BULL. AUSTRAL. MATH. SOC. 16A70
VOL. 8 (1973), 181-185. (16A38)

A generalization of a theorem of Wedderburn

Steve Ligh

Outcalt and Yaqub have extended the Wedderburn Theorem which states that a finite division ring is a field to the case where R is a ring with identity in which every element is either nilpotent or a unit. In this paper we generalize their result to the case where R has a left identity and the set of nilpotent elements is an ideal. We also construct a class of non-commutative rings showing that our generalization of Outcalt and Yaqub's result is real.

1. Introduction

Wedderburn's Theorem, asserting that a finite division ring is necessarily commutative, has been generalized in several directions [*1*, *6*, *7*, *8*]. A survey on a few papers concerning commutativity theorems for rings revealed that the two non-commutative rings defined on the Klein group $(G, +)$ have been cited quite often as counterexamples to show that certain hypotheses cannot be deleted. In particular, one of them has the following multiplication: if $x \neq 0$ is in G, then $xg = 0$ or $xg = g$ for all g in G. One of the purposes of this note is to characterize the class of abelian groups which admit such multiplications. Then this class of rings is used to obtain a generalization of a theorem of Outcalt and Yaqub [*8*].

2. A class of non-commutative rings

DEFINITION. Let $(G, +)$ be an abelian group and H a proper subset

Received 12 October 1972.

of G such that $0 \notin H$. Define a multiplication $*$ on G as follows: $h * g = g$ and $x * g = 0$ for each $h \in H$, $x \in G-H$ and $g \in G$. The multiplication $*$ is called "trivial" if and only if $(G, +, *)$ is a ring.

We now give a complete characterization of the class of abelian groups which admit a trivial multiplication.

THEOREM 1. *An abelian group* $(G, +)$ *admits a trivial multiplication if and only if each element of* G *is of order two.*

Proof. Suppose $(G, +, *)$ is a ring and $*$ is trivial. Then there is $h \neq 0$ in G such that $hg = g$ for each g in G. Let $y \neq 0$ be an element of G. Since $(h+h)y = 0$ or y, it follows that $(h+h)y = 0$. Hence $0 = (h+h)y = y + y$. Thus each element of $(G, +)$ is of order two.

Conversely, let $(G, +)$ be an abelian group of which each element is of order two. Thus G can be considered as a vector space over Z_2. Hence G has a basis B. For each $x \neq 0$ in G, there is a positive integer n such that $x = b_1 + b_2 + \ldots + b_n$, $b_i \in B$. Let $H = \{x \in G : n \text{ is odd}\}$. Define a multiplication $*$ on G as follows: if $h \in H$, $h * g = g$ for each $g \in G$. For $x \in G-H$, $x * g = 0$ for each $g \in G$. It follows from [5] or can be verified easily that $(G, +, *)$ satisfies all the axioms of a ring except perhaps the right distributive law. That the right distributive law also holds can be checked easily. Hence $(G, +, *)$ is a ring and the multiplication defined is trivial. This completes the proof of the theorem.

Observe that the class of rings just constructed has the following properties: the set of nilpotent elements is an ideal, left identities exist, $(xy)^n = x^n y^n$ for each element x and y. This class of non-commutative rings serves as counterexamples to show that certain hypotheses of various commutativity theorems cannot be omitted. For example, see [3, 4, 6, 7, 8].

3. A commutativity theorem

Many generalizations of the famous Wedderburn's Theorem have appeared

https://doi.org/10.1017/S0004972700042416 Published online by Cambridge University Press

recently. In [8] Outcalt and Yaqub provided the following:

THEOREM (Outcalt and Yaqub). *Let* R *be a ring with identity in which each element is either nilpotent or a unit in* R . *Then*

(a) the set N *of nilpotent elements in* R *is an ideal;*

(b) if (i) R/N *is finite and (ii)* $x \equiv y \pmod N$ *implies that* $x^2 = y^2$ *or both* x *and* y *commute with all elements of* N ,

then R *is commutative.*

We now extend the above result to a much larger class of rings. We simply assume the ring has a left identity and the set of nilpotent elements is an ideal. But first we state another generalization of Wedderburn's Theorem given by Herstein [1].

