGREE 3

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Abstract

Let ξ be an $SO(n)$-bundle over a simply connected manifold M with a spin structure $Q \to M$. The string class is an obstruction to lift the structure group $LSpin(n)$ of the loop group bundle $LQ \to LM$ to the universal central extension of $LSpin(n)$ by the circle. We prove that the string class vanishes if and only if $1/2$ the first Pontrjagin class of ξ vanishes when M is a compact simply connected homogeneous space of rank one, a simply connected 4-dimensional manifold or a finite product space of those manifolds. This result is deduced by using the Eilenberg–Moore spectral sequence converging to the mod p cohomology of LM whose E_2-term is the Hochschild homology of the mod p cohomology algebra of M. The key to the consideration is existence of a morphism of algebras, which is injective below degree 3, from an important graded commutative algebra into the Hochschild homology of a certain graded commutative algebra.

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1. Introduction

Let X be a simply connected space and LX the space of continuous closed paths on X. If M is a simply connected manifold, then we regard LM as the space of smooth free loops on M. Throughout this paper, the map $\int_{S^1} \circ ev^* : H^*(X; \mathbf{Z}) \to H^{*-1}(LX; \mathbf{Z})$ will be denoted by $\mathscr{D}_X$ and called the *D-map of X*, where $\int_{S^1} : H^*(S^1 \times LX; \mathbf{Z}) \to H^{*-1}(LX; \mathbf{Z})$ is the integration map along S^1 and $ev : S^1 \times LX \to X$ is the evaluation map. We say the *D*-map $\mathscr{D}_X$ is *good* if $\mathscr{D}_X : H^4(X; \mathbf{Z}) \to H^3(LX; \mathbf{Z})$ is a monomorphism.

Let ξ be an $SO(n)$-bundle over a simply connected manifold M with a spin structure $Q \to M$. In [8], McLaughlin defined the string class $\mu(Q)$ in $H^3(LX; \mathbf{Z})$ which

is an obstruction to lift the structure group of the $LSpin(n)$-bundle $LQ \to LM$ to $\widehat{LSpin(n)}$, where $\mathbf{T} \to \widehat{LSpin(n)} \to LSpin(n)$ is the universal central extension of $LSpin(n)$ by the circle and $n \geq 5$. It is asserted in [8, Lemma 2.2] that the first Pontrjagin class $p_1(\xi)$ is two times the pullback of the generator ι of $H^4(BSpin(n); \mathbf{Z})$ by the classifying map of a spin structure $Q \to M$ for ξ. Following [8], we denote the pullback of ι by $\frac{1}{2}p_1(\xi)$. The argument of the proof of [8, Theorem 3.1] enables us to conclude that the D-map $\mathscr{D}_M : H^4(M; \mathbf{Z}) \to H^3(LX; \mathbf{Z})$ carries $\frac{1}{2}p_1(\xi)$ to the string class $\mu(Q)$. Therefore if the D-map of M is good, then $\mu(Q)$ vanishes if and only if $\frac{1}{2}p_1(\xi)$ vanishes for any $SO(n)$-bundle ξ with a spin structure $Q \to M$. In this case we can deduce that the $LSpin(n)$-bundle over the infinite dimensional manifold LM has a string structure if and only if $1/2$ the first Pontrjagin class of the $SO(n)$-bundle over the finite dimensional manifold M vanishes. Our goal is to study which manifolds have a good D-map. In [8, Theorem 3.1], it has been proved that every 2-connected manifold has a good D-map. Recently, Kuribayashi has proved

THEOREM A ([4, Theorem 1]). *Let M be a simply connected manifold. If $H^4(M; \mathbf{Z})$ is torsion free and $\dim H^2(M; \mathbf{R}) \leq 1$. Then the D-map of M is good. Therefore, in this case, $\frac{1}{2}p_1(\xi)$ vanishes if the string class $\mu(Q)$ vanishes.*

We can deduce from Theorem A that the complex Grassmann manifold has a good D-map.

In this paper, the following theorem will be proved. As a consequence we can obtain many manifolds whose D-maps are good.

THEOREM 1.1. *Let X be a simply connected space. Suppose that*

(1.1) $H^4(X; \mathbf{Z}) \cong \mathbf{Z} \oplus \cdots \oplus \mathbf{Z} \oplus \mathbf{Z}/p_1 \oplus \cdots \mathbf{Z}/p_k$, *where p_i is prime for any i, and*

(1.2) $x^2 = 0$ *for any element $x \in H^2(X; \mathbf{Z}/2)$ if $H^4(X; \mathbf{Z})$ has 2-torsion.*

Then the D-map of X is good.

Theorem 1.1 is a generalization of Theorem A. Applying Theorem 1.1 to simply connected 4-manifolds and compact, simply connected homogeneous spaces of rank one, we have

THEOREM 1.2. *Let M be a simply connected 4-dimensional manifold, compact simply connected homogeneous spaces of rank one or a finite product of those manifolds. Then the D-map of M is good. Therefore the string class $\mu(Q)$ vanishes if and only if $\frac{1}{2}p_1(\xi)$ vanishes.*

The classification of compact, simply connected, homogeneous space of rank one has been made by Oniscik [9]. In [7], McCleary and Ziller have determined the mod p cohomology of the homogeneous spaces completely for any prime p. These results are also used to prove Theorem 1.2. Following [7, p. 767], we now list such homogeneous spaces which are not diffeomorphic to spheres or projective spaces:

(1) $(SO(2n + 1), SO(2n − 1) \times SO(2), 1)$,   (2) $(SO(2n + 1), SO(2n − 1), 1)$,
(3) $(SU(3), SO(3), 4)$,   (4) $(Sp(2), SU(2), 10)$,
(5) $(G_2, SO(4), (1, 3))$,   (6) $(G_2, U(2), 3)$,
(7) $(G_2, SU(2), 3)$,   (8) $(G_2, SO(3), 4)$,
(9) $(G_2, SO(3), 28)$,

where the triple (G, H, i) consisting of the Lie group G, the subgroup H and the integer or a pair of integers means the homogeneous space G/H of G by the subgroup H with the index i. Here the index of the subgroup H of G is that of *the subalgebra Lie(H) of the Lie algebra Lie(G) of G* in the sense of Dynkin [2]. If a Lie group G has n simple factors, then $\pi_3(G)$ is isomorphic to a free abelian group of rank n, that is, $\pi_3(G) = \oplus^n \mathbf{Z}$. Therefore in the above cases we can regard $j_* : \pi_3(H) \to \pi_3(G)$ as multiplication by an integer n or a pair of integers (n, m) associated to the inclusion $j : H \to G$. The assertion of [9, Lemma 4] guarantees that the index of the subgroup H of G can be interpreted as the above integer or pair of integers determined by the inclusion j.

