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AN OSCILLATION CRITERION FOR nth ORDER NON-LINEAR DIFFERENTIAL EQUATIONS WITH FUNCTIONAL ARGUMENTS

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ABSTRACT. An oscillation criterion for an even order equation: $x^{(n)}+q(t)f(x(t)), x[g(t)])=0$ is provided. This criterion is an extension of a result established by Yeh for the second order equation $\ddot{x}+q(t)f(x(t)), x[g(t)])=0$.

Conditions are given here, under which all solutions of the equation $x^{(n)}(t)+q(t)f(x(t), x[g(t)])=0$ are oscillatory, where n is even, $n\geq 2$.

In a recent paper Cheh-Chih Yeh [1] established an oscillation criterion for the second order non-linear differential equation

$$\ddot{x}+q(t)f(x(t), x[g(t)])=0, \tag{1}$$

The purpose of this note is to extend Yeh's criterion to the following nth order equation,

$$x^{(n)}+q(t)f(x(t), x[g(t)])=0, \quad n \text{ even.} \tag{2}$$

without imposing any additional restrictions on the functions involved. Examples are provided to illustrate our results.

We assume in the sequel the following conditions due to Yeh:

(i) $q, g\in C[t_0,\infty)$, $f\in(R\times R)$, $R=(-\infty,\infty)$, and $f(y_1, y_2)$ has the sign of y_1 and y_2 when they have the same sign;

(ii) there exists a function $\sigma\in C[t_0,\infty)$ such that $\sigma(t)\leq g(t)$ and $0<k\leq \dot{\sigma}(t)\leq 1$;

(iii) there exist positive constants M and c such that $x\geq M$ implies

$$\liminf_{|y|\to\infty}\left|\frac{f(x, y)}{y}\right|\geq c>0;$$

(iv) $q(t)\geq 0$; and

$$\limsup_{t\to\infty}\frac{1}{t^{m-1}}A_m(t)=\infty,$$

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where

$$A_m(t)=\frac{1}{m!}\int_{t_0}^{t}(t-u)^{m-1}q(u)\,du$$

is the mth primitive of q for some $m>2$. Travis [9] has recently demonstrated that all solutions of (1) are oscillatory under the conditions (i)–(iii), and

(iv)′ $q(t)\geq 0$, and

$$\limsup_{t\to\infty} t\int_{t}^{\infty} q(s)\,ds=\infty.$$

Wintner [10] considered the linear differential equation

$$\ddot{x}+q(t)x=0, \tag{3}$$

and showed that the condition

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}du\int_{t_0}^{t}q(s)\,ds=\infty \tag{v}$$

is sufficient for equation (3) to be oscillatory, even when q is not assumed to be positive. Hartman [5] has shown that the limit cannot be replaced by the upper limit in the condition (v). Also, the integral criterion given in (iv) includes that of Travis [9] and the one by Wintner [10].

In what follows we consider only non-trivial solutions of (2) which are indefinitely continuable to the right. A solution $x(t)$ of (2) is said to be oscillatory if it has arbitrarily large zeros, and non-oscillatory if it is eventually of constant sign. Equation (2) is said to be oscillatory if every solution of (2) is oscillatory.

We will have an occasion to use the following Lemmas given in [4].

LEMMA 1. *Let u be a positive and n times differentiable function on $[t_0,\infty)$. If $u^{(n)}(t)$ is of constant sign and not identically zero in any interval $[t_1,\infty)$, then there exist a $t_u\geq t_0$ and an integer l, $0\leq l\leq n$ with $n+l$ even for $u^{(n)}\geq 0$ or $n+l$ odd for $u^{(n)}\leq 0$ and such that $l>0$ implies that $u^{(k)}(t)>0$ for $t\geq t_u$, $(k=0,1,\ldots,l-1)$ and $l\leq n-1$ implies that $(-)^{l+k}u^{(k)}(t)>0$ for $t\geq t_u$, $(k=l,l+1,\ldots,n-1)$.*

