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R. E. ALLARDICE, Esq., M.A., Vice-President, in the Chair.

On a certain expression for a spherical harmonic, with some extensions.

By JOHN DOUGALL, M.A.

The object of this paper is to show how the leading properties of Spherical Harmonic Functions may be readily deduced by employing as a typical harmonic a certain simple algebraical expression, which obviously satisfies Laplace's equation ; and to extend a similar method to the case of any number of variables.

Some well-known expressions of Spherical Harmonics by means of Definite Integrals are readily arrived at by this method.

The most important properties of Spherical Harmonics, perhaps, are those connected with the integration of the product of two harmonics over the surface of a sphere, and it will be seen that this integral takes a somewhat remarkable form when the harmonics are of the type I have referred to.

Consider the function of x, y, z

$$u \equiv (ax + by + cz)^n$$

We have $\dfrac{d^2u}{dx^2} + \dfrac{d^2u}{dy^2} + \dfrac{d^2u}{dz^2} = (a^2 + b^2 + c^2)(ax + by + cz)^{n-2}$.

Hence if $a^2 + b^2 + c^2 = 0$, u is a harmonic of degree n.

Let $a = f + \iota f'$, $b = g + \iota g'$, $c = h + \iota h'$; $\iota = \sqrt{-1}$

The condition $a^2 + b^2 + c^2 = 0$ gives

$$f^2 + g^2 + h^2 = f'^2 + g'^2 + h'^2$$
$$ff' + gg' + hh' = 0$$

f, g, h ; f', g', h' being real.

Hence by transformation of rectangular axes, we may reduce u to the form $(x + \iota y)^n$, or, in polar co-ordinates, $r^n \sin^n\theta(\cos n\phi + \iota \sin n\phi)$; so that the real and imaginary parts of u are two conjugate sectorial harmonics. In what follows, then, we consider the general harmonic as made up of a sum of sectorial harmonics.

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Observe that, by a well known proposition,

(1) $r^{2n+1}(ax+by+cz)^{-n-1}$ is another harmonic of degree n, where $r^2=x^2+y^2+z^2$.

Now let us consider the integral of the product of two harmonics of the type u, taken over the surface of a sphere of radius ρ.

If the degrees of the two harmonics are positive, but different, Green's Theorem shows at once that that integral is zero.

Let then $u_n \equiv (a_1x+b_1y+c_1z)^n$

$$v_n \equiv (a_2x+b_2y+c_2z)^n$$

n being a positive integer.

By Green's Theorem

$$\iint \frac{n}{\rho} u_n v_n dS$$

$$= \iiint \left(\frac{du_n}{dx}\,\frac{dv_n}{dx} + \frac{du_n}{dy}\,\frac{dv_n}{dy} + \frac{du_n}{dz}\,\frac{dv_n}{dz}\right) dV$$

and n being a positive integer, we may take the latter integral through the whole volume of the sphere.

But the function to be integrated through the volume being homogeneous of degree $2n-2$, we have

its volume integral = the product of its surface integral into

$$\int_0^\rho \frac{r^{2n-2}}{\rho^{2n-2}} \cdot \frac{r^2}{\rho^2} \,.\, dr.$$

This last integral $= \dfrac{\rho}{2n+1}$.

$$\text{Hence } \iint u_n v_n dS = \frac{\rho^2 n}{2n+1}(a_1a_2+b_1b_2+c_1c_2) \times \iint u_{n-1}v_{n-1}dS$$

Continuing this method of reduction, we get finally

$$(2)\quad \iint u_n v_n dS = \frac{n(n-1)\,\ldots\,1}{(2n+1)(2n-1)\ldots 3}.4\pi\rho^{2n+2}(a_1a_2+b_1b_2+c_1c_2)^n.$$

And we may also write for reference

$$(3)\quad \iint u_n v_m dS = 0 \quad \text{when, } m,\ n \text{ are different.}$$

In order to apply this result at once, let us find an expression for a Zonal Harmonic. We see that $(\iota\cos a.x+\iota\sin a.y+z)^n$ is a case of our form, and $\therefore \displaystyle\int_0^{2\pi}(\iota\cos.ax+\iota\sin ay+z)^n da$ is a harmonic.

