GENERATING FUNCTIONS FOR BESSEL AND RELATED POLYNOMIALS

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1. Introduction. Krall and Frink [**4**] aroused interest in what they term Bessel polynomials. They studied in some detail what may, in hypergeometric form, be written as

$$(1) \qquad y_n(x, a, b) = {}_2F_0(-n, a-1+n; —; -x/b).$$

The simple Bessel polynomial is the special case $a = b = 2$. In essence the polynomials considered by Krall and Frink, and by others [**1**; **3**], are the terminating ${}_2F_0$'s with numerator parameters $(-n)$ and $(c+n)$, n a non-negative integer. We shall therefore work with

$$(2) \qquad {}_2F_0(-n, c+n; —; -x)$$

for brevity. From (2), the polynomial (1) is easily obtained.

Burchnall [**1**] produced a convergent generating function for $y_n(x, a, b)$. It is the purpose of this note to obtain relations which yield in simple special instances two generating functions, one of them Burchnall's, for Bessel polynomials.

2. Generating functions of Burchnall type. Consider the polynomials

$$(3) \quad \psi_n(x) = {}_{q+2}F_p\left[\begin{matrix} -n, c+n, 1-\beta_1-n, 1-\beta_2-n, \ldots, 1-\beta_q-n; \\ 1-\alpha_1-n, 1-\alpha_2-n, \ldots, 1-\alpha_p-n; \end{matrix} \; (-1)^{p+q+1}x\right]$$

in which the c and all the α's and β's are independent of n, and in which n is a non-negative integer. The α's and β's are not to be non-positive integers, but are otherwise unrestricted.

If in (3) no α's and no β's are used, the resulting polynomial is that of (2), the generalized Bessel polynomial.

We now prove that

$$(4) \quad (1-4xt)^{-\frac{1}{2}}\left[\frac{2}{1+\sqrt{(1-4xt)}}\right]^{c-1} {}_pF_q\left[\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q;\end{matrix} \; \frac{2t}{1+\sqrt{(1-4xt)}}\right] = \sum_{n=0}^{\infty} \frac{\psi_n(x)(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_p)_n t^n}{n!\,(\beta_1)_n(\beta_2)_n \ldots (\beta_q)_n},$$

in which $(1-4xt)^{\frac{1}{2}} \to 1$ as $t \to 0$. Burchnall's [**1**] generating function is a

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special case of (4) in which there are no α's and no β's, x is to be replaced by (x/b) and t by $(\frac{1}{2}bt)$. In Burchnall's paper the misprint, $\{\frac{1}{2} - \frac{1}{2}(1 - 2xt)^{\frac{1}{2}}\}^{2-a}$ where $\{\frac{1}{2} + \frac{1}{2}(1 - 2xt)^{\frac{1}{2}}\}^{2-a}$ was intended, should be corrected.

For our proof of (4) we need a simple result from the theory of the ${}_2F_1$.

LEMMA.

$$ {}_2F_1(\gamma, \gamma + \tfrac{1}{2}; 2\gamma; 4z) = (1 - 4z)^{-\frac{1}{2}}\left[\frac{2}{1 + \sqrt{(1 - 4z)}}\right]^{2\gamma-1}. $$

To obtain the result in the lemma, use Gauss's formula

$$ {}_2F_1(a, b; 2b; 4x(1 + x)^{-2}) = (1 + x)^{2a}\, {}_2F_1(a, a + \tfrac{1}{2} - b; b + \tfrac{1}{2}; x^2), $$

for which see Magnus and Oberhettinger [**5**], put $b = \gamma$, $s = \gamma + \frac{1}{2}$, $x(1 + x)^{-2} = z$, and use the fact that a ${}_1F_0$ is a binomial.

Since $(1 - \alpha - n)_k = (-1)^k(\alpha)_n/(\alpha)_{n-k}$, it follows that

$$ \frac{\psi_n(x)(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_p)_n}{n!\,(\beta_1)_n(\beta_2)_n \ldots (\beta_q)_n} = \sum_{k=0}^{n} \frac{(c)_{n+k}(\alpha_1)_{n-k}(\alpha_2)_{n-k} \ldots (\alpha_p)_{n-k}\, x^k}{(c)_n(\beta_1)_{n-k}(\beta_2)_{n-k} \ldots (\beta_q)_{n-k}(n - k)!\,k!}. $$