LEMMA 2. *If the function u is as in Lemma 1 and*

$$u^{(n-1)}(t)u^{(n)}(t)\leq 0 \quad \text{for} \quad t\geq t_u,$$

then for every λ, $0<\lambda<1$, there exists a $M_1>0$ such that

$$u(\lambda t)\geq M_1 t^{n-1}\,|u^{(n-1)}(t)| \quad \text{for all large } t.$$

THEOREM 1. *Under the conditions (i)–(iv) with $m>2$ all solutions of (2) are oscillatory.*

Proof. Let $x(t)$ be a non-oscillatory solution of (2). Assume that $x(t)>0$ for

$t \ge t_0$ and choose a $t_1 \ge t_0$ so that $g(t) \ge t_0$ for $t \ge t_1$. By Lemma 1, there exist a $t_2 \ge t_1$ such that $x^{(n-1)}(t) > 0$ and $\dot{x}(t) > 0$ for $t \ge t_2$. Choose a $t_3 \ge t_2$ so that $\sigma(t) \ge 2t_2$ for $t \ge t_3$. It is easy to check that we can apply Lemma 2 for $u = \dot{x}$, $\lambda = \frac{1}{2}$ and conclude that there exist $M_1 > 0$ and $t_4 \ge t_3$ such that

$$\dot{x}[\tfrac{1}{2}\sigma(t)] \ge M_1 \sigma^{n-2}(t) x^{(n-1)}[\sigma(t)] \ge M_1 \sigma^{n-2}(t) x^{(n-1)}(t) \quad \text{for} \quad t \ge t_4. \tag{4}$$

Let $w(t) = x^{(n-1)}(t)/x[\frac{1}{2}\sigma(t)]$. Thus $w(t)$ satisfies

$$\dot{w}(t) = -q(t)\frac{f(x(t), x[g(t)])}{x[\frac{1}{2}\sigma(t)]} - \tfrac{1}{2}\dot{\sigma}(t)w(t)\frac{\dot{x}[\frac{1}{2}\sigma(t)]}{x[\frac{1}{2}\sigma(t)]}.$$

Since $\dot{x}(t) > 0$ for $t \ge t_4$, $\lim_{t\to\infty} x(t)$ exists either as a finite or infinite limit. If $\lim_{t\to\infty} x(t) = b$ is finite, then

$$\lim_{t\to\infty} \frac{f(x(t), x[g(t)])}{x[\frac{1}{2}\sigma(t)]} = \frac{f(b, b)}{b} > 0.$$

If $\lim_{t\to\infty} x(t) = \infty$, then by (iii) we have

$$\frac{f(x(t), x[g(t)])}{x[\frac{1}{2}\sigma(t)]} \ge \frac{f(x(t), x[g(t)])}{x[g(t)]} \ge c > 0, \quad \text{for} \quad t \ge t_4. \tag{5}$$

In either case (5) holds for $t \ge t_4$. Since $x(t)$ is increasing for $t \ge t_4$, we have

$$q(t)\frac{f(x(t), x[g(t)])}{x[\frac{1}{2}\sigma(t)]} \ge q(t)\frac{f(x(t), x[g(t)])}{x[g(t)]} \ge cq(t) \tag{6}$$

and

$$\tfrac{1}{2}\dot{\sigma}(t)w(t)\frac{\dot{x}[\frac{1}{2}\sigma(t)]}{x[\frac{1}{2}\sigma(t)]} \ge \frac{k}{2}w(t)\left[M_1\sigma^{n-2}(t)\frac{x^{(n-1)}(t)}{x[\frac{1}{2}\sigma(t)]}\right] = \tfrac{1}{2}Mk\sigma^{n-2}(t)w^2(t).$$

Using (ii) we get

$$\tfrac{1}{2}\dot{\sigma}(t)w(t)\frac{\dot{x}[\frac{1}{2}\sigma(t)]}{x[\frac{1}{2}\sigma(t)]} \ge \tfrac{1}{2}k^{n-1}M_1t^{n-2}w^2(t).$$

