Canad. Math. Bull. Vol. **48** (3), 2005 pp. 409–413

The Existence of Universal Inner Functions on the Unit Ball of $\mathbb{C}^n$

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Abstract. It is shown that there exists an inner function I defined on the unit ball $\mathbf{B}^n$ of $\mathbb{C}^n$ such that each function holomorphic on $\mathbf{B}^n$ and bounded by 1 can be approximated by "non-Euclidean translates" of I.

In 1929, G. D. Birkhoff [3] established the existence of an entire function g with the remarkable property that its translates are dense in the space of all entire functions. That is, for each entire function f, there is a sequence $\{a_j\}$ such that $g(z + a_j) \to f(z)$ uniformly on compact sets. W. Seidel and J. L. Walsh [8] proved an analogous result in the unit disc, replacing Euclidean translations by non-Euclidean ones. Such functions, whose "translates" are dense, came to be called universal functions (see the survey [2]). In 1955, M. Heins [6] showed the existence of a Blaschke product which is universal in the family of functions holomorphic in the unit disc which are bounded by 1. There is no difficulty in extending the results of Birkhoff, Seidel and Walsh to several variables, but the result of Heins is quite different.

Let $H(\mathbf{B}^n)$ denote the space of holomorphic functions in the open unit ball $\mathbf{B}^n$ of $\mathbb{C}^n$, endowed with the topology of uniform convergence on compact subsets. The subfamily consisting of functions bounded by 1 will be called the closed unit "*ball*" of $H(\mathbf{B}^n)$, which we place in quotation marks for two reasons. First of all, $H(\mathbf{B}^n)$ is not a normed space. Secondly, two balls recur in this paper: the "ball" in $H(\mathbf{B}^n)$ and the ball $\mathbf{B}^n$ in $\mathbb{C}^n$ In 1979, P. S. Chee [4] proved that there is a universal function with respect to the closed unit "ball" of $H(\mathbf{B}^n)$, that is, a function in the "ball" whose non-Euclidean translates are dense in the "ball". Since then, it seems there has been no improvement of Chee's result. In this note, we apply A. B. Aleksandrov's density theorem [1] to show that there is such a universal function which is in fact *inner* .

We recall that a function in the "ball" of $H(\mathbf{B}^n)$ has radial limits at almost every point of the unit sphere $\mathbf{S}^n$ and such a function is said to be *inner* if these radial limits are almost surely of modulus 1.

Inner functions in one variable, which were conceived and nurtured by Seidel and baptized by Beurling, have played a fundamental role in the study of the Hardy spaces H^p. They are also of interest in engineering in the guise of H^∞ control theory. Our motivation in ferreting out a universal function in higher dimensions is threefold. First, we are fascinated by the universality phenomenon itself. Secondly, there was a time (not so long ago) when (non-constant) inner functions on the ball $\mathbf{B}^n$, $n > 1$,

Received by the editors April 30, 2002.
Research supported by NSERC(Canada)
AMS subject classification: Primary: 32A35, 30D50; secondary: 47B38.
Keywords: universal inner functions.

were thought not to exist. This was repudiated by Aleksandrov who showed that there is in fact a wealth of inner functions on the ball (see below). Our quest strongly supports this thesis. Thirdly, there is a need (and a hope) for extending H^∞ control theory to higher dimensions (see for example [7]). The natural role of inner functions in such applications seems undiminished in passing from one to several dimensions.

Here is our result.

Theorem *There exists an inner function I on $\mathbf{B}^n$ and a sequence Φ_k of automorphisms of $\mathbf{B}^n$ such that to each function f in the closed unit "ball" of $H(\mathbf{B}^n)$ there corresponds a subsequence of $\{I \circ \Phi_k\}$ which converges to f.*

The proof of the theorem will be completed by a combination of the following five lemmas. The first lemma will be invoked repeatedly. It follows easily from a normal family argument and the maximum principle.