In order to prove Theorem 1.1, we need to consider the injectivity of the D-map $\mathscr{D}_X : H^4(X; \mathbf{Z}) \to H^3(LX; \mathbf{Z})$. To this end, we study the algebra structure of $H^*(LX; \mathbf{Z}/p)$ and the injectivity of the mod p D-map $\mathscr{D}_{X,p} = \int_{S^1} \circ ev^* : H^4(X; \mathbf{Z}/p) \to H^3(LX; \mathbf{Z}/p)$ for any prime p. The behavior of $\mathscr{D}_{X,p}$ in $H^4(X; \mathbf{Z}/p)$ is determined by Theorem 1.3 and Theorem 1.4. We note that the D-map $\mathscr{D}_X$ is a derivation (see [5, Section 3]). More precisely, $\mathscr{D}_X(xy) = \mathscr{D}_X(x)y + (−1)^{\deg x} x \mathscr{D}_X(y)$. Here $H^*(LX; \mathbf{Z})$ is regarded as a two sided $H^*(X; \mathbf{Z})$-module via the homomorphism induced from the evaluation map $\pi : LX \to X$ at zero.

NOTATION.   Let Λ be a graded algebra and S a subset of Λ. Then the ideal of Λ generated by elements of S will be denoted by $(S)_\Lambda$. For any graded vector space $V = \oplus_{i \geq 0} V^i$, $V^{\leq n}$ means $\oplus_{0 \leq i \leq n} V^i$. We denote the commutative algebra with the 2-simple system of generators $\{z_j\}_{j=1,\ldots,n}$ by $\Delta(z_1, \ldots z_n)$. Let T be a subset of a vector space W over a field $\mathbf{k}$. We denote the subspace of W generated by elements of T by $\mathbf{k}\{T\}$.

THEOREM 1.3.   *Suppose that X is a simply connected space and that there exists a morphism of algebras*

$$\phi : B = \Lambda(y_1, \ldots, y_l) \otimes \mathbf{Z}/p[x_1, \ldots, x_n]/(\rho_1, \ldots, \rho_m) \to H^*(X; \mathbf{Z}/p),$$

which is an isomorphism below degree 4, where $\rho_1, \ldots, \rho_n$ are decomposable elements with degree 4, $\deg x_j = 2$ or 4, $\deg y_i = 3$ and $l = 0$, $\deg x_j = 2, 3$ or 4 if $p = 2$. We regard $H^*(LX; \mathbf{Z}/p)$ as a B-module via the composition map $\pi^*\phi$. Then there exists a morphism of algebras and of B-modules

$$\psi : \Lambda(y_1, \ldots, y_l) \otimes \Gamma[\bar{y}_1, \ldots, \bar{y}_l] \otimes \{A_p/(d(\omega_1), \ldots, d(\omega_m))_{A_p}\} \to H^*(LX; \mathbf{Z}/p)$$

which is a monomorphism below degree 3, *where* $d(\omega_i) = \sum_{j=1}^{n}(\partial \rho_i/\partial x_j)\bar{x}_j$, $\deg \bar{x}_j = \deg x_j - 1$, $\deg \bar{y}_i = 2$, $\deg \omega_i = \deg \rho_i - 2$ *and* $A_p = \mathbf{k}_p[x_1, \ldots, x_n]/(\rho_1, \ldots, \rho_m) \otimes \Lambda(\bar{x}_1, \ldots, \bar{x}_n)$ *if* $p \neq 2$, *and* $A_2 = \mathbf{k}_2[x_1, \ldots, x_n]/(\rho_1, \ldots, \rho_m) \otimes \Delta(\bar{x}_1, \ldots, \bar{x}_n)$ *if* $p = 2$, *where* $\Delta(\bar{x}_1, \ldots, \bar{x}_n)$ *is the commutative algebra with the* 2-*simple system of generators* $\{\bar{x}_j\}_{j=1,\ldots,n}$.

We identify the elements y_j and x_i with $\phi(y_j)$ and $\phi(x_i)$, respectively. Let σ^* : $H^*(X; \mathbf{Z}/p) \to H^{*-1}(\Omega X; \mathbf{Z}/p)$ be the cohomology suspension and $i : \Omega X \to LX$ the inclusion map.

THEOREM 1.4. *One can choose the elements* $\bar{y}_i$ *and* $\bar{x}_j$ *in Theorem* 1.3 *so that* $i^*(\bar{y}_i) = \sigma^*(y_i)$, $i^*(\bar{x}_j) = \sigma^*(x_j)$ *and*

$$\mathscr{D}_{X,p}|_{H^4(X;\mathbf{Z}/p)} = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\bar{x}_j.$$

Notice that, for any simply connected space X, one can construct an algebra and a morphism of algebras ϕ satisfying the condition of Proposition 1.3 by using indecomposable elements x_j and y_i in $H^*(X; \mathbf{Z}/p)$.

This paper is organized as follows. In section 2, Theorem 1.2 is proved by applying Theorem 1.1. Our main tool to prove Theorem 1.3 is the Eilenberg-Moore spectral sequence converging to $H^*(LX; \mathbf{Z}/p)$ whose E_2-term is isomorphic to the bigraded Hochschild homology of $H^*(X; \mathbf{Z}/p)$. We will determine the indecomposable elements in $H^*(X; \mathbf{Z}/p)$ with degree below 3 and relations between the elements via the E_2-term of the spectral sequence. To this end, Section 3 is devoted to studying the Hochschild homology below degree 3 of a commutative algebra. In section 4, Theorem 1.3 and Theorem 1.4 are proved. Moreover we study the structure of the kernel of the mod pD-map $\mathscr{D}_{X,p} : H^4(X; \mathbf{Z}/p) \to H^3(LX; \mathbf{Z}/p)$ as a vector space. Finally, by using Theorem 1.3 and Theorem 1.4, we prove Theorem 1.1.

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2. Proof of Theorem 1.2

Let X and Y be simply connected spaces satisfying the condition (1.1). By the Universal Coefficient Theorem, we see that $H^2(X; \mathbf{Z})$ and $H^2(Y; \mathbf{Z})$ are torsion free. Hence it follows from the Künneth Theorem that $H^4(X \times Y; \mathbf{Z})$ is isomorphic to $\oplus_{i+j=4}H^i(X; \mathbf{Z}) \otimes H^j(Y; \mathbf{Z})$. As a consequence, the product space $X \times Y$ also satisfies the condition (1.1). It is clear that if X and Y satisfy the condition (1.2) then $X \times Y$ also satisfy the condition (1.2). Let M be a simply connected 4-dimensional

manifold. Since $H^4(M; \mathbf{Z})$ is isomorphic to $\mathbf{Z}$, it follows that M satisfies the condition (1.1). Thus, in order to prove Theorem 1.2, it suffices to show that any compact, simply connected homogeneous space of rank one satisfies the conditions (1.1) and (1.2).

PROPOSITION 2.1.   *Any compact, simply connected homogeneous space of rank one satisfies the conditions* (1.1) *and* (1.2).

PROOF.   It is clear that spheres and projective spaces satisfy (1.1) and (1.2). We will show that the nine homogeneous spaces listed in Section 1 satisfy (1.1) and (1.2).

From the computation of the cohomology of the homogeneous spaces M by Mc-Cleary and Ziller [7, Theorem 1], one can conclude that $H^4(M; \mathbf{Z})$ is torsion free for the cases (1), (3), (5), (6) and (8).