Thus

$$\dot{w}(t) \le -cq(t) - c_1t^{n-2}w^2(t),$$

where $c_1 = \frac{1}{2}k^{n-1}M_1$. Whence it follows that

$$\int_{t_4}^{t} (t-u)^{m-1}\dot{w}(u)\,du \le -\int_{t_4}^{t} c(t-u)^{m-1}q(u)\,du - c_1\int_{t_4}^{t} (t-u)^{m-1}u^{n-2}w^2(u)\,du.$$

Since

$$\int_{t_4}^{t} (t-u)^{m-1}\dot{w}(u)\,du=(m-1)\int_{t_4}^{t}(t-u)^{m-2}w(u)\,du-w(t_4)(t-t_4)^{m-1},$$

we get

$$\frac{c}{t^{m-1}}\int_{t_4}^{t}(t-u)^{m-1}q(u)\,du$$

$$\leq w(t_4)\left(\frac{t-t_4}{t}\right)^{m-1}-\frac{c_1}{t^{m-1}}\int_{t_4}^{t}(t-u)^{m-1}u^{n-2}w^2(u)\,du$$

$$-\frac{m-1}{t^{m-1}}\int_{t_4}^{t}(t-u)^{m-2}w(u)\,du$$

$$=w(t_4)\left(\frac{t-t_4}{t}\right)^{m-1}+\frac{(m-1)^2}{4c_1}\frac{1}{t^{m-1}}\int_{t_4}^{t}\frac{(t-u)^{m-3}}{u^{n-2}}\,du$$

$$-\frac{1}{t^{m-1}}\int_{t_4}^{t}\left[(c_1u^{n-2})^{\frac{1}{2}}w(u)(t-u)^{(m-1)/2}-\frac{(m-1)(t-u)^{(m-3)/2}}{2(c_1u^{n-2})^{1/2}}\right]^2du$$

$$\leq w(t_4)\left(\frac{t-t_4}{t}\right)^{m-1}+\frac{(m-1)^2}{4c_1}\frac{1}{t^{m-1}}\int_{t_4}^{t}\frac{t^{m-3}}{u^{n-2}}\,du$$

$$=w(t_4)\left(\frac{t-t_4}{t}\right)^{m-1}+\frac{(m-1)^2}{4c_1}\frac{1}{t^2}\left[\frac{1}{3-n}(t^{3-n}-t_4^{3-n})\right]\to w(t_4)\quad\text{as}\quad t\to\infty,$$

which contradicts condition (iv). A similar proof holds if $x(t)<0$ for $t\geq t_0$.

THEOREM 2. *Let in Theorem* 1, *the condition* (iii) *be replaced by*:

(iii)′ $f(y_1, y_2)$ *is a continuously differentiable function with respect to* y_1 *and* y_2, *and*

$$\frac{\partial f(y_1,y_2)}{\partial y_i}\geq k_1>0\quad \textit{for}\quad y_i\neq 0,\qquad i=1,2$$

Then the conclusion of Theorem 1 holds.

Proof. Let $x(t)$ be a nonoscillatory solution of (2). Assume that $x(t)>0$ for $t\geq t_0$, $t_0\geq 0$. It follows, as in the proof of Theorem 1, that there exists $t_4\geq t_0$ so that

$$\dot{x}(t)>0,\qquad x^{(n-1)}(t)>0,$$

$$\dot{x}[\tfrac{1}{2}t]\geq M_1t^{n-2}x^{(n-1)}(t),$$

and

$$\dot{x}[\tfrac{1}{2}\sigma(t)]\geq M_1\sigma^{n-2}(t)x^{(n-1)}(t);\quad\text{for}\quad t\geq t_4.$$

