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ARC COMPONENTS OF CERTAIN CHAINABLE CONTINUA

BY
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ABSTRACT. It is shown that if a chainable continuum has exactly two arc components, then one of them is an arc and the other is a half-ray.

1. **Introduction.** We first give a brief discussion of how the author became interested in the topic which is the title of this paper. In [6] we proved that the only arcwise connected inverse limit (with onto bonding maps) of arcs is an arc. It was previously known [3] that the only locally connected inverse limit of arcs (respectively, simple closed curves) is an arc (respectively, simple closed curve). The question arose as to what are the arcwise connected inverse limits of simple closed curves? It was clear to the author that there are at least two such inverse limits, namely a simple closed curve and a sin $(1/x)$-circle (see Figure 1).

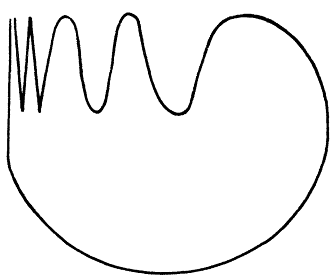

FIGURE 1.

It follows by using some of the results in [6] that an arcwise connected inverse limit of simple closed curves can be thought of as composed of two pieces (which fit together nicely), one an arc and the other a chainable continuum with at most two arc components. Hence, the problem of determining the arcwise connected inverse limits of simple closed curves is really the problem of determining those chainable continua which have exactly two arc components. The author thought for a long time that there were only two such chainable continua, namely the objects in Figure 2 and Figure 3.

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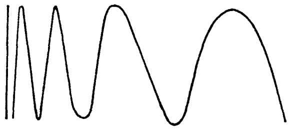

FIGURE 2.

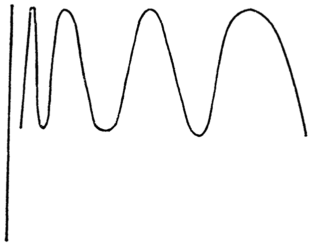

FIGURE 3.

This faulty intuition was the result of recognizing that certain chainable continua (such as those in Figure 4 and Figure 5) which looked quite different than the object in Figure 2 were, in fact, homeomorphic to it.

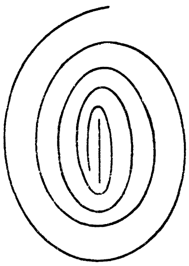

FIGURE 4.

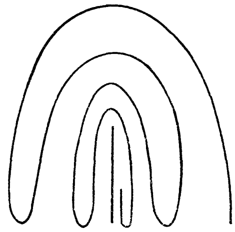

FIGURE 5.

However, a very simple modification (see Figure 6) of the object in Figure 2 yields a chainable continuum with exactly two arc components which is not homeomorphic to any of the chainable continua in Figures 2 through 5.

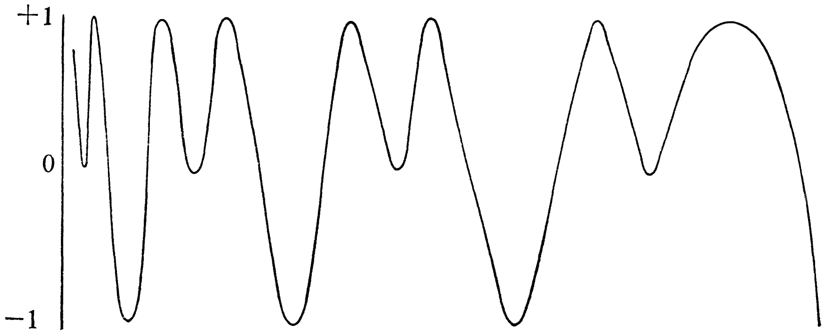

FIGURE 6.

By using simple modifications of the construction in Figure 6 and using Theorem 2 (which was called to the author's attention by K. Kuratowski) of [5] it is easy to see that there are uncountably many topologically different chainable continua with exactly two arc components. Thus, the question arose—what can one say about the structure of the chainable continua which have exactly two arc components?