We consider the case (9). Let $\pi : Spin(n) \to SO(n)$ be the universal covering. By the Hurewicz theorem, $(j\pi)_* : H_3(Spin(3); \mathbf{Z}) \to H_3(G_2; \mathbf{Z})$ is multiplication by 28. In order to prove that $j_* : H_3(SO(3); \mathbf{Z}) \to H_3(G_2; \mathbf{Z})$ is multiplication by 14, we will show that $\pi_* : H_3(Spin(3); \mathbf{Z}) \to H_3(SO(3); \mathbf{Z})$ is multiplication by 2. Let us consider the homology Leray-Serre spectral sequence $\{\hat{E}^r_{*,*}, \hat{d}^r\}$ of the universal $SO(n)$-bundle. Since $\hat{E}^2_{2,1} = H_2(BSO(3); H_1(SO(3); \mathbf{Z})) = H_2(BSO(3); \mathbf{Z}/2) = \mathbf{Z}/2$ and $\hat{E}^2_{4,0} = H_4(BSO(3); \mathbf{Z}) = \mathbf{Z}$, it follows that $\hat{E}^3_{4,0} = 2\mathbf{Z}$. Therefore we can deduce $\hat{d}^3 : \hat{E}^3_{4,0} \to \hat{E}^3_{0,3}$ is multiplication by $1/2$. Note that $\hat{E}^3_{0,3} = H_3(SO(3); \mathbf{Z}) = \mathbf{Z}$. The index of the map $B(\pi)^* : H^4(BSO(3); \mathbf{Z}) = \mathbf{Z} \to H^4(BSpin(3); \mathbf{Z}) = \mathbf{Z}$ is 4. From the Universal Coefficient Theorem, it follows that the index of the map $B(\pi)_* : H_4(BSpin(3); \mathbf{Z}) \to H_4(BSO(3); \mathbf{Z})$ is 4 also. Thus the naturality of the differential in the spectral sequence enables us to conclude that the index of π_* is 2. Hence we see $j_* : H_3(SO(3); \mathbf{Z}) \to H_3(G_2; \mathbf{Z})$ is multiplication by 14. To prove that the homogeneous space $M = (G_2, SO(3), 28)$ satisfies the condition (1.1), we consider the homology Leray-Serre spectral sequence $\{E^r_{*,*}, d^r\}$ of the fibration $SO(3) \to G_2 \to G_2/SO(3) = M$. Let $\{F_p H_*\}_{p\geq 0}$ be the filtration of $H_*(G_2; \mathbf{Z})$ which comes from the spectral sequence $\{E^r_{*,*}, d^r\}$. Notice that j_* coincides with the boundary homomorphism

$$H_3(SO(3); \mathbf{Z}) = E^2_{0,3} \to E^\infty_{0,3} \cong E^0_{0,3} = F_0 H_3 \subset F_1 H_3 \subset F_2 H_3 \subset F_3 H_3 = H_3(G_2; \mathbf{Z}).$$

Since $E^2_{0,2} = H_2(SO(3); \mathbf{Z}) = 0$ and $E^2_{1,1} = 0$, we obtain $H_3(M; \mathbf{Z}) = E^2_{3,0} = E^\infty_{3,0} \cong F_3 H_3 / F_2 H_3$. From [7, Theorem 1 (9)], it follows that $E^2_{2,1} = \mathbf{Z}/2$ and $H_4(M; \mathbf{Z})$ does not have a 2-torsion part and a free part. Therefore we see $\mathbf{Z}/2 = E^2_{2,1} \cong E^\infty_{2,1} \cong F_2 H_3 / F_1 H_3$. The fact that the index of j is non-zero and $E^0_{0,3}$ is a subgroup of $H_3(G_2; \mathbf{Z}) = \mathbf{Z}$ allows us to deduce $E^2_{0,3} \cong E^\infty_{0,3}$. Since M is simply connected, $E^2_{1,2} = 0$. Hence j_* coincides with the inclusion $\mathbf{Z} = H_3(SO(3); \mathbf{Z}) = E^2_{0,3} \cong E^\infty_{0,3} \cong E^0_{0,3} = F_0 H_3 = F_1 H_3 \subset F_2 H_3 \subset F_3 H_3 = H_3(G_2; \mathbf{Z})$. The above argument yields

that the inclusion $\mathbf{Z} = F_1H_3 \to F_2H_3 = \mathbf{Z}$ is multiplication by 2. Since the index of j_* is 14, we have the inclusion $\mathbf{Z} = F_2H_3 \to F_3H_3 = \mathbf{Z}$ is multiplication by 7. It turns out that $H_3(M; \mathbf{Z}) = \mathbf{Z}/7$. By using the Universal Coefficient Theorem, we see the manifold $M = G_2/SO(3)$ satisfies the condition (1.1).

The same argument works in cases (2), (4) and (7). From [7, Theorem 1], we obtain that the only case where $H^4(M; \mathbf{Z})$ has 2-torsion is (4): $(G, H, i) = (Sp(2), SU(2), 10)$. It is clear that the manifold satisfies the condition (1.2) since the manifold $Sp(2)/SU(2)$ is 2-connected. $\square$

REMARK 2.2. For the manifolds M in the cases (1), (3), (5), (6) and (8), $H^4(M; \mathbf{Z})$ is torsion free. Therefore, by virtue of Theorem A, we can deduce that the D-maps of these manifolds are good. Since the manifolds in the case (2), (4) and (7) are 2-connected, it follows from [8, Theorem 3.1] that the manifolds have a good D-map. However, we cannot conclude that the manifold in the case (9) and product spaces of compact, simply connected homogeneous spaces of rank one have a good D-map by applying Theorem A or [8, Theorem 3.1].

3. The Hochschild homology below degree 3

The purpose of this section is to prepare the proof of Theorem 1.3. In order to consider the algebra structure of $H^*(LX; \mathbf{Z}/p)$, we use the Eilenberg-Moore spectral sequence converging to $H^*(LX; \mathbf{Z}/p)$ whose E_2-term is isomorphic to the Hochschild homology of $H^*(X; \mathbf{Z}/p)$. Before we begin calculating this spectral sequence, we give an available complex to determine the algebra structure of the Hochschild homology of a certain commutative algebra. For details of the Hochschild homology of commutative differential graded algebras, see [1]. For the rest of this paper, a commutative algebra A will mean a positive graded commutative algebra over $\mathbf{Z}/p$ such that $A^0 = \mathbf{Z}/p$ and $A^1 = 0$. Let Λ be a commutative algebra $\Lambda(y_1, \ldots, y_l) \otimes \mathbf{Z}/p[x_1, \ldots, x_n]/(\rho_1, \ldots, \rho_m)$, where ρ_i is decomposable for any i. We will suppose that $2 \le \deg x_1 \le \cdots \le \deg x_n$, $3 \le \deg y_1 \le \cdots \le \deg y_l$, $\deg \rho_1 \le \cdots \le \deg \rho_m$ and $l = 0$ if $p = 2$. If $\rho_1, \ldots, \rho_m$ is a regular sequence, the Koszul-Tate complex (see [11], [3, Proposition 1.1]) of Λ is a complex for computing the Hochschild homology $HH_*(\Lambda)$. In the general case, we can also obtain a complex for computing $HH_*(\Lambda)$ by extending the Koszul-Tate complex.