Letting

$$w(t)=\frac{x^{(n-1)}(t)}{f\left(x\left[\frac{t}{2}\right],x\left[\frac{\sigma(t)}{2}\right]\right)},$$

we have

$$\dot{w}(t) = -q(t)\frac{f(x(t), x[g(t)])}{f\left(x\left[\frac{t}{2}\right], x\left[\frac{\sigma(t)}{2}\right]\right)} - \frac{1}{2}\frac{w(t)}{f\left(x\left[\frac{t}{2}\right], x\left[\frac{\sigma(t)}{2}\right]\right)}$$

$$\times\left(\frac{\partial f\left(x\left[\frac{t}{2}\right], x\left[\frac{\sigma(t)}{2}\right]\right)}{\partial x\left[\frac{t}{2}\right]}\cdot \dot{x}\left[\frac{t}{2}\right] + \frac{\partial f\left(x\left[\frac{t}{2}\right], x\left[\frac{\sigma(t)}{2}\right]\right)}{\partial x\left[\frac{\sigma(t)}{2}\right]}\,\dot{x}\left[\frac{\sigma(t)}{2}\right]\dot{\sigma}(t)\right).$$

Since $x(t)$ is increasing for $t \geq t_4$, we have

$$\dot{w}(t) \leq -q(t) - \tfrac{1}{2}w^2(t)[k_1M_1t^{n-2} + k_1kM_1\sigma^{n-2}(t)].$$

Using (ii), we get $\dot{w}(t) \leq -q(t) - ct^{n-2}w^2(t)$, where $c = \frac{1}{2}k_1M_1(1+k^{n-1})$, and the remaining of the proof follows exactly that of Theorem 1.

Remarks

1. If $n=2$, then Theorem 1 becomes Theorem 1 in [1].

2. If $n=2$, $f(x,y)=F(x)$, $xF(x)>0$ and $F'(x) \geq k > 0$ $('=d/dx)$ for $x \neq 0$, then $q(t)$ need not be a positive function to ensure the oscillation of (2), (see [2, 3, 8]). In that case Theorem 2 in [1] is included in our Theorem 2.

3. We can verify that if condition (v) holds, then (iv) will also hold for $m=3$, and from Remark 2, $q(t)$ need not be a positive function. Thus the oscillation criterion of Wintner [10] is a special case of our Theorem 2.

The following examples are illustrative.

Example 1. Consider the equation

$$\text{(a)}\qquad x^{(n)} + f\left(x\left[\frac{t}{2}\right]\right) = 0, \quad n \text{ even}, \quad t > 0,$$

where

$$f(x) = \begin{cases} x\exp(x[1+\sin x]), & \text{for } x \geq 0. \\ x, & \text{for } x \leq 0. \end{cases}$$

Here $q(t)=1$, $g(t)=\sigma(t)=t/2$ and $\dot{\sigma}(t)=\frac{1}{2}$. It is easy to check that the hypotheses of Theorem 1 are satisfied. Hence all solutions of (a) are oscillatory. We may add that the oscillation criteria presented in the majority of papers, concerned with the case when f is a nondecreasing function (see the recent survey paper by Kartsatos [6] and the references contained therein), and hence cannot be applied to equation (a), since f is not a monotone function. Also the oscillation criteria, obtained by Mahfoud [7] (Theorem 3 and Corollaries 1–3), cannot be applied to equation (a) with $n > 2$, since the condition

$$q(t) \geq r(t)g^{n-2}(t)\dot{g}(t),$$

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where r is a positive, nondecreasing continuous function on $(0, \infty)$, is not satisfied.

EXAMPLE 2. Consider the equation

$$\text{(b)} \qquad x^{(n)} + \sin hx[t] + \sin hx\left[\frac{t}{2}\right] = 0, \quad n \text{ even}, \quad t > 0.$$

Here $f(x, y) = \sin hx + \sin hy$, $q(t) = 1$, $g(t) = \sigma(t) = t/2$, and $\dot{\sigma}(t) = \frac{1}{2}$.

$$\frac{\partial f(x, y)}{\partial x} = \cos hx \geq 1 > 0 \quad \text{and} \quad \frac{\partial f(x, y)}{\partial y} = \cos hy \geq 1 > 0 \quad \text{for} \quad x, y \neq 0.$$

Thus the hypotheses of Theorem 2 are satisfied and equation (b) is oscillatory. We note once again that results of Mahfoud are not applicable to equation (b).

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