Lemma 1 *If a sequence of holomorphic functions on a domain is bounded by M and converges at some point to a value of modulus M, then the sequence converges to this value uniformly on compact subsets of the domain.*

Let ζ lie on the boundary $\mathbf{S}^n$ of $\mathbf{B}^n$ and let $\epsilon > 0$. By an ϵ-lens neighbourhood of the radius at ζ, we mean an ϵ-neighbourhood of this radius in the Bergman distance. In this note, we shall say that a mapping F, holomorphic in $H(\mathbf{B}^n)$, has non-tangential limit η at $\zeta \in \mathbf{S}^n$ if $F(z)$ tends to η as z approaches η in each lens neighbourhood of the radius at ζ.

Since the non-euclidean distance is contracted by holomorphic mappings, we have the following multidimensional analog of the Lindelöf Theorem.

Lemma 2 *Let F be a holomorphic mapping from $\mathbf{B}^n$ into $\mathbf{B}^m$. If at some $\zeta \in \mathbf{S}^n$, F has a radial limit $\eta \in \mathbf{S}^m$, then F also has non-tangential limit η at ζ.*

We also require a density lemma, which follows from a theorem of Aleksandrov.

Lemma 3 *There exists a sequence I_k of inner functions on $\mathbf{B}^n$ such that, to each function f in the closed unit "ball" of $H(\mathbf{B}^n)$, corresponds a subsequence of I_k which converges to f.*

Indeed, by a result of Aleksandrov [1] (see also [2]), the inner functions are dense in the closed unit "ball" of $H(\mathbf{B}^n)$. Since the latter space is separable (with respect to the compact-open topology), the lemma follows.

For convenience, we shall write $\pm\mathbf{1} = (\pm 1, 0, \dots, 0)$. Since each of the inner functions I_k has radial limits of modulus 1 almost everywhere on the unit sphere $\mathbf{S}^n$ and the unitary group acts transitively on $\mathbf{S}^n$, we may, by precomposing with a small unitary rotation, assume that I_k has radial limits of modulus 1 at the points $\pm\mathbf{1}$. We denote these limits respectively by $I_k(\pm\mathbf{1})$. From Lemma 2, these same two limits are also attained non-tangentially.

Next, we shall modify the inner functions I_k so that they have the limit 1 at $\pm\mathbf{1}$. For this, we need the following one-variable result.

https://doi.org/10.4153/CMB-2005-038-4 Published online by Cambridge University Press

Lemma 4 *Given complex numbers $b, c \in \mathbf{B}^1$ and $\alpha^{\pm}, \eta^{\pm} \in \mathbf{S}^1$, with $\eta^+ = \eta^-$ if $\alpha^+ = \alpha^-$, there exists φ which is holomorphic in $\overline{\mathbf{B}}^1$, is inner, and satisfies*

$$\varphi(b) = c, \quad \varphi(\alpha^+) = \eta^+, \quad \varphi(\alpha^-) = \eta^-.$$

We may suppose that $b = 0 = c$. If $\eta^+/\alpha^+ = \eta^-/\alpha^-$, we may take φ to be a rotation. If $\eta^+/\alpha^+ \neq \eta^-/\alpha^-$, we may take $\varphi(z) = z\psi(z)$, where ψ is an automorphism of $\mathbf{B}^1$ with $\psi(\alpha^+) = \eta^+/\alpha^+$ and $\psi(\alpha^-) = \eta^-/\alpha^-$.

Let I_0 be a non-constant inner function on $\mathbf{B}^n$ having non-tangential limits at $\pm\mathbf{1}$ which we denote by $\alpha^{\pm}$ and set $b = I_0(0)$. In Lemma 4, for each natural number k let

$$c = 1 - 2^{-k}, \quad \eta^+ = \frac{1}{I_k(+\mathbf{1})}, \quad \eta^- = \frac{1}{I_k(-\mathbf{1})},$$

and φ_k be the associated function. Furthermore, set $\Psi_k = \varphi_k \circ I_0$. Then $|\Psi_k| \leq 1$ and $\Psi_k(0) = 1 - 2^{-k}$. Hence, Lemma 1 tells us that the sequence $\{\Psi_k\}$ converges to 1 uniformly on compact subsets of $\mathbf{B}^n$. Now set $G_k = \Psi_k \cdot I_k$. Then $G_k(\pm\mathbf{1}) = 1$ and to each function f in the closed unit "ball" of $H(\mathbf{B}^n)$, corresponds a subsequence of G_k which converges to f in the compact-open topology of $\mathbf{B}^n$.