PROPOSITION 3.1. *The Hochschild homology of Λ is calculable as the homology of the following complex $(\mathcal{E}, d)$:*

$$\mathcal{E} := \Lambda \otimes \Gamma[\bar{y}_1, \ldots, \bar{y}_l] \otimes \Lambda(\bar{x}_1, \ldots, \bar{x}_n) \otimes \Gamma[\omega_1, \ldots, \omega_m] \otimes \mathcal{C},$$

$d(\omega_i) = \sum_{j=1}^{n}(\partial\rho_i/\partial x_j)\bar{x}_j$, $d(\lambda) = d(\bar{y}_i) = d(\bar{x}_j) = 0$ *for* $\lambda \in \Lambda$, $i = 1, \ldots, l$, $j = 1, \ldots, n$ *and* bideg $\lambda = (0, \deg\lambda)$ *for* $\lambda \in \Lambda$, bideg $\bar{x}_j = (-1, \deg x_j)$, bideg $\bar{y}_i = (-1, \deg y_i)$, bideg $\omega_i = (-2, \deg\rho_i)$. *Here $\mathscr{C}$ is a suitable differential graded algebra which is a tensor product of an exterior algebra and a divided power algebra. Moreover, the differential d satisfies*

$$\{d\mathscr{E} \cap \Lambda \otimes \Gamma[\bar{y}_1, \ldots, \bar{y}_l] \otimes \Lambda(\bar{x}_1, \ldots, \bar{x}_n)\}^{\leq 3}$$
$$= \{d(\Lambda \otimes \Gamma[\bar{y}_1, \ldots, \bar{y}_l] \otimes \Lambda(\bar{x}_1, \ldots, \bar{x}_n) \otimes \Gamma[\omega_1, \ldots, \omega_m])$$
$$\cap \Lambda \otimes \Gamma[\bar{y}_1, \ldots, \bar{y}_l] \otimes \Lambda(\bar{x}_1, \ldots, \bar{x}_n)\}^{\leq 3}.$$

PROOF. Let A and B denote the commutative algebra $\Lambda(y_1, \ldots, y_l)$ and $\mathbf{Z}/p[x_1, \ldots, x_n]/(\rho_1, \ldots, \rho_m)$ respectively. If $\rho_1, \ldots, \rho_m$ is a regular sequence, then there exists the following proper projective resolution $\mathscr{F} \xrightarrow{\mu} B \longrightarrow 0$ of B as a left $B \otimes B$-module ([11], [3, Proposition 1.1]):

$$\mathscr{F} = B \otimes B \otimes \Lambda(\bar{x}_1, \ldots, \bar{x}_n) \otimes \Gamma[\omega_1, \ldots, \omega_m],$$

$\mu : B \otimes B \to B$ is the multiplication of B, $d(\bar{x}_j) = x_j \otimes 1 - 1 \otimes x_j$ and $d(\omega_i) = \sum_{j=1}^{n} \zeta_{ij}\bar{x}_j$, where ζ is an element in $B \otimes B$ satisfying $\rho_i \otimes 1 - 1 \otimes \rho_i = \sum_{j=1}^{n}\zeta_{ij}(x_j \otimes 1 - 1 \otimes x_j)$ and $\mu(\zeta) = \partial\rho_i/\partial x_j$.

In particular, we can choose the element $\sum_{k=1}^{n}\mu_{jk}^{(i)}x_k \otimes 1 + \sum_{k=1}^{n} 1 \otimes \mu_{kj}^{(i)}x_k$ as the element ζ_{ij} mentioned above if $\rho_i = \sum_{j=1}^{n}\sum_{k=1}^{n}\mu_{jk}^{(i)}x_k x_j$.

Let us consider the general case where $\rho_1, \ldots, \rho_m$ are decomposable elements. By modifying the method to construct a minimal model of a differential graded algebra, we obtain the required differential graded algebra $\mathscr{E}$. The argument of [11, Lemma 3.3] enables us to deduce that $H^{i,j}(\mathscr{F}) = 0$. When $i + j = 1$, every element in $\mathscr{F}^{i,j}$ can be written by a linear combination of elements $\bar{x}_1, \ldots, \bar{x}_n$. Therefore, $H^{i,j}(\mathscr{F}) = 0$ for $i + j = 1$. Suppose that $i + j = 2$ and $H^{i,j}(\mathscr{F}) \neq 0$. Then, from the definition of the differential d, we obtain $(i, j) = (-2, 4)$. If the element $u = \sum_{i<j}a_{ij}\bar{x}_i\bar{x}_j$ is in $\text{Ker}\, d^{-2,4}$, then $0 = du = \sum_{i<j} a_{ij}(x_i \otimes 1 - 1 \otimes x_i)\bar{x}_j - \sum_{i<j}a_{ij}(x_j \otimes 1 - 1 \otimes x_j)\bar{x}_i$. Thus we see that $a_{1n}(x_1 \otimes 1 - 1 \otimes x_1) + \cdots + a_{n-1 \, n}(x_{n-1} \otimes 1 - 1 \otimes x_{n-1}) = 0$ and hence $a_{in} = 0$ for any $i < n$. Inductively, we have $a_{ij} = 0$ for any i and j. From this fact, we can conclude that each element of a basis $\{z_1, \ldots, z_s\}$ for $H^{-2,4}(\mathscr{F})$ represents an element $\sum a_{ij}\bar{x}_i\bar{x}_j + \sum b_k\omega_k$, where b_k is nonzero for some k. The element z_α and its representative element will be denoted by the same notation. We define the differential graded algebra $(\mathscr{F}_1, d)$ by $\mathscr{F}_1 = \mathscr{F} \otimes \Lambda(\bar{z}_\alpha)$ and $d(\bar{z}_\alpha) = z_\alpha$, where bideg $z_\alpha = (-3, 4)$. Clearly, $H^{i,j}(\mathscr{F}_1) = 0$ for $i + j = 2$. From the form of a representative element of z_α, it follows that $d(\Lambda(\bar{z}_\alpha)) \cap B \otimes B \otimes \Lambda(\bar{x}_1, \ldots, \bar{x}_n) = 0$. Consider the case where $i + j = 3$. It is easy to verify that $\text{Ker}\, d^{-2,5} \cap \mathbf{Z}/p\{\bar{x}_i\bar{x}_j; 1 \leq i, j \leq n\} = 0$ in the only case where $p = 2$ and that $\text{Ker}\, d^{-3,6} \cap \mathbf{Z}/p\{\bar{x}_i\bar{x}_j\bar{x}_k; 1 \leq i, j, k \leq n\} = 0$. We define the elements $\bar{v}_\beta$ with total degree 3 corresponding to representative elements

v_β of a basis of $H^{i,j}(\mathscr{F}_1)$ for $(i,j) = (-2,5)$ and $(-3,6)$. Put $\mathscr{F}_2 = \mathscr{F}_1 \otimes \Gamma[\bar{v}_\beta]$ and extend the differential d by demanding that $d(\bar{v}_\beta) = v_\beta$. The elements of $\operatorname{Ker} d \cap \mathscr{F}_1^{-1,4}$ are characterized as follows:

LEMMA 3.2.   *Let u be an element of $\operatorname{Ker} d \cap \mathscr{F}_1^{-1,4}$. Then u can be written as* $\sum_{j=1}^n (\sum_{k=1}^n \lambda_{jk} x_k \otimes 1 + \sum_{k=1}^n 1 \otimes \lambda_{kj} x_k) \bar{x}_j$ *with coefficients λ_{jk} satisfying $\lambda_{jj} = \alpha^{(1)} \mu_{jj} + \cdots + \alpha^{(m)} \mu_{jj}$ and $(\lambda_{jk} + \lambda_{kj}) = \alpha^{(1)}(\mu_{jk}^{(1)} + \mu_{kj}^{(1)}) + \cdots + \alpha^{(m)}(\mu_{jk}^{(m)} + \mu_{kj}^{(m)})$ for some $\alpha^{(i)}$ (for the notation $\mu_{jk}^{(l)}$ see the definition of the above resolution $\mathscr{F}$). Therefore, the element $1 \otimes_{\Lambda \otimes \Lambda} u$ in $\Lambda \otimes_{\Lambda \otimes \Lambda} \mathscr{F}_1$ belongs to $\mathbf{Z}/p\{d(\omega_1), \ldots, d(\omega_m)\}$.*

Let $\{u_\alpha\}$ be a basis for $H^{-1,4}(\mathscr{F}_1)$. We extend the complex $\mathscr{F}_2$ to $\mathscr{F}_3 = \mathscr{F}_2 \otimes \Gamma[\bar{u}_\alpha] = \mathscr{F}_1 \otimes \Gamma[\bar{v}_\beta] \otimes \Gamma[\bar{u}_\alpha]$ with the differential defined by $d(\bar{u}_\alpha) = u_\alpha$. From this construction, we see that $H^{i,j}(\mathscr{F}_2) = 0$ for $i + j = 3$ and $d(\Gamma[\bar{v}_\beta]) \cap B \otimes B \otimes \Lambda(\bar{x}_1, \ldots, \bar{x}_n) = 0$. By continuing the same process above total degree 4, we can get a proper projective resolution $\widetilde{\mathscr{E}}_B$ of B as a $B \otimes B$-module : $\widetilde{\mathscr{E}}_B = \mathscr{F} \otimes \mathscr{C}$. By virtue of [11, Lemma 3.2], we conclude that the differential graded algebra $(\widetilde{\mathscr{E}}_A, d)$, defined by $\widetilde{\mathscr{E}}_A = A \otimes A \otimes \Gamma[\bar{y}_1, \ldots \bar{y}_l]$ and $d(\bar{y}_i) = y_i \otimes 1 - 1 \otimes y_i$, is a proper projective resolution of A as an $A \otimes A$-module. Therefore, the differential graded algebra $\widetilde{\mathscr{E}} = \widetilde{\mathscr{E}}_A \otimes \widetilde{\mathscr{E}}_B$ is a proper projective resolution of Λ as a $\Lambda \otimes \Lambda$-module. Thus the Hochschild homology $HH_{*,*}(\Lambda) = Tor_{\Lambda \otimes \Lambda}^{*,*}(\Lambda, \Lambda)$ is obtained as the homology of the complex $(\mathscr{E}, d) = (\Lambda \otimes_{\Lambda \otimes \Lambda} \widetilde{\mathscr{E}}, 1 \otimes d)$. From Lemma 3.2, it follows that $d(\bar{u}_\alpha) = 1 \otimes_{\Lambda \otimes \Lambda} d(\bar{u}_\alpha) = 1 \otimes_{\Lambda \otimes \Lambda} u_\alpha$ is in $\mathbf{Z}/p\{d(\omega_1), \ldots, d(\omega_m)\}$. This fact and the definitions of $d(\bar{z}_\alpha)$ and $d(\bar{v}_\beta)$ allow us to deduce that

$$
\{d\mathscr{E} \cap \Lambda \otimes \Gamma[\bar{y}_1, \ldots, \bar{y}_l] \otimes \Lambda(\bar{x}_1, \ldots, \bar{x}_n)\}^{\leq 3}
$$
$$
= \{d(\Lambda \otimes_{\Lambda \otimes \Lambda} (\widetilde{\mathscr{E}}_A \otimes \mathscr{F})) \cap \Lambda \otimes \Gamma[\bar{y}_1, \ldots, \bar{y}_l] \otimes \Lambda(\bar{x}_1, \ldots, \bar{x}_n)\}^{\leq 3}.
$$

This completes the proof of Proposition 3.1.   □

PROOF OF LEMMA 3.2.   For any element $u \in \operatorname{Ker} d \cap \mathscr{F}_1^{-1,4}$, we can write $u = \sum_j \xi_j \bar{x}_j$, where $\xi_j = \sum_{k=1}^n \lambda_{jk} x_k \otimes 1 + \sum_{k=1}^n 1 \otimes \lambda'_{jk} x_k$. Since $d(u) = 0$ in $\mathscr{F}_1^{0,4} = \mathscr{F}^{0,4}$, it follows that the element

$$
du = \sum_{j=1}^n \sum_{k=1}^n \lambda_{jk} x_k x_j \otimes 1 - \sum_{j=1}^n \sum_{k=1}^n \lambda_{jk} x_k \otimes x_j + \sum_{j=1}^n \sum_{k=1}^n \lambda'_{jk} x_j \otimes x_k
$$
$$
- \sum_{j=1}^n \sum_{k=1}^n 1 \otimes \lambda'_{jk} x_k x_j
$$

belongs to the ideal $(\rho_i \otimes 1, 1 \otimes \rho_i; 1 \leq i \leq m)$ in $\mathbf{Z}/p[x_1, \ldots, x_n] \otimes \mathbf{Z}/p[x_1, \ldots, x_n]$. This fact enables us to conclude that $\sum_{j=1}^n \sum_{k=1}^n \lambda_{jk} x_k \otimes x_j = \sum_{j=1}^n \sum_{k=1}^n \lambda'_{jk} x_j \otimes x_k$ and so $\lambda_{jk} = \lambda'_{kj}$. Moreover we see $\sum_{j=1}^n \sum_{k=1}^n \lambda_{jk} x_k x_j = \alpha^{(1)} \rho_1 + \cdots + \alpha^{(m)} \rho_m =$