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In this paper we determine the structure of such continua in the following sense. We prove (see Theorem 1 below) that any two chainable continua, each having exactly two arc components, have the same two topological types of arc components: one arc component is an arc and the other is a half-way (i.e., homeomorphic to $[0, \infty)$).

The applications mentioned above of the results in this paper to inverse limits will appear in another paper.

I express my gratitude to D. G. Paulowich for his many valuable comments concerning the material in this paper.

Throughout this paper a *continuum* will mean a compact connected metric space with more than one point. The symbol $\bar{S}$ will mean the closure of S.

2. **Results.** An *arc component* of a continuum X is defined to be a maximal arcwise connected subset of X.

The main purpose of this section is to prove Theorem 1 which says that if a chainable continuum has exactly two arc components, then one of them is an arc and the other is a half-ray. The proof of Theorem 1 is accomplished in essentially two steps. The first (Lemma 3) is to show that every one-to-one continuous mapping of $[0, \infty)$ into a certain type of space is a homeomorphism. The second step (Lemma 6) is to show that one of the arc components is compact—this compactness of one arc component is then used (see the proof of Theorem 1) to show there exists a one-to-one continuous mapping of $[0, \infty)$ onto the other arc component which, by the first step, is a homeomorphism.

Though the content of the first lemma below is well-known and has been used in the literature (see for example [2]), we state it here as a formal lemma because it will be used quite often throughout this section. The proof is easy and is omitted.

LEMMA 1. *If K is a subcontinuum of an hereditarily unicoherent continuum X and $Y \subset X$ is arcwise connected, then $K \cap Y$ is arcwise connected.*

LEMMA 2. *Let X be an hereditarily decomposable, hereditarily unicoherent continuum and let $f\colon [0, \infty) \to X$ be a one-to-one continuous function. Then there does not exist a sequence $\{t_n\}_{n=1}^{\infty}$ in $[0, \infty)$ such that $t_n \to \infty$ and $\{f(t_n)\}_{n=1}^{\infty}$ converges to some point $p \in f([0, \infty))$.*

Proof. Let $S = \{p \in f([0, \infty))$: there is a sequence $\{t_n\}_{n=1}^{\infty}$ in $[0, \infty)$ such that $t_n \to \infty$ and $\{f(t_n)\}_{n=1}^{\infty}$ converges to $p\}$ and suppose S is nonempty. It is easy to see that S is closed in $f([0, \infty))$. We now show that

(*) if $x = f(s)$ and $y = f(t)$, with $s < t$, are each in S, then $f([s, t]) \subset S$.

To see this first note that since $x \in S$, $x \in \overline{f([t, \infty))}$. Hence, x and y are in $\overline{f([t, \infty))}$ so, by Lemma 1, $f([s, t]) \subset \overline{f([t, \infty))}$. Since f is one-to-one, it now follows that $f([s, t]) \subset S$. This completes the proof of (*). It is now easy to see that S is of the form $f([s, t])$ or $f([s, \infty))$ for some s and t, $0 \leq s \leq t < \infty$. If S were of the form

$f([s, t])$ for some s and t, $0 \le s \le t < \infty$, then $\overline{f([t+1, \infty))} \cap f([s, t+1])$ would equal $S \cup \{f(t+1)\}$ which is not connected, contradicting the hereditary unicoherence of X. Thus, S is of the form $f([s, \infty))$ for some $s \in [0, \infty)$. Let $M = \overline{f([s, \infty))}$. Note that because $f([s, \infty)) = S$, no set of the form $f([s, t])$ has interior in M and, hence, $f([s, \infty))$ is of the first category in M. Hence, $f([s, \infty))$ does not contain a nonempty open subset of M. Thus, M is an hereditarily unicoherent continuum which contains a dense subset of $f([s, \infty))$ that is a one-to-one continuous image of a half-open interval but that contains no interior points relative to M. We may now apply Theorem 4 of [7] to conclude that M is indecomposable, which contradicts the assumption that X is hereditarily decomposable. Therefore, S is empty.