Therefore,

$$ \begin{aligned} \sum_{n=0}^{\infty} & \frac{\psi_n(x)(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_p)_n t^n}{n!\,(\beta_1)_n(\beta_2)_n \ldots (\beta_q)_n} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(c)_{n+k}(\alpha_1)_{n-k}(\alpha_2)_{n-k} \ldots (\alpha_p)_{n-k}\, x^k t^n}{(c)_n(\beta_1)_{n-k}(\beta_2)_{n-k} \ldots (\beta_q)_{n-k}(n - k)!\,k!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(c)_{n+2k}(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_p)_n\, x^k t^{n+k}}{(c)_{n+k}(\beta_1)_n(\beta_2)_n \ldots (\beta_q)_n\, n!\,k!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(c + n)_{2k}(xt)^k}{(c + n)_k\, k!} \cdot \frac{(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_p)_n t^n}{(\beta_1)_n(\beta_2)_n \ldots (\beta_q)_n\, n!} \\ &= \sum_{n=0}^{\infty} {}_2F_1(\tfrac{1}{2}c + \tfrac{1}{2}n, \tfrac{1}{2}c + \tfrac{1}{2}n + \tfrac{1}{2}; c + n; 4xt)\, \frac{(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_p)_n\, t^n}{(\beta_1)_n(\beta_2)_n \ldots (\beta_q)_n\, n!} \\ &= \sum_{n=0}^{\infty} (1 - 4xt)^{-\frac{1}{2}}\left[\frac{2}{1 + \sqrt{(1 - 4xt)}}\right]^{c+n-1} \frac{(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_p)_n\, t^n}{(\beta_1)_n(\beta_2)_n \ldots (\beta_q)_n\, n!} \\ &= (1 - 4xt)^{-\frac{1}{2}}\left[\frac{2}{1 + \sqrt{(1 - 4xt)}}\right]^{c-1} {}_pF_q\left[\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q;\end{matrix}\ \frac{2t}{1 + \sqrt{(1 - 4xt)}}\right], \end{aligned} $$

which completes the derivation of equation (4). The method used is nothing like that in Burchnall's work.

If $p \leqslant q + 1$, as is true for the generalized Bessel polynomials, the generating function series, the ${}_pF_q$, even has a region of convergence near $t = 0$. For generating function purposes the luxury of convergence is, fortunately, not at all necessary.

3. Another generating function for Bessel polynomials. Consider next the polynomials

$$\sigma_n(x) = {}_{p+2}F_q\left[\begin{matrix} -n, c+n, \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} -x\right], \tag{5}$$

of which the Bessel polynomials are again a special instance, namely that in which no α's and no β's are used. This extension of the generalized Bessel polynomials is, of course, distinct from that in equation (3).

We shall obtain the generating function relation

$$(1-t)^{-c}{}_{p+2}F_q\left[\begin{matrix} \tfrac{1}{2}c, \tfrac{1}{2}c+\tfrac{1}{2}, \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} \frac{4xt}{(1-t)^2}\right] = \sum_{n=0}^{\infty} \frac{\sigma_n(x)(c)_n t^n}{n!}, \tag{6}$$

which contains a divergent generating function, a ${}_2F_0$, for the Bessel polynomials, namely the choice of no α's and no β's. The series in (6) converge near $t = 0$ if $p \leqslant q-1$.

We find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\sigma_n(x)(c)_n t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-n)_k(c+n)_k(c)_n(\alpha_1)_k(\alpha_2)_k \ldots (\alpha_p)_k(-x)^k t^n}{(\beta_1)_k(\beta_2)_k \ldots (\beta_q)_k\, k!\, n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(c)_{n+k}(\alpha_1)_k(\alpha_2)_k \ldots (\alpha_p)_k\, x^k t^n}{(\beta_1)_k(\beta_2)_k \ldots (\beta_q)_k\, k!\, (n-k)!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(c)_{n+2k}(\alpha_1)_k(\alpha_2)_k \ldots (\alpha_p)_k\, x^k t^{n+k}}{(\beta_1)_k(\beta_2)_k \ldots (\beta_q)_k\, k!\, n!} \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(c+2k)_n\, t^n (c)_{2k}(\alpha_1)_k(\alpha_2)_k \ldots (\alpha_p)_k(xt)^k}{n!\, (\beta_1)_k(\beta_2)_k \ldots (\beta_q)_k\, k!} \\
&= \sum_{k=0}^{\infty} \frac{(c)_{2k}(\alpha_1)_k(\alpha_2)_k \ldots (\alpha_p)_k\, (xt)^k}{(\beta_1)_k(\beta_2)_k \ldots (\beta_q)_k\, k!\, (1-t)^{c+2k}} \\
&= (1-t)^{-c}{}_{p+2}F_q\left[\begin{matrix} \tfrac{1}{2}c, \tfrac{1}{2}c+\tfrac{1}{2}, \alpha_1, \alpha_2, \ldots \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} \frac{4xt}{(1-t)^2}\right],
\end{aligned}$$

which completes the derivation of equation (6).

For $c = 1$, the equivalent of equation (6) has already appeared in Sister Celine's work [**2**].

References

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