By putting $x = \sqrt{x^2+y^2}\cos\phi$

$$y = \sqrt{x^2+y^2}\sin\phi$$

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this becomes $\int_0^{2\pi}(\iota\sqrt{x^2+y^2}\cos\overline{a-\phi}+z)^n da$

$$=\int_0^{2\pi}(\iota\sqrt{x^2+y^2}\cos\beta+z)^n d\beta\cdot$$

This function is symmetrical about Oz, and considering x, y, z as co-ordinates of a point on a sphere of radius unity, we see that when $x=0$ $y=0$ $z=1$, the value of the integral $=2\pi$.

Hence the zonal harmonic with axis Oz,

$$\text{say}\quad P_n(z)=\frac{1}{2\pi}\int_0^{2\pi}(\iota\sqrt{x^2+y^2}\cos\beta+z)^n d\beta.$$

Now our result (2) shows that in order to find the surface integral of the product of this and another harmonic $(ax+by+cz)^n$, we may substitute a, b, c for x, y, z, respectively in the expression for $P_n(z)$ and multiply the result by a numerical coefficient which we may call k.

$$\text{We thus get }\frac{k}{2\pi}\int_0^{2\pi}(\iota\sqrt{a^2+b^2}\cos\beta+c)^n d\beta.$$

$$=\frac{k}{2\pi}\quad\int_0^{2\pi}(1-\cos\beta)^n d\beta$$

$$=\frac{4\pi}{2n+1}.c^n$$

Now c^n is the value of $(ax+by+cz)^n$ at the pole of $P_n(z)$; and by adding together forms like $(ax+by+cz)^n$ we may get any harmonic.

Hence the well known important result which we may write

$$(4)\quad \iint P_n V_n dS=\frac{4\pi}{2n+1}\cdot V'_n.$$

Now, considering further the form $(\iota x\cos a+\iota y\sin a+z)^n$, we see by Fourier's Theorem that if we expand it in cosines and sines of multiples of a, the coefficient of $\cos sa$ is $\frac{1}{\pi}\int_0^{2\pi}(\iota x\cos a+\iota y\sin a+z)^n$ $\cos sada$. This is obviously a harmonic, and by putting

$$x=\sqrt{x^2+y^2}\cos\phi$$
$$y=\sqrt{x_2+y^2}\sin\phi$$

and reducing, we see that it is a tesseral harmonic of type s.

The other of the same type is got by writing $\sin sa$ instead of $\cos sa$ in the integral.

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Now consider the surface integral ($r=1$) of the product of the above harmonic under the $\int$ with another $\int_0^{2\pi}(\iota x\cos\beta+\iota y\sin\beta+z)^{n'}$ $\cos s'\beta d\beta$.

If n, n' are different, (3) shows that the result is zero. If $n'=n$ but s and s' different, we see writing out its value from (2) that the result is again zero. If $n'=n$, $s'=s$, the result is

$$k\int_0^{2\pi}\int_0^{2\pi}(1-\cos\overline{\alpha-\beta})^n\cos s\alpha\cos s\beta d\alpha d\beta.$$

We may expand $(1-\cos\overline{\alpha-\beta})^n$ in cosines of multiples of $\overline{\alpha-\beta}$, and the only term which contributes anything to the integral is that containing $\cos s(\alpha-\beta)$.

Expanding by ordinary trigonometry, and integrating, we get

$$(5)\ \frac{k}{2^{n-1}}(-1)^s\pi^2\frac{2n!}{(n-s)!(n+s)!}=(-1)^s\frac{(2\pi)^3n!n!}{(2n+1)(n-s)!(n+s)!}$$

In order to obtain expansions of harmonics in polar co-ordinates,

$$\text{put}\ \begin{matrix}\xi=x+\iota y\\ \eta=x-\iota y\end{matrix}\Big\}\quad(x^2+y^2+z^2=1).$$

$$(ax+by+cz)^n\ \text{becomes}\ \left(\frac{a(\xi+\eta)}{2}+\frac{b(\xi-\eta)}{2\iota}+cz\right)^n$$

$$=\left(\frac{\iota c}{2}\right)^n(\alpha\xi+\frac{1}{\alpha}\eta-2\iota z)^n\ \text{say.}$$

Now taking $\alpha\xi+\frac{1}{\alpha}\eta$ as a single term, we may expand this by the Binomial Theorem, and pick out those terms in which the power indices of ξ and η differ by s.