$\alpha^{(1)}(\sum_{j=1}^{n}\sum_{k=1}^{n}\mu_{jk}^{(1)}x_kx_j)+\cdots+\alpha^{(m)}(\sum_{j=1}^{n}\sum_{k=1}^{n}\mu_{jk}^{(m)}x_kx_j)$ in $\mathbf{Z}/p[x_1,\ldots,x_n]$. Thus we have

$$\sum_{j=1}^{n}\lambda_{jj}x_j^2+\sum_{j<k}(\lambda_{jk}+\lambda_{kj})x_kx_j=\alpha^{(1)}\left(\sum_{j=1}^{n}\mu_{jj}^{(1)}x_j^2+\sum_{j<k}(\mu_{jk}^{(1)}+\mu_{kj}^{(1)})x_kx_j\right)+\cdots$$
$$+\alpha^{(m)}\left(\sum_{j=1}^{n}\mu_{jj}^{(m)}x_j^2+\sum_{j<k}(\mu_{jk}^{(m)}+\mu_{kj}^{(m)})x_kx_j\right)$$

in $\mathbf{Z}/p[x_1,\ldots,x_n]$. Therefore, the required relations for λ_{jk} are obtained. Let μ be the multiplication of B. Since $\mu(\zeta_{ij})=\sum_{k=1}^{n}(\mu_{jk}^{(i)}+\mu_{kj}^{(i)})x_k$, it follows that

$$1\otimes_{\Lambda\otimes\Lambda}u=\sum_{j=1}^{n}\left(\sum_{k=1}^{n}(\lambda_{jk}+\lambda_{kj})x_k\right)\bar{x}_j$$
$$=\sum_{j=1}^{n}\alpha^{(1)}\left(\sum_{k=1}^{n}(\mu_{jk}^{(1)}+\mu_{kj}^{(1)})x_k\right)\bar{x}_j+\cdots+\sum_{j=1}^{n}\alpha^{(m)}\left(\sum_{k=1}^{n}(\mu_{jk}^{(m)}+\mu_{kj}^{(m)})x_k\right)\bar{x}_j$$
$$=\sum_{j=1}^{n}\alpha^{(1)}\mu(\zeta_{1j})\bar{x}_j+\cdots+\sum_{j=1}^{n}\alpha^{(m)}\mu(\zeta_{mj})\bar{x}_j$$
$$=\alpha^{(1)}d(\omega_1)+\cdots+\alpha^{(m)}d(\omega_m).$$

Thus we have Lemma 3.2. $\square$

Applying Proposition 3.1, we can partially know the algebra structure of the Hochschild homology of the graded algebra Λ.

PROPOSITION 3.3. *Let Λ be the graded algebra $\Lambda(y_1,\ldots,y_l)\otimes\mathbf{Z}/p[x_1,\ldots,x_n]/(\rho_1,\ldots,\rho_m)$. Then there exists a morphism of algebras*

$$\phi:\Lambda(y_1,\ldots,y_l)\otimes\Gamma[\bar{y}_1,\ldots,\bar{y}_l]\otimes\{A/(d\omega_1,\ldots,d\omega_m)_A\}\to HH_{*,*}(\Lambda)$$

which is a monomorphism below total degree 3, where $A=\mathbf{Z}/p[x_1,\ldots x_n]/(\rho_1,\ldots,\rho_m)\otimes\Lambda(\bar{x}_1,\ldots,\bar{x}_n)$.

PROOF.   Let us consider the following commutative diagram:

$$
\begin{array}{ccccc}
\mathscr{K} \otimes \mathbf{Z}/p\{\omega_1, \ldots, \omega_m\} & \xrightarrow{\bar{d}} & \mathscr{K} := \Lambda \otimes \Gamma[\bar{y}_1, \ldots, \bar{y}_l] \otimes \Lambda(\bar{x}_1, \ldots, \bar{x}_n) & \longrightarrow & 0 \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} & & \downarrow \\
\mathscr{E} & \xrightarrow{\quad d \quad} & \mathscr{E} & \xrightarrow{\quad d \quad} & \mathscr{E}
\end{array}
$$

where i and j are the inclusion maps and $\bar{d}$ is the restriction of d to $\mathscr{K} \otimes \mathbf{Z}/p\{\omega_1, \ldots, \omega_m\}$. Suppose that $H(i)[z] = 0$ and the total degree of $[z]$ is below 3. Then $i(z)$ belongs to $\{d(\mathscr{E}) \cap \mathscr{K}\}^{\leq 3}$. From Proposition 3.1, we can see that $i(z)$ is in the vector space $\{\operatorname{Im} d \circ j\}^{\leq 3}$. Therefore z is an element in $\operatorname{Im} \bar{d}$. The map $\phi = H(i)$ is the demanded homomorphism.   □

4. Kernel of the D-map $\mathscr{D}_{X,p}$

Let X be a simply-connected space. In order to study the structure of the kernel of $\mathscr{D}_{X,p} = \int_{S^1} \circ ev^* : H^4(X; \mathbf{Z}/p) \to H^3(LX; \mathbf{Z}/p)$, it is important to consider the ring structure of $H^*(LX; \mathbf{Z}/p)$, in particular, to clarify indecomposable elements with degree below 3 in $H^*(LX; \mathbf{Z}/p)$ and relations between their elements in $H^3(LX; \mathbf{Z}/p)$. To this end, we use the Eilenberg-Moore spectral sequence converging to $H^*(LX; \mathbf{Z}/p)$ ([11],[3]) whose E_2-term is the Hochschild homology of $H^*(X; \mathbf{Z}/p)$:

$$
E_2^{*,*} \cong Tor_{H^*(X;\mathbf{Z}/p) \otimes H^*(X;\mathbf{Z}/p)}^{*,*}(H^*(X; \mathbf{Z}/p), H^*(X; \mathbf{Z}/p)) = HH(H^*(X; \mathbf{Z}/p)).
$$

Proposition 3.3 plays an important role in explaining the algebra structure of $H^*(LX; \mathbf{Z}/p)$. Before we prove Theorem 1.3, we prepare a lemma.

LEMMA 4.1.   *Let C_1 and C_2 be commutative algebras. Suppose that there exists a morphism of algebras $\theta : C_1 \to C_2$ which is an isomorphism below degree s. Then $Tor_{\theta \otimes \theta}(\theta, \theta) : Tor_{C_1 \otimes C_1}^{i,j}(C_1, C_1) \to Tor_{C_2 \otimes C_2}^{i,j}(C_2, C_2)$ is an isomorphism if $i = 0$ and $j \leq s$, $i = -1$ and $j \leq s - 1$ or $i < -1$ and $i + j < s - i - 2$.*

PROOF.   Let $Bar^{*,*}(C_1)$ be the complex obtained from the bar resolution of C_1 as an $C_1 \otimes C_1$-module and $Bar^{*,*}(C_2)$ the similar complex constructed from C_2. Let α be an element of $Bar^{-i,*}(C_1)$. We can write $u = a[b_1| \cdots |b_i]c$, where a and c are elements of A and b_l is an element of $\overline{C_1 \otimes C_1}$. If there exists an element b_l such that $\deg b_l > s$, then $\deg \alpha = \deg a + \deg b_1 + \cdots + \deg b_l + \cdots + \deg b_i + \deg c - i$

$> 0 + 2 + \cdots + 2 + s + 2 + \cdots + 2 + 0 - i = s + i - 2$ when $i \neq 0$. Thus we see that the morphism $Bar^{i,j}(\theta) : Bar^{i,j}(C_1) \to Bar^{i,j}(C_2)$, which is induced from θ, is an isomorphism if $-i + j \leq s + i - 2$ and $i \neq 0$. It is clear that $B^{0,j}(\theta)$ is an isomorphism if $j \leq s$. Therefore we have Lemma 4.1. $\square$