Lemma 3. *Let X be an hereditarily decomposable, hereditarily unicoherent continuum. If $f\colon [0, \infty) \to X$ is a one-to-one continuous function, then f is a homeomorphism* (*of* $[0, \infty)$ *onto* $f([0, \infty))$).

Proof. We need only show that the image under f of an open subset of $[0, \infty)$ is open relative to $f([0, \infty))$. Let $s, t \in [0, \infty)$ such that $s < t$. Since f is one-to-one, $f([0, \infty)) - f((s, t)) = f([0, s]) \cup f([t, \infty))$. Any point $p \in f([0, \infty))$ which is a limit point of $f([t, \infty))$ would be the limit of a sequence $\{f(t_n)\}_{n=1}^{\infty}$ where $t_n \ge t$ for each $n = 1, 2, \ldots$. By Lemma 2, $t_n \not\to \infty$ so there is a subsequence $\{t_{n_i}\}_{i=1}^{\infty}$ which converges to a number $t_0 \ge t$.

Since $\{f(t_{n_i})\}_{i=1}^{\infty}$ is a subsequence of $\{f(t_n)\}_{n=1}^{\infty}$, $\{f(t_{n_i})\}_{i=1}^{\infty}$ converges to p. But $\{t_{n_i}\}_{i=1}^{\infty}$ converging to t_0 implies $\{f(t_{n_i})\}_{i=1}^{\infty}$ converges to $f(t_0)$. Hence, $f(t_0) = p$ so $p \in f([t, \infty))$. Hence, $f([0, \infty)) - f((s, t))$ is closed relative to $f([0, \infty))$ which implies $f((s, t))$ is open relative to $f([0, \infty))$. A similar argument shows that any set of the form $f([0, t))$ is open relative to $f([0, \infty))$. The lemma follows.

Remark. We could have stated a stronger result than the one given in Lemma 3; namely that any set of the form $f((s, t))$ or $f([0, t))$ is open relative to $\overline{f([0, \infty))}$. However, this stronger fact does not seem to be useful here.

Lemma 4. *Let X be an a-triodic and hereditarily unicoherent continuum, let $f\colon [0, \infty) \to X$ be one-to-one and continuous such that $\overline{f([0, \infty))}$ is not arcwise connected, and let $p \notin f([0, \infty))$ be in the same arc component of X as $f([0, \infty))$. Let α be an arc from p to a point of $f([0, \infty))$, α given by a homeomorphism $h\colon [0, 1] \to \alpha$ with $h(0) = p$. If*

$$r_0 = g.l.b.\ \{r \in [0, 1] \colon h(r) \in f([0, \infty))\},$$

then $h(r_0) = f(0)$.

Proof. By Lemma 1, $\alpha \cap f([0, \infty))$ is arcwise connected (and obviously closed) in $f([0, \infty))$. If $\alpha \cap f([0, \infty))$ were of the form $f([s, \infty))$ for some $s \in [0, \infty)$, then $f([0, \infty))$ would be contained in an arc so that $\overline{f([0, \infty))}$ would be arcwise connected (by Lemma 1). Hence, $\alpha \cap f([0, \infty))$ is of the form $f([s, t])$ for some s and t,

$0 \leq s \leq t < \infty$. Therefore, $h(r_0) \in f([0, \infty))$. If $h(r_0) = f(t_0)$ with $t_0 > 0$, then $f([0, t_0+1]) \cup \alpha$ would be a triod. Hence, $h(r_0) = f(0)$.