Let r and s be nonnegative real numbers with $r^2 + s^2 = 1$ and put

$$\Phi(z) = (\phi_1(z), \ldots, \phi_n(z))$$

where

$$\phi_1(z) = \frac{z_1 + r}{1 + rz_1}, \quad \phi_j(z) = \frac{sz_j}{1 + rz_1}, \quad 2 \leq j \leq n.$$

Then Φ is an automorphism of $\mathbf{B}^n$, and Φ^{-1} can be obtained by replacing r by $-r$ in the definition of Φ. Now let r_k and s_k be sequences of positive numbers with $r_k^2 + s_k^2 = 1$ and let Φ_k be the sequence of associated automorphisms of $\mathbf{B}^n$ as above.

We now construct an inner function meeting the requirements of the theorem. To this end, for those positive real numbers r_k and s_k with $r_k^2 + s_k^2 = 1$, set

$$J_k = \{z \in \mathbf{B}^n : \|z\| \leq r_k\}$$

and

$$V_k = \{z \in \mathbf{B}^1 : |1 - z| < 2^{-k}\}.$$

Set $g_k = G_k \circ \Phi_k^{-1}$. The sequence Φ_k^{-1} converges to $-\mathbf{1}$ uniformly on compact subsets of $\mathbf{B}^n$. Indeed, the first component converges to -1 by Lemma 1 and the other components trivially converge to zero. From Lemma 2, we have the non-tangential limits $\Phi_k^{-1}(\pm\mathbf{1}) = \pm\mathbf{1}$, $G_k(\pm\mathbf{1}) = 1$ and $g_k(\pm\mathbf{1}) = 1$. Hence, choosing r_1 arbitrarily between 0 and 1, we may inductively choose r_k increasing to 1 so rapidly that

$$g_l(r_{k+1}) \in V_{k+1}, \quad l = 1, 2, \ldots, k,$$

and

$$g_{k+1} \colon J_k \to V_{k+1}.$$

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It follows from the latter condition that the product

$$I = \prod_{j=1}^{\infty} g_j$$

converges uniformly on compact subsets of $\mathbf{B}^n$. From the following lemma, we have that I is in fact an inner function.

Lemma 5 *The product of inner functions is again an inner function.*

In Lemma 5, the product of infinitely many inner functions $\{g_k\}$ is allowed, where of course we assume that the product converges uniformly on compact sets. Once we notice that there is a common set E of measure zero on the unit circle, outside of which *each* of the functions g_k has radial limits of modulus 1, the proof of Lemma 5 proceeds in exactly the same manner as the proof that a Blaschke product is inner, and it works just as well in several variables.

There remains only to verify that some subsequence of $I \circ \Phi_k$ tends to f in the compact-open topology. In fact, it suffices to show that $G_k - I \circ \Phi_k$ converges to 0 on compact subsets of $\mathbf{B}^n$. However,

$$G_k - I \circ \Phi_k = \Bigl(1 - \prod_{j \neq k} g_j \circ \Phi_k\Bigr) G_k, \quad \Phi_k(0) = r_k,$$

and so the desired convergence will follow from Lemma 1, provided we prove

$$\lim_{k \to \infty} \prod_{j \neq k} g_j \circ \Phi_k(0) = \lim_{k \to \infty} \prod_{j \neq k} g_j(r_k) = 1.$$

Let $g_j(r_k) = 1 + u_{j,k}$. Then the conditions on g_k imply that $|u_{j,k}| < 2^{-k}$ if $j < k$ and $|u_{j,k}| < 2^{-j}$ if $j > k$. Consequently,

$$\begin{aligned} \Bigl| \prod_{j \neq k} g_j(r_k) - 1 \Bigr| &\leq \prod_{j \neq k} \bigl(1 + |u_{j,k}|\bigr) - 1 \\ &\leq (1 + 2^{-k})^{k-1} \prod_{j=k+1}^{\infty} (1 + 2^{-j}) - 1 \\ &\to 0 \quad \text{as} \quad k \to \infty. \end{aligned}$$

The proof is complete.

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https://doi.org/10.4153/CMB-2005-038-4 Published online by Cambridge University Press