Doing this we get $(\alpha\xi+\frac{1}{\alpha}\eta-2\iota z)^n$

$$=\Sigma\{(ae^{\iota\phi})^s+(ae^{\iota\phi})^{-s}\}\frac{n!}{s!(n-s)!}(-2\iota)^{n-s}\theta_n^{(s)}$$

$\theta_n^{(s)}$ being the function given in Thomson and Tait's Natural Philosophy, Vol. I., p. 205. Taking as a particular case the form $(\iota x\cos\alpha+\iota y\sin\alpha+z)^n$, we easily find from this

$$\int_0^{2\pi}(\overline{\iota x\cos\alpha+y\sin\alpha}+z)^n\cos s\alpha d\alpha=\frac{\iota^{2n-s}}{2^{s-1}}\frac{n!}{s!(n-s)!}\theta_n^{(s)}\cos s\phi$$

Another mode of expansion leads to other well-known forms. Observe $(\alpha\xi+\frac{1}{\alpha}\eta-2\iota z)^n$ may be written in either of the forms

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$$\frac{1}{(a\xi)^n}\left\{1-(z+\iota a\xi)^2\right\}^n$$

$$\left(\frac{a}{\eta}\right)^n\left\{1-\left(z+\frac{\iota\eta}{a}\right)^2\right\}^n$$

Expanding these in powers of a by Taylor's Theorem, we get the expressions for the harmonics in terms of differential co-efficients of $(1-z^2)^n$.

The expansion of the Biaxal surface Harmonic may be deduced at once from (4) and (5).

The definite integral expressions for the elementary harmonics are particular cases of expressions as the sum of a finite number of terms.

Thus let $V=(\iota x\cos a+\iota y\sin a+z)^n=\frac{1}{2}H_0+H_1\cos a+H_2\cos 2a+\ldots.$
$+K_1\sin a+K_2\sin 2a+\ldots.$

and take a_1, $a_2,\ldots\ldots a_p$ a series of angles in equi-different progression, the common difference being $2\pi/p$.

Then $V_1=(\iota x\cos a_1+\iota y\sin a_1+z)^n=\frac{1}{2}H_0+H_1\cos a_1+H_2\cos a_2+$ &c., and it is easy to show that

$$V_1\cos sa_1+V_2\cos a_2+\ldots\ldots V_p\cos sa_p=\frac{p}{2}H_s.$$

the other terms on the right hand side vanishing, provided p is not less than $2n+1$.

In this way we may express any harmonic as a sum of sectorial harmonics.

In particular,

$$V_1+V_2+\ldots V_p=\tfrac{1}{2}pH_0.$$

Taking $p=\infty$, we get the definite integral

$$\frac{1}{2\pi}\int_0^{2\pi}(\iota x\cos a+\iota y\sin a+z)^n da$$

as before.

We may interpret this integral in a way that will lead us naturally to one or two others of similar form.

Suppose we have a function of x, y, z, the variables being connected by a relation, say $x^2+y^2+z^2=\rho^2$; so that to any particular value of z correspond in general an infinite system of values of x, y, say x_1y_1, x_2y_2, &c.

Then, obviously, the sum of *all* the values which the function takes for this particular value of z, say $f(x_1, y_1, z)+f(x_2, y_2, z)+$ &c.

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(multiplied by a suitable infinitesmal) is a function of z and ρ alone. In particular cases, of course, it may be zero, or infinite.

To treat the matter perfectly generally, we should have to consider x, y, z as complex variables, but we may get a sufficient view under the following restrictions, viz.: we consider z as real, but x and y capable of taking values either *purely* real or purely imaginary.

We may consider surface values, *i.e.*, suppose $x^2+y^2+z^2=1$.