PROOF OF THEOREM 1.3.   Let $\{E_r, d_r\}$ be the Eilenberg-Moore spectral sequence converging to $H^*(LX; \mathbf{Z}/p)$. By virtue of Proposition 3.3 and Lemma 4.1, we have a homomorphism ψ from $B := \Lambda(y_1, \dots, y_l) \otimes \Gamma[\bar{y}_1, \dots, \bar{y}_l] \otimes \{A_p/(d(\omega_1), \dots, d(\omega_m))_{A_p}\}$ to $E_2^{*,*}$ which is a monomorphism below degree 3. From [10, Proposition 4.2], it is seen that $\pi^*(y_j) = y_j \in F^0 H^*(LX; \mathbf{Z}/p)$ and $\pi^*(x_i) = x_i \in F^0 H^*(LX; \mathbf{Z}/p)$. Therefore the injectivity of π^* allows us to conclude that $d_2 : E_2^{-2,*} \to E_2^{0,*}$ is trivial. Since $E_2^{i,j} = 0$ if $q < -2p$, it follows that the elements in $E_2^{-1,j}$ survive in the E_∞-term if $j \leq 4$. Thus we have an monomorphism $\psi : B^{\leq 3} \to (Tot E_0^{*,*})^{\leq 3}$. In order to complete the proof of Proposition 1.3, we must solve extension problems below degree 3. More precisely, we need to consider whether it is true that $\sum \lambda_{i,j} x_i \bar{x}_j = 0$ in $H^*(LX; \mathbf{Z}/p)$ when $\sum \lambda_{ij} x_i \bar{x}_j \in \mathbf{Z}/p\{d(\omega_1), \dots, d(\omega_m)\}$. Note that the element $\bar{x}_j$ in $E_0^{-1,*}$ and its representative element in $H^*(LX; \mathbf{Z}/p)$ are denoted by the same notation. Since the generators with degree 3 and filtration degree 0 are the elements $y_1, \dots, y_l$, we can write the element $\sum \lambda_{ij} x_i \bar{x}_j$ as $\sum \mu_k y_k$ with some constants μ_k. Let $\pi : LX \to X$ be the fibration defined by $\pi(\gamma) = \gamma(0)$. From [10, Proposition 4.2], we see that the element y_k in $E_0^{0,*}$ is identified with the element $\pi^*(y_k)$. Hence the given equality is written as $\sum \lambda_{ij} x_i \bar{x}_j = \sum \mu_k \pi^*(y_k)$. Let s be the section of the fibration $LX \to X$ defined by $s(x) = C_x$, where C_x is the constant loop at x. Since we can choose a representative element of $\bar{x}_i$ so that $s^*(\bar{x}_i) = 0$ in $H^*(X; \mathbf{Z}/p)$, it follows that $\sum \mu_k y_k = s^*(\sum \lambda_{ij} x_i \bar{x}_j) = 0$ in $H^*(X; \mathbf{Z}/p)$ and hence $\mu_k = 0$ for any k. Thus $\sum \lambda_{i,j} x_i \bar{x}_j = 0$ in $H^*(LX; \mathbf{Z}/p)$ when $\sum \lambda_{ij} x_i \bar{x}_j \in \mathbf{Z}/p\{d(\omega_1), \dots, d(\omega_m)\}$. We have proved Theorem 1.3. $\square$

REMARK 4.2.   In the case $p = 2$, we cannot solve extension problems completely by using the usual argument on total degrees and column degrees of the associated bigraded algebra $E_0^{*,*}$. For example, it may be possible that $\bar{x}_i^2$ is equal to some $\bar{y}_j$. However some information about the squaring operations in $H^*(X; \mathbf{Z}/2)$ allows us to determine whether or not $\bar{x}_i^2$ is equal to $\bar{y}_j$. To be exact, if $Sq^1 x_i = \varepsilon y_j$ then $\bar{x}_i^2 = \varepsilon \bar{y}_j$, where $\varepsilon = 0$ or 1. For details, see the proof of [6, Theorem 2.5].

PROOF OF THEOREM 1.4.   Let $\{E_r, d_r\}$ be the Eilenberg-Moore spectral sequence used in the proof of Theorem 1.3. By applying [10, Proposition 4.5] and the same argument as the proof of [3, Lemma 1.3], we can show that $i^*(\bar{y}_i) = \sigma^*(y_i)$ and $i^*(\bar{x}_j) = \sigma^*(x_j)$ with any choice of representative elements of $\bar{y}_i$ and $\bar{x}_j$ in $E_0^{-1,*}$. To proceed with the proof, we need the following lemma:

LEMMA 4.3. *For each indecomposable element $\bar{x}_j$ in $E_0^{-1,*}$, one can choose its representative element $\tilde{x}_j$ in $H^*(LX; \mathbf{Z}/p)$ so that $\mathcal{D}_{X,p}(x_j) = \bar{x}_j$.*

From Lemma 4.3 and the fact that the D-map $\mathcal{D}_{X,p}$ is a derivation, we can get $\mathcal{D}_{X,p} = \sum_{j=1}^{n} \tilde{x}_j \partial/\partial x_j$. This completes the proof.   □