LEMMA 5. *Let X be a chainable continuum with two arc components C and D such that $C \cap \bar{D} \neq \varnothing$. If $c \in C \cap \bar{D}$ and $d \in D$, then there is a one-to-one continuous function f: $[0, \infty) \to D$ such that $f(0) = d$ and $c \in \overline{f([0, \infty))}$.*

Proof. Let $\{d_i\}_{i=1}^{\infty}$ be a sequence of points in D converging to c and assume $d_i \neq d$ for all i. For each $i = 1, 2, \ldots$, let A_i be the (unique) arc in D from d to d_i and, for $n = 1, 2, \ldots$, let $\alpha_n \bigcup_{i=1}^{n} A_i$. Since X is chainable, α_n is an arc for each $n = 1, 2, \ldots$. Let $x \in \alpha_1 - \{d, d_1\}$. Then, for each n, x divides α_n into two subarcs α_n' and α_n'' (with x a noncut point of each), the primes being chosen so that, for each n, $\alpha_n' \subset \alpha_{n+1}'$ (and $\alpha_n'' \subset \alpha_{n+1}''$) (see the proof of Theorem 1 of [7]). Since infinitely many terms of the sequence $\{d_i\}_{i=1}^{\infty}$ belong to $\bigcup_{n=1}^{\infty} \alpha_n'$ or $\bigcup_{n=1}^{\infty} \alpha_n''$, $c \in \overline{\bigcup_{n=1}^{\infty} \alpha_n'}$ or $c \in \overline{\bigcup_{n=1}^{\infty} \alpha_n''}$. Without loss of generality assume $c \in \overline{\bigcup_{n=1}^{\infty} \alpha_n'}$. Clearly $\bigcup_{n=1}^{\infty} \alpha_n'$ is a one-to-one continuous image of $[0, \infty)$. If $d \in \bigcup_{n=1}^{\infty} \alpha_n'$, then letting $H = (\bigcup_{n=1}^{\infty} \alpha_n' - \beta) \cup \{d\}$ where β denotes the unique arc in $\bigcup_{n=1}^{\infty} \alpha_n'$ from x to d, it is obvious that H is a one-to-one continuous image of $[0, \infty)$. If $d \notin \bigcup_{n=1}^{\infty} \alpha_n'$, then letting $H = (\bigcup_{n=1}^{\infty} \alpha_u') \cup \gamma$, where γ is the unique arc in D from d to x, it follows from Lemma 4 that H is a one-to-one continuous image of $[0, \infty)$. This completes the proof.

LEMMA 6. *If X is a chainable continuum with exactly two arc components A and B, then A is compact or B is compact.*

Proof. We first note that X is hereditarily decomposable. To see this let K be a nondegenerate subcontinuum of X. Then, by Lemma 1, K has at most two arc components, namely $K \cap A$ and $K \cap B$. If K were arcwise connected, then K would be an arc. Thus, we may assume K has exactly two arc components. Each arc component of K is obviously contained in a composant of K so K does not have uncountably many mutually disjoint composants. Hence, K is decomposable (see, for example, Theorem 3–46 and Theorem 3–47 of [4]) which completes the proof that X is hereditarily decomposable. Now suppose that neither A nor B is compact, let $a \in A \cap \bar{B}$, and let $b \in B \cap \bar{A}$. Applying Lemma 5 twice we obtain two one-to-one continuous functions f_1: $[0, \infty) \to B$ and f_2: $[0, \infty) \to A$ such that $f_1(0) = b$, $a \in \overline{f_1([0, \infty))}$, $f_2(0) = a$, and $b \in \overline{f_2([0, \infty))}$. Let $H_1 = f_1([0, \infty))$ and let $H_2 = f_2([0, \infty))$. We now show that $\bar{H}_1 \not\supset H_2$. Suppose $\bar{H}_1 \supset H_2$. Since $b \in (\bar{H}_2 - H_2)$ there exists a sequence $\{s_n\}_{n=1}^{\infty}$ in $[0, \infty)$ such that $s_n \to \infty$ and the sequence $\{f_2(s_n)\}_{n=1}^{\infty}$ converges to b. Since $(\bar{H}_1 - H_1) \supset H_2$ a sequence $\{t_n\}_{n=1}^{\infty}$ in $[0, \infty)$ can be produced such that $t_n \to \infty$ and $d(f_1(t_n), f_2(s_n)) < 1/n$, where d denotes the distance for X. Hence, $t_n \to \infty$ and $\{f_1(t_n)\}_{n=1}^{\infty}$ converges to the point $b \in f_1([0, \infty))$. This contradicts Lemma 2 ($\bar{H}_1$, as a subcontinuum of the hereditarily decomposable continuum X, is hereditarily decomposable) and completes the proof that $\bar{H}_1 \not\supset H_2$. Similarly, $\bar{H}_2 \not\supset H_1$. Now let $x \in (H_2 - \bar{H}_1)$, $x = f_2(s)$ for some $s \in [0, \infty)$. By Lemma