Take then the harmonics $(z+\iota x)^n$, $(z+\iota x)^{-n-1}$, both of which by (1) give us surface harmonics of the n^{th} degree, and we get the following cases:—

I. $z<1$

(i) $x=\sqrt{1-z^2}\cos a$
$y=\sqrt{1-z^2}\sin a$

Harmonics are $\displaystyle\int_0^{2\pi}(z+\iota\sqrt{1-z^2}\cos a)^n da$

$$\int_0^{2\pi}(z+\iota\sqrt{1-z^2}\cos a)^{-n-1}\,da$$

(ii) $x=\sqrt{1-z^2}\cosh a$
$y=\sqrt{1-z^2}\,\iota\sinh a$

Harmonic $\displaystyle\int_{-\infty}^{+\infty}(z+\iota\sqrt{1-z^2}\cosh a)^{-n-1}da$

II. $z>1$

(i) $x=\iota\sqrt{z^2-1}\cos a$
$y=\iota\sqrt{z^2-1}\sin a$

Harmonic $\displaystyle\int_0^{2\pi}(z-\sqrt{z^2-1}\cos a)^n da$

$$\int_0^{2\pi}(z-\sqrt{z^2-1}\cos a)^{-n-1}da$$

(ii) $x=\iota\sqrt{z^2-1}\cosh a$
$y=\sqrt{z^2-1}\sinh a$

Harmonic $\displaystyle\int_{-\infty}^{+\infty}(z-\sqrt{z^2-1}\cosh a)^{-n-1}da.$

These are well known expressions for Zonal Harmonics of the first and second kinds. By similar considerations we might also obtain integrals for the harmonics which are not zonal.

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We go on to sketch briefly the application of the preceding methods to functions of p variables.

The function $v \equiv (a_1x_1 + a_2x_2 + \ ..a_px_p)^n$ satisfies the equation $\frac{d^2v}{dx_1^2} + \frac{d^2v}{dx_2^2} + \ldots + \frac{d^2v}{dx_p^2} = 0$, provided $a_1^2 + a_2^2 + \ldots + a_p^2 = 0$.

The fundamental proposition about the integral of the product of two harmonics can be extended to the general case. To avoid circumlocution, we speak of points, lines, surfaces, volumes, by a well understood extension of the language of Analytical Geometry.

Now we may extend Green's Theorem to the case of p variables, the proof following exactly the same lines as in the case of three variables. The only difficulty is about the meaning of the element of surface, but we may define it in this way.

Let the surface be $F(x_1, x_2 \ldots) = 0$, and transform the variables linearly and orthogonally to a new set ξ_1, ξ_2, &c., of which $\xi_1 = l_1x_1 + l_2x_2 +$ &c., where l_1, l_2, &c., are proportional to the values of $\frac{dF}{dx_1}$, $\frac{dF}{dx_2}$, &c., at the point x_1', x_2', &c., and are such that $l_1^2 + l_2^2 + \ldots = 1$.

We have $d\xi_1 d\xi_2 \ldots d\xi_p = dx_1 dx_2 \ldots dx_p$, and we take $d\xi_2 \ldots d\xi_p$ as the element of surface. We have then, putting dV for $dx_1dx_2 \ldots dx_p$, and dS for the element of surface

$$\int\left(\frac{du}{dx_1}\frac{dv}{dx_1} + \ldots\right)dV = \int v\frac{du}{d\xi_1}dS - \int v\left(\frac{d_2u}{dx_1^2} + \ldots\right)dV$$
$$= \int u\frac{dv}{d\xi_1}dS - \int u\left(\frac{d^2v}{dx_1^2} + \ldots\right)dV_1$$

We may, as before, apply this to finding the surface integral of the product of two harmonics $(a_1x_1 + a_2x_2 + \ldots)^n$, $(b_1x_1 + b_2x_2 + \ldots)_n$ over the surface $x_1^2 + x_2^2 + \ldots = \rho^2$.

The only difference in the general case is that the element of surface varies as r^{p-1} instead of as r^2. Unless $m = n$, the integral vanishes. If $m = n$, the result is

(remembering that $\frac{d}{d\rho}\int\int \ldots\ldots dx_1dx_2 \ldots dx_p = \frac{p\rho^{p-1}\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2}+1)}$),

$$(6) \quad \rho^{2n+p-1} \cdot \frac{\pi^{\frac{p}{2}}\Gamma(n+1)}{2^n\Gamma(n+\frac{p}{2})}(a_1b_1 + a_2b_2 \ldots)^n.$$

Let us now find an expression for the rational integral harmonic, which is a function of x_p and $x_1^2 + x_2^2 + \ldots + x_p^2$.