THEOREM (Herstein). *Let* R *be a ring such that for every element* x *in* R *there exists an integer* $n = n(x)$, *and a polynomial* $P(t) = P_x(t)$ *with integer coefficients, such that* $x^{n+1}P(x) = x^n$. *If all the nilpotent elements of* R *are in the center of* R , *then* R *is commutative.*

THEOREM 2. *Let* R *be a ring with a left identity* e *and let the set of nilpotent elements be an ideal. If*

(i) R/N *is finite and*

(ii) $x \equiv y \pmod N$ *implies that* $x^2 = y^2$ *or both* x *and* y *commute with all elements of* N ,

then R *is commutative.*

Proof. First we show that e is also a right identity by demonstrating that e is unique. Suppose there exists w in R such that $wr = r$ for each r in R . Since R/N is commutative, we see that $ew - we = w - e$ is an element of N . Thus $w \equiv e \pmod N$ implies that $w = w^2 = e^2 = e$ or both w and e commute with all elements of N . In the latter case, we see that $e(w-e) = (w-e)e$ implies that $e = w$. Since e is unique it follows [2, p. 55] that e is the identity of R .

Next we wish to show that each element of N is in the center of R . Since R/N is finite, it is a direct sum of fields:

https://doi.org/10.1017/S0004972700042416 Published online by Cambridge University Press

$$R/N = R_1/N \oplus R_2/N \oplus \ldots \oplus R_j/N \ .$$

Using Lemmas 1 and 2 in [8], we see that if $b_i + N$ is in R_i/N, then $ab_i = b_ia$ for each a in N. Now let $a \in N$ and $b \in R$. Then $b = b_1 + b_2 + \ldots + b_j + n$, $n \in N$. Thus $ab = ba$, since by Lemma 1 in [8] N is a commutative subring of R and $ab_i = b_ia$ for $i = 1, \ldots, j$. This shows that N is a subset of the center of R.

Since R/N is finite and has no nonzero nilpotent elements, it follows that for each x in R there is an integer $n = n(x)$ such that $x^n - x$ is in N. Hence there is an integer k such that $x^k = x^{k+1}P(x)$. Now by Herstein's Theorem R is commutative.

Recall that the class of non-commutative rings constructed in Section 2 satisfies the hypotheses of Theorem 2 except *(ii)*. Hence we see that Outcalt and Yaqub's result is a corollary of our theorem.

Finally we remark that in Theorem 2 one can assume a right identity instead of a left identity. However, it is not known whether or not one can drop that assumption.

References

[1] I.N. Herstein, "A note on rings with central nilpotent elements", *Proc. Amer. Math. Soc.* 5 (1954), 620.

[2] Nathan Jacobson, *Lectures in abstract algebra, Volume I* (Van Nostrand, New York, Toronto, London, 1951).

[3] E.C. Johnsen, D.L. Outcalt and Adil Yaqub, "An elementary commutativity theorem for rings", *Amer. Math. Monthly* 75 (1968), 288-289.

[4] J. Luh, "A commutativity theorem for primary rings", *Acta Math. Acad. Sci. Hungar.* 22 (1971), 211-213.

[5] J.J. Malone, Jr, "Near-rings with trivial multiplications", *Amer. Math. Monthly* 74 (1967), 1111-1112.

https://doi.org/10.1017/S0004972700042416 Published online by Cambridge University Press

[6] H.G. Moore, "On commutativity in certain rings", *Bull. Austral. Math. Soc.* 2 (1970), 107-115.

[7] D.L. Outcalt and Adil Yaqub, "A generalization of Wedderburn's Theorem", *Proc. Amer. Math. Soc.* 18 (1967), 175-177.

[8] D.L. Outcalt and Adil Yaqub, "A commutativity theorem for rings", *Bull. Austral. Math. Soc.* 2 (1970), 95-99.

Department of Mathematics,
University of Southwestern Louisiana,
Lafayette,
Louisiana,
USA.

https://doi.org/10.1017/S0004972700042416 Published online by Cambridge University Press