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HIGH ORDER DERIVATIONS AND HIGH ORDER LIE-LIKE ELEMENTS

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Introduction. We can define high order derivations of an algebra into the ground field by diagrams. Then consider the same diagrams in the category of coalgebras. By reversing all the arrows in these diagrams, we come to a new notion – high order Lie-like elements of a coalgebra. These elements are useful in the study of the structure of coalgebras and sequences of divided powers.

We assume the knowledge of [**7**] and use the same conventions and notations.

1. Definitions and connections. Nakai [**6**] defined high order derivations from a commutative algebra A to an A-module M and derived some properties of them. If A has an augmentation α, the ground field K becomes an A-module. If we restrict ourselves to high order α-differentiations (high order derivations of A to K), the commutativity of A is no longer necessary. Now we give the dual definition – high order Lie-like elements and state some dual properties (for direct proofs see [**1**]).

Let (C, Δ, ϵ) be a coalgebra. For any integer $n \geqq 2$, $\Delta_n = (I \otimes \Delta)\Delta_{n-1} = (\Delta \otimes I)\Delta_{n-1}$. Further, let $\Delta_0 = I$. For any positive integer q, $\otimes^q C = C \otimes C \otimes \ldots \otimes C$ (q times) and for any element x in C, $\otimes^q x = x \otimes x \otimes \ldots \otimes x$ is in $\otimes^q C$. Then for any integers

$$1 \leqq i_1 < i_2 < \ldots < i_s \leqq q + 1 \qquad (\text{where } 1 \leqq s \leqq q),$$

we define a "twist" map $\tau_{i_1 \ldots i_s}$ from $\otimes^{q+1} C$ into itself by

$$\tau_{i_1 \ldots i_s}(x_1 \otimes \ldots \otimes x_{q+1}) = x_{i_1} \otimes \ldots \otimes x_{i_s} \otimes x_1 \ldots \otimes \hat{x}_{i_1} \otimes \ldots \otimes \hat{x}_{i_s} \otimes \ldots \otimes x_{q+1}.$$

Let $T_{i_1 \ldots i_s}$ be the inverse of $\tau_{i_1 \ldots i_s}$.

Definition. For any $\alpha \in G(C)$ (i.e., a group-like element of C) and any positive integer q, an element $x \in C$ is called a q-th *order Lie-like element of C relative to* α if and only if

$$\Delta_q x = \sum_{s=1}^{q} (-1)^{s-1} \sum_{i_1 < \ldots < i_s} T_{i_1 \ldots i_s}(\otimes^s \alpha \otimes \Delta_{q-s} x).$$

We denote by $L_\alpha{}^q(C)$ the set of q-th order Lie-like elements of C relative to α and put $L_\alpha{}^\infty(C) = \bigcup_{q=1}^{\infty} L_\alpha{}^q(C)$.

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For $\alpha \in G(C)$, $x \in C$ and $f \in C^*$, we set

$$[x, f]_\alpha = x < f - f(\alpha)x - f(x)\alpha \quad (\in C)$$

where $x < f = (f \otimes I)\Delta x = \sum_{(x)} f(x_{(1)})x_{(2)}$. We shall drop the subscript α if it is clear from the context. For simplicity we shall write $[x, f_1, \ldots, f_q]$ for $[[\ldots [x, f_1], \ldots], f_q]$.

THEOREM 1. *Let $\alpha \in G(C)$ and let D be any dense subset of C^*. Then an element x of C is in $L_\alpha^q(C)$ if and only if $[x, f_1, \ldots, f_q] = 0$ for any elements $f_1, \ldots, f_q$ in D.*

COROLLARY 1. *For any integer $q \geqq 2$, $x \in L_\alpha^q(C)$ if and only if $[x, f] \in L_\alpha^{q-1}(C)$ for every $f \in D$.*

COROLLARY 2. *For any integer $q \geqq 1$, $x \in L_\alpha^q(C)$ implies $x \in L_\alpha^n(C)$ for any integer $n \geqq q$.*

PROPOSITION 1. *For any integer $q \geqq 1$, $x \in L_\alpha^q(C)$ implies $\epsilon(x) = 0$.*

PROPOSITION 2. *Let $(x_0, x_1, \ldots, x_q, \ldots)$ be a sequence of divided powers in C. Then $x_0 \in G(C)$ and $x_q \in L_{x_0}^q(C)$ for any positive integer q.*