1, $\overline{H}_1 \cap H_2$ is arcwise connected. Also, since $f_2(0)=a$, $f_2(0) \in \overline{H}_1 \cap H_2$. Suppose there is a point $y \in \overline{H}_1 \cap H_2$ such that $y=f_2(t)$ with $t>s$. Then there is an arc σ from $f_2(0)$ to y such that $\sigma \subset \overline{H}_1 \cap H_2$. Since $x \notin \sigma$, $\sigma \cap f_2([0, t])$ is not connected, a contradiction. Therefore, $\overline{H}_1 \cap H_2 \subset f_2([0, s])$. Similarly, $\overline{H}_2 \cap H_1 \subset f_1([0, r])$ for some $r \in [0, \infty)$. Let $M=f_2([s+1, s+2])$ and let

$$N = \overline{H}_1 \cup f_2([0, s+1]) \cup \overline{f_2([s+2, \infty))}.$$

Since $(\overline{H}_1 \cap \overline{H}_2)-H_2$ is nonempty (because the point b is in it) and contained in $\overline{H}_1 \cap \overline{f_2([s+2, \infty))}$, we have that $\overline{H}_1 \cap \overline{f_2([s+2, \infty))} \neq \varnothing$. It now follows that N is connected. Hence, M and N are subcontinua of X. But $M \cap N = \{f_2(s+1), f_2(s+2)\}$ which is not connected. This contradicts the hereditary unicoherence of X. Therefore, A is compact or B is compact.

THEOREM 1. *If X is a chainable continuum with exactly two arc components, then one of them is an arc and the other is homeomorphic to* $[0, \infty)$.

Proof. Let A and B denote the arc components of X. By Lemma 6 one of them, say A, is compact. Hence, A is an arcwise connected chainable metric continuum so A is an arc [6]. We now show that there is a point $b \in B$ such that $B-\{b\}$ is arcwise connected. Let $x \in B$ such that $B-\{x\}$ is not arcwise connected. Since X is a-triodic and B is arcwise connected, $B-\{x\}$ has exactly two arc components denoted by B_1 and B_2. Clearly $x \in \overline{B}_1 \cap \overline{B}_2 \cap B$. Thus, since B_1 and B_2 are disjoint open subsets of B and $B_1 \cup B_2 = B-\{x\}$, we have $\overline{B}_1 \cap \overline{B}_2 \cap B = \{x\}$. Now suppose $\overline{B}_1 \not\subset B$ and $\overline{B}_2 \not\subset B$ so that $\overline{B}_1 \cap A \neq \varnothing$ and $\overline{B}_2 \cap A \neq \varnothing$. Then, since

$$\overline{B} \cap A = (\overline{B_1 \cup B_2}) \cap A = (\overline{B}_1 \cup \overline{B}_2) \cap A = (\overline{B}_1 \cap A) \cup (\overline{B}_2 \cap A)$$