PROOF OF LEMMA 4.3.   Let $f_j : X \to K_j = K(\mathbf{Z}/p, n_j)$ be the map representing the element x_j of $H^{n_j}(X; \mathbf{Z}/p)$, where $n_j = \deg x_j$. From the naturality of the D-map $\mathcal{D}_{X,p}$, to prove Lemma 4.2, it suffices that $\mathcal{D}_{K_j,p}(\iota_j) = \bar{\iota}_i$, where ι_j is the fundamental element of $H^{n_j}(K_j; \mathbf{Z}/p)$ and $\bar{\iota}_j$ is the element corresponding to ι_j (see Theorem 1.3).   We denote the map $\int_{S^1} \circ ev^* : H^*(K_j; \mathbf{Z}/p) \to H^*(S^1 \times \Omega K_j; \mathbf{Z}/p) \to H^{*-1}(\Omega K_j; \mathbf{Z}/p)$ by $\tilde{\mathcal{D}}$. Here ΩX means the space of continuous loops on X which map $1 \in S^1$ to the base point of X. Let $\sigma^* : H^*(K_j; \mathbf{Z}/p) \to H^{*-1}(\Omega K_j; \mathbf{Z}/p)$ be the cohomology suspension.   Since $i^*(\bar{\iota}_j) = \sigma^*(\iota_j)$ and $\sigma^*(\iota_j)$ is the fundamental element in $H^{n_j-1}(\Omega K_j; \mathbf{Z}/p) = H^{n_j-1}(K(\mathbf{Z}/p, n_j - 1); \mathbf{Z}/p)$, we see that if $\tilde{\mathcal{D}}(\iota_j) = \sigma^*(\iota_j)$ then $\mathcal{D}_{K_j,p}(\iota_j) = \bar{\iota}_j$. We now prove $\tilde{\mathcal{D}}(\iota_j) = \sigma^*(\iota_j)$.   Let $f : (I^{n_j}, \partial I^{n_j}) \to (K_j, *)$ be a continuous map and $g : I^{n_j-1} \to \Omega K_j$ a map defined by $g(t)(s) = f(t, s)$ for $t \in I^{n_j-1}$ and $s \in I$.   The argument of the proof of [8, Proposition 2.1] enables us to deduce that the element dual$[f]$ in dual$(\pi_{n_j}(K_j) \otimes \mathbf{Z}/p)$ is mapped to dual$[g]$ in dual$(\pi_{n_j-1}(\Omega K_j) \otimes \mathbf{Z}/p)$ by $\tilde{\mathcal{D}}$ under the isomorphisms $H^{n_j}(K_j; \mathbf{Z}/p) \cong$ dual$(H_{n_j}(K_j; \mathbf{Z}/p)) \cong$ dual$(\pi_{n_j}(K_j) \otimes \mathbf{Z}/p)$ and $H^{n_j-1}(\Omega K_j; \mathbf{Z}/p) \cong$ dual$(\pi_{n_j-1}(\Omega K_j) \otimes \mathbf{Z}/p)$. Let $\varepsilon_1 : PK_j \to K_j$ be the path-loop fibration defined by $\varepsilon_1(\gamma) = \gamma(1)$. Then the homology suspension $H_{n_j-1}(\Omega K_j; \mathbf{Z}) \xleftarrow{\cong} H_{n_j}(PK_j, \Omega K_j; \mathbf{Z}) \xrightarrow{\varepsilon_{1*}} H_{n_j}(K_j; \mathbf{Z})$ is regarded as the homomorphism $\pi_{n_j-1}(\Omega K_j) \xleftarrow{\cong} \pi_{n_j}(PK_j, \Omega K_j) \xrightarrow{\varepsilon_{1*}} \pi_{n_j}(K_j)$ under the identification with the Hurewicz maps. We define the map $\tilde{f} : (I^{n_j}, \partial I^{n_j}, I^{n_j-1} \times 0 \cup \partial I^{n_j-1} \times I) \to (PK_j, \Omega K_j, *)$ by $\tilde{f}(t, s)(u) = *$ if $0 \leq u \leq 1/(s+1)$ and $\tilde{f}(t, s)(u) = f(t, u(s+1) - 1)$ if $1/(s+1) \leq u \leq 1$. Then we can deduce that $\varepsilon_{1*}(\tilde{f}) = f$ and that $\partial(\tilde{f})$ is homotopic to g by the homotopy $H : (I^{n_j-1} \times I, \partial I^{n_j-1} \times I) \to (\Omega K_j, *)$ defined by $H(t, l)(u) = *$ if $0 \leq u \leq l/2$ and $H(t, l)(u) = f(t, (2u - l)/(2 - l))$ if $l/2 \leq u \leq 1$. Thus it follows that cohomology suspension σ^* maps dual$[f]$ to dual$[g]$. This completes the proof.   □

We can determine the structure of the kernel of the D-map $\mathcal{D}_{X,p} : H^4(X; \mathbf{Z}/p) \to H^3(LX; \mathbf{Z}/p)$ completely. Let $\eta_p : H^4(X; \mathbf{Z}) \to H^4(X; \mathbf{Z}/p)$ be the mod p reduction. Then we have

PROPOSITION 4.4.   *Suppose that X is a simply connected space and that there exists a morphism of algebras ψ to $H^*(X; \mathbf{Z}/p)$ from an algebra $\Lambda(y_1, \ldots, y_l) \otimes \mathbf{Z}/p[x_1, \ldots, x_n]/(\rho_1, \ldots, \rho_m)$ which is an isomorphism below degree 4, where $\deg \rho_i = 4$. If x is an element in the kernel of the D-map $\mathcal{D}_X = \int_{S^1} \circ ev^* : H^4(X; \mathbf{Z}) \to H^3(LX; \mathbf{Z})$, then $\eta_2(x) = \sum \lambda_i x_i^2$ for some constant λ_i and $\eta_p(x) = 0$ if $p \neq 2$.*

PROOF.   By virtue of Theorem 1.3 and Theorem 1.4, we see that $\mathscr{D}_{X,p}$ coincides with the operator $\sum_{i=1}^{n} \bar{x}_i \partial/\partial x_i$. Therefore, the image of the map $\mathscr{D}_{X,p}$ is included in the image of ψ of Theorem 1.3. Hence, we can deduce that if $\mathscr{D}_{X,p}(\alpha) = 0$ for some $\alpha \in H^4(X; \mathbf{Z}/p)$, then

$$\sum_{i=1}^{n} \frac{\partial \alpha}{\partial x_i} \bar{x}_i \in (d(\omega_1), \ldots, d(\omega_m))_{A_p}.$$

We can write $\alpha = \alpha_1 + \alpha_2$ by using elements α_1 and α_2 which are linear combinations of $x_i x_j$ and x_k respectively. From the definition of $d(\omega_j)$, it follows that $\alpha_2 = 0$ and

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \bar{x}_i (\alpha_1 - \sum \xi_j \rho_j) = 0$$

in A_p for some ξ_j in $\mathbf{Z}/p$. Since $\deg \rho_i = 4$ and $\deg \partial(\alpha_1 - \sum \xi_j \rho_j)/\partial x_i = 2$ for any element x_i with degree 2, one can conclude that $\partial(\alpha_1 - \sum \xi_j \rho_j)/\partial x_i = 0$ in $\mathbf{Z}/p[x_1, \ldots, x_n]$ for any i. Thus, in $\mathbf{Z}/p[x_1, \ldots, x_n]/(\rho_1, \ldots, \rho_m)$, $\alpha_1 = 0$ if $p \neq 2$ and $\alpha_1 = \sum \lambda_i x_i^2$ if $p = 2$.   □

PROOF OF THEOREM 1.1.   For any element x in $\mathrm{Ker}\{\mathscr{D}_M : H^4(X; \mathbf{Z}) \to H^3(LX; \mathbf{Z})\}$, its mod p reduction $\eta_p(x)$ is zero if $p \neq 2$ by Proposition 4.4. Therefore it follows from (1.1) that the free part and odd torsion part of x is zero. Moreover, the condition (1.2) enables us to deduce that the 2-torsion part of x is zero.   □

REMARK 4.5.   So far, we have considered the string class of an $SO(n)$-bundle in the case where $n \geq 5$. The case $n = 4$ must be treated separately as mentioned in [8, Remark, page 150] because the universal central extension of $LSpin(n)$ is an extension by a 2-torus. The fact that $SO(4)$ is not simple causes the difference. In the case where $n = 3$, since $SO(3)$ is simple, we can define the string class of an $SO(3)$-bundle with a spin structure in similar fashion to the case $n \geq 5$. However, the index of the homomorphism $B\pi^* : H^4(BSO(3); \mathbf{Z}) = \mathbf{Z} \to H^4(BSpin(3); \mathbf{Z}) = \mathbf{Z}$ is 4, not 2, where $\pi : Spin(3) \to SO(3)$ is the universal covering. This fact is proved by using the same argument as the proof of [8, Lemma 2.2, page 148]. Notice that $H^5(BSO(3); \mathbf{Z})$ is zero though $H^5(BSO(n); \mathbf{Z}) = \mathbf{Z}/2$ for $n \geq 5$. Thus the string class $\mu(Q)$ of an $SO(3)$-bundle ξ with a spin structure $Q \to M$ can be regarded as the image of 1/4 *the Pontrjagin class* of ξ by the *D*-map $\mathscr{D}_M : H^4(M; \mathbf{Z}) \to H^3(LM; \mathbf{Z})$.

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