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If $f_1, f_2, \ldots f_{p-1}$ be real quantities such that the sum of their squares $=1$, then $(f_1x_1+f_2x_2+\ldots\ldots+f_{p-1}x_{p-1}+\iota x_p)^n$ is a harmonic, and if we take the sum of functions of this form for all real values of the f's such that the sum of their squares $=1$, we shall get what we want. This mode of derivation of the symmetrical harmonic is somewhat different from our former method, but it is easily seen to lead to the same result.

Consider first the integral

$$\iint\ldots(f_1x_1+\ldots\ \ldots+f_{p-1}x_{p-1}+\iota x_p)^n df_1\ldots\ldots df_{p-1}$$

with the integration extending to all real values of the f's for which the sum of their squares is not greater than ρ^2.

Change the variables f_1, f_2, &c., by a linear and orthogonal substitution to a new set, ϕ_1, ϕ_2, &c., of which

$$\sqrt{(x_1^2+\ldots+x_{p-1}^2)}\phi_1=f_1x_1+\ldots\ldots+f_{p-1}x_{p-1}$$

The integral is then

$$\iint\cdots(\sqrt{x_1^2+\ldots x_{p-1}^2}\phi_1+\iota x_p)^n d\phi_1 d\phi_2\ldots d\phi_{p-1}$$

and the limits are not changed.

Integrating with respect to $\phi_2\ldots\phi_{p-1}$ first, and disregarding numerical factors, we get

$$\int_{-\rho}^{+\rho}(\rho^2-\phi_1^2)^{\frac{p-2}{2}}(\sqrt{x_1^2+\ldots x_{p-1}^2}\phi_1+\iota x_p)^n d\phi_1.$$

Now differentiate this with respect to ρ, and in the result put $\rho=1$.

This evidently gives us what we started to find.

If we put $\phi_1=\cos\theta$, the result takes the form

$$\int_0^\pi(x_p+\iota\sqrt{x_1^2+x_2^2+\ldots x_{p-1}^2}\cos\theta)^n\sin^{p-3}\theta d\theta.$$

Now the surface integral of the product of this and a harmonic $(a_1x_1+\ldots+a_px_p)^n$ is obtained by substituting a_1 for x_1, a_2 for x_2, &c. in the above expression and multiplying by a numerical factor K.

This integral is $\therefore$ $\mathrm{K}(a_p)^n\int_0^\pi(1-\cos\theta)^n\sin^{p-3}\theta d\theta=\mathrm{M}(a_p)^n$, where M is a numeric, easily found.

By adding any number of harmonics of the typical form, we get a result which we may write

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(7) $\int Q_n V_n dS = MV_n'$, *where* V_n' is the value of V_n at the pole of Q.

The surface value of the above symmetrical harmonic is

$$\int_0^{\pi}(x + \sqrt{x^2 - 1}\cos\theta)^n \sin^{p-3}\theta d\theta.$$

We may consider this as a function of the single variable x, and it is easy from the above results to show that

$$\int_{-1}^{+1} Q_n Q_m (1 - x^2)^{\frac{p-3}{2}} dx = 0$$

when m, n are different, but = a certain finite numerical quantity, easily found, when $m = n$.

The functions we have arrived at include, of course, as particular cases, simple and zonal Harmonic Functions. They are discussed at the end of Vol. I. of Heine's *Kugelfunctionen*.

The method of this paper is not mentioned in Heine, but I have found since I had it worked out that it is not new. In Professor Cayley's collected Works, Vol. I., page 397, will be found a short paper in which he proves (6) and (7). His proofs, however, are quite different from those I had arrived at, and have given above.

A Method of Teaching Electrostatics in School.

By J. T. Morrison, M.A., B.Sc.

The object of the paper was to suggest for the teaching of electrostatics a leading idea, which should *readily* co-ordinate all the facts, introduce no misleading inferences, and guide the course of learners in the direction of the most recent investigations—in all which respects the notion of attraction and repulsion is at least a partial failure. The leading idea or fact referred to is, that almost all electrostatic distributions, however complex, can be analysed into one or more repetitions of a certain simple system, which is called in the paper "an electrostatic system," and which may be described as follows :—Two equally and oppositely electrified conducting surfaces, facing each other, separated by any dielectric, and insulated from each other. A complete study of one system of this kind, and

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