is connected and $(\overline{B}_1 \cap \overline{B}_2) \cap A = (\overline{B}_1 \cap A) \cap (\overline{B}_2 \cap A)$, $(\overline{B}_1 \cap \overline{B}_2) \cap A \neq \varnothing$. Now since $\overline{B}_1 \cap \overline{B}_2 = [(\overline{B}_1 \cap \overline{B}_2) \cap A] \cup [(\overline{B}_1 \cap \overline{B}_2) \cap B]$, since $(\overline{B}_1 \cap \overline{B}_2) \cap B = \{x\}$, $\overline{B}_1 \cap \overline{B}_2$ is not connected. This contradiction establishes that $\overline{B}_1 \subset B$ or $\overline{B}_2 \subset B$. Without loss of generality we assume $\overline{B}_1 \subset B$. Then $\overline{B}_1$ is arcwise connected and, thus, is an arc [6]. Let $b \in B$ be the noncut point of $\overline{B}_1$ different from x. It is easy to verify (by considering arcs between points in B_1 and/or in B_2) that $B-\{b\}$ is arcwise connected. We now show that there is a homeomorphism of $[0, \infty)$ onto B. Let $a \in \overline{B} \cap A$. By Lemma 5 there is a one-to-one continuous function f: $[0, \infty) \to B$ such that $f(0)=b$ and $a \in \overline{f([0, \infty))}$. We show that f maps onto all of B; then (X being hereditarily decomposable) Lemma 3 implies f is a homeomorphism of $[0, \infty)$ onto B. Suppose f is not onto B so that there is a point $y \in B$ such that $y \notin f([0, \infty))$. Since $B-\{b\}$ is arcwise connected, there is an arc $\alpha \subset B-\{b\}$ from y to a point of $f((0, \infty))$. Since $f(0) \notin \alpha$, this contradicts Lemma 4. Hence, f is onto B.

REMARK. If a continuum is hereditarily indecomposable, then each arc component is a point. Thus, each arc component of the pseudo-arc is a point. However, there are hereditarily decomposable chainable continua such that each arc component must be a point (such an example may be found in [1]).

REMARK. One might conjecture that a slightly more general form of Lemma 6 is valid, namely that if X is an hereditarily unicoherent continuum with exactly two arc components, then one of them is compact (in other words, a-triodicity might not seem to play a crucial role in Lemma 6). However, such a conjecture is false even for continua in the plane as the example in Figure 7 shows. The example is due to G. S. Young. Descriptively, X is a triod T (contained in the x and y axes with noncut points $(-1, 0)$, $(0, 1)$, and $(0, -1)$) together with a half-ray $H=\{(x, \sin(1/x)): 0<x\leq 1\}$ and a sequence of arcs $A_0, A_1, A_2, \ldots$ such that A_n has one noncut point at $(-1/2^n, 0)$ and such that each point of $\{(x, \sin(1/x)): 1/2^n\leq x\leq 1\}$ is within $1/2^n$ of some point of A_n. The two arc components of X are H and $T\cup(\bigcup_{n=0}^{\infty} A_n)$.

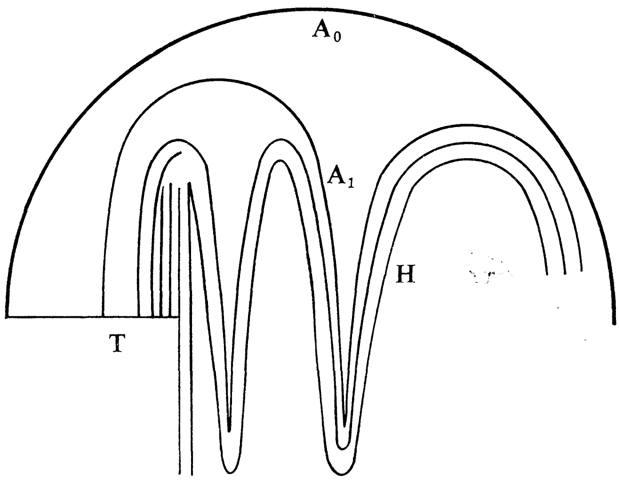

FIGURE 7.

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