Let A be an algebra, A^0 its dual coalgebra [**7**, Proposition 6.0.2] and $\mathrm{Diff}_\alpha^q(A)$ the set of q-th order α-differentiations of A, i.e.,

$$\mathrm{Diff}_\alpha^q(A) = \{\delta \in A^* | \delta(x_0 x_1 \ldots x_q) = \sum_{s=1}^{q} (-1)^{s-1} \sum_{i_1 < \ldots < i_s} \alpha(x_{i_1}) \ldots \alpha(x_{i_s})\delta(x_0 \ldots \hat{x}_{i_1} \ldots \hat{x}_{i_s} \ldots x_q) \text{ for all } x_0, x_1, \ldots, x_q \in A\}$$

With the help of Proposition 6.0.3 of [**7**] it is easy to see that $\mathrm{Diff}_\alpha^1 A = L_\alpha^1(A^0)$. But for $q > 1$, $\mathrm{Diff}_\alpha^q A \neq L_\alpha^q(A^0)$ in general.

PROPOSITION 3. *$L_\alpha^q(A^0) \subset \mathrm{Diff}_\alpha^q(A)$. For any element $\delta \in \mathrm{Diff}_\alpha^q(A)$, we have $\delta \in L_\alpha^q(A^0)$ if and only if $\delta \in A^0$.*

Proof.

$$L_\alpha^q(A^0) = \{\delta \in A^0 | \Delta_q\delta = \sum_{s=1}^{q} (-1)^{s-1} \sum_{i_1 < \ldots < i_s} T_{i_1 \ldots i_s}(\otimes^s \alpha \otimes \Delta_{q-s}\delta)\}.$$

$$\begin{aligned}
\mathrm{Diff}_\alpha^q(A) &= \{\delta \in A^* | \delta(x_0 x_1 \ldots x_q) = \sum_{s=1}^{q} (-1)^{s-1} \sum_{i_1 < \ldots < i_s} \alpha(x_{i_1}) \ldots \\
&\qquad \alpha(x_{i_s})\delta(x_0 \ldots \hat{x}_{i_1} \ldots \hat{x}_{i_s} \ldots x_q) \text{ for all } x_0, x_1, \ldots, x_q \in A\} \\
&= \{\delta \in A^* | (\Delta_q\delta)(x_0 \otimes x_1 \otimes \ldots \otimes x_q) = \sum_{s=1}^{q} (-1)^{s-1} \sum_{i_1 < \ldots < i_s} \\
&\qquad T_{i_1 \ldots i_s}(\otimes^s \alpha \otimes \Delta_{q-s}\delta)(x_0 \otimes x_1 \otimes \ldots \otimes x_q) \\
&\qquad \text{for all } x_0, x_1, \ldots, x_q \in A\} \\
&= \{\delta \in A^* | \Delta_q\delta = \sum_{s=1}^{q} (-1)^{s-1} \sum_{i_1 < \ldots < i_s} T_{i_1 \ldots i_s}(\otimes^s \alpha \otimes \Delta_{q-s}\delta)\},
\end{aligned}$$

where, by abuse of notation Δ_q also denotes the map $A^* \to (\otimes^{q+1} A)$ which is the dual map of the multiplication $\otimes^{q+1} A \to A$ (where $\otimes^{q+1} A$ means $A \otimes \ldots \otimes A$ $q + 1$ times). The rest of the proof is clear.

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Thus to show $\mathrm{Diff}_\alpha{}^q A \neq L_\alpha{}^q(A^0)$, it is enough to find an element $\delta \in \mathrm{Diff}_\alpha{}^q(A)$ which is not in A^0. We now give such an example.

Example. Let $A = K[x_i]_{i=1}^\infty$ be a polynomial algebra in infinitely many indeterminates $x_1, x_2, \ldots$ over a field K. Let P be the subset of A consisting of zero and all polynomials of positive degree. Then $A = K \oplus P$. Let α be the projection of A onto K. Then α is an augmentation of A with augmentation ideal P. Since

$$\{1, x_i, x_ix_j, x_ix_jx_k, \ldots\}_{i,j,k=1}^\infty$$

is a basis of A over K, we can pick $\delta \in A^*$ such that $\delta(x_i{}^2) = 1$ for $i = 1, 2, \ldots$ and δ kills any other element in this basis. By straight forward computation one shows $\delta \in \mathrm{Diff}_\alpha{}^2(A)$. But we show now that $\delta \notin A^0$. By Proposition 6.0.3 of [**7**], it suffices to show that $\delta < A$ is infinite dimensional (where, for each $a \in A$, $\delta < a$ is defined by $\delta < a(x) = \delta(ax)$ for all $x \in A$). Consider $\delta < x_i$. We have $\delta < x_i(x_i) = \delta(x_i{}^2) = 1$ and if $y \neq x_i$ is any other element in the basis, $\delta < x_i(y) = \delta(x_iy) = 0$. That is, $\delta < x_i = x_i{}^*$ is the element in A^* dual to x_i. Hence $\{\delta < x_i\}_{i=1}^\infty$ is a linearly independent set. Thus $\delta < A$ is infinite dimensional.

PROPOSITION 4. *Let A be an algebra with an augmentation α such that the augmentation ideal* $\mathrm{Ker}(\alpha)$ *is finitely generated. Then* $\mathrm{Diff}_\alpha{}^q(A) = L_\alpha{}^q(A^0)$ *for every positive integer q.*

Proof. For any $\delta \in \mathrm{Diff}_\alpha{}^q(A)$, we have to show $\delta \in A^0$. Let $\{x_1, \ldots, x_n\}$ be a set of generators of $\mathrm{Ker}(\alpha)$. For any monomial $x_{j_0}x_{j_1}\ldots x_{j_q}\ldots x_{j_{q+t}}$ in the x_i's of length $> q$, put $y_i = x_{j_i}$ for $i < q$ and $y_q = x_{j_q}\ldots x_{j_{q+t}}$. Now

$$\begin{aligned}\delta(x_{j_0}x_{j_1}\ldots x_{j_q}\ldots x_{j_{q+t}}) &= \delta(y_0y_1\ldots y_q)\\ &= \textstyle\sum_{s=1}^q (-1)^{s-1} \sum_{i_1<\ldots<i_s}\alpha(y_{i_1})\ldots\alpha(y_{i_s})\delta(y_0\ldots\hat{y}_{i_1}\ldots\hat{y}_{i_s}\ldots y_q)\\ &= 0\end{aligned}$$

since $y_0, y_1, \ldots, y_q \in \mathrm{Ker}(\alpha)$. Let S be the subset of $\mathrm{Ker}(\alpha)$ consisting of all elements in $\mathrm{Ker}(\alpha)$ expressible as a linear combination of monomials in the x_i's of length $> q$. Then clearly S is an ideal of A (since $A = K \oplus \mathrm{Ker}(\alpha)$) and S is contained in the kernel of δ. Since the number of monomials in the x_i's of length $\leqq q$ is finite, S is a cofinite subspace of $\mathrm{Ker}(\alpha)$. Thus S is a cofinite ideal of A contained in $\mathrm{Ker}(\delta)$, hence $\delta \in A^0$.

Remark. One can find an example to show that $\mathrm{Ker}(\alpha)$ being finitely generated is not a necessary condition.

Recall that a higher differentiation of an algebra A is a sequence $(\delta_0, \delta_1, \ldots, \delta_n)$ of elements in A^* (where n may be infinite) such that for $x, y \in A$,

$$\delta_j(xy) = \textstyle\sum_{i=0}^j \delta_i(x)\delta_{j-i}(y), \quad j = 0, 1, 2, \ldots, n.$$

The following proposition shows that higher differentiations of A are nothing but sequences of divided powers in A^0.

PROPOSITION 5. *Let A be an algebra. Then a sequence $(\delta_0, \delta_1, \ldots, \delta_n)$ of elements in A^* is a higher differentiation of A if and only if it is a sequence of divided powers in A^0.*

Proof. Suppose $(\delta_0, \delta_1, \ldots, \delta_n)$ is a higher differentiation of A. Then for any x and y in A,

$$\Delta\delta_j(x \otimes y) = \delta_j(xy) = \sum_{i=0}^{j} \delta_i(x)\delta_{j-i}(y) = \sum_{i=0}^{j} \delta_i \otimes \delta_{j-i}(x \otimes y)$$

for $j = 0, 1, \ldots, n$. Thus

$$\Delta\delta_j = \sum_{i=0}^{j} \delta_i \otimes \delta_{j-i}; j = 0, 1, \ldots, n.$$

Therefore $(\delta_0, \delta_1, \ldots, \delta_n)$ is a sequence of divided powers in A^0 provided each δ_j is in A^0. But this is true by Proposition 6.0.3 of [7]. Conversely if $(\delta_0, \delta_1, \ldots, \delta_n)$ is a sequence of divided powers in A^0. Then for any $x, y \in A$,

$$\delta_j(xy) = \Delta\delta_j(x \otimes y) = \sum_{i=0}^{j} \delta_i \otimes \delta_{j-i}(x \otimes y) = \sum_{i=0}^{j} \delta_i(x)\delta_{j-i}(y)$$

for $j = 0, 1, \ldots, n$. That is, $(\delta_0, \delta_1, \ldots, \delta_n)$ is a higher differentiation of A.

Now we reverse our direction. We shall connect high order Lie-like elements (respectively sequences of divided powers) of a coalgebra C with high order differentiations (respectively higher differentiations) of the algebra C^*.

Let C be a coalgebra and $\alpha \in G(C)$. Then the map $x \to x^0$ (where $x^0(f) = f(x)$ for all $f \in C^*$) embeds the set $L_\alpha{}^q(C)$ into the set $\mathrm{Diff}_{\alpha^0}{}^q(C^*)$ for every positive integer q. It also embeds the set of sequences of divided powers in C into the set of higher differentiations of C^*.

PROPOSITION 6. *If C is finite dimensional and $\alpha \in G(C)$, then $L_\alpha{}^q(C) \cong \mathrm{Diff}_{\alpha^0}{}^q(C^*)$ as sets for every positive integer q and the set of sequences of divided powers in C is isomorphic to the set of higher differentiations of C^*.*

Proof. Since α is a group like element of C, it is easy to see that α^0 is an augmentation of C^*. Since C is finite dimensional, $\mathrm{Ker}(\alpha^0)$ is finite dimensional. We can apply Proposition 4. Now

$$\mathrm{Diff}_{\alpha^0}{}^q(C^*) = L_{\alpha^0}{}^q(C^{*0}) = L_{\alpha^0}{}^q(C^{**}) \cong L_\alpha{}^q(C).$$

Finally the set of higher differentiations of C^* is the set of sequences of divided powers in $C^{*0} = C^{**} \cong C$ (Proposition 5), which is isomorphic to the set of sequences of divided powers in C.

2. Sequences of divided powers.

PROPOSITION 7. *Let $(x_0, x_1, \ldots, x_i, \ldots)$ and $(y_0, y_1, \ldots, y_i, \ldots)$ be two sequences of divided powers in a coalgebra C. If $x_i = y_i$ for $i = 0, 1, \ldots, N$, then*

$$x_{N+j} - y_{N+j} \in L_{x_0}{}^j(C) \quad \text{for} \quad j = 1, 2, \ldots .$$

Proof. We prove by induction on j. For $j = 1$,

$$\begin{aligned}
\Delta(x_{N+1} - y_{N+1})
&= \Delta(x_{N+1}) - \Delta(y_{N+1}) \\
&= \sum_{i=0}^{N+1} x_i \otimes x_{N+1-i} - \sum_{i=0}^{N+1} y_i \otimes y_{N+1-i} \\
&= \sum_{i=0}^{N+1} x_i \otimes x_{N+1-i} - \left(\sum_{i=1}^{N} x_i \otimes x_{N+1-i} + x_0 \otimes y_{N+1} + y_{N+1} \otimes x_0\right) \\
&= x_0 \otimes x_{N+1} + x_{N+1} \otimes x_0 - x_0 \otimes y_{N+1} - y_{N+1} \otimes x_0 \\
&= x_0 \otimes (x_{N+1} - y_{N+1}) + (x_{N+1} - y_{N+1}) \otimes x_0.
\end{aligned}$$

Thus $x_{N+1} - y_{N+1} \in L_{x_0}{}^1(C)$. Now assume the proposition is true up to $j - 1$ and prove $x_{N+j} - y_{N+j} \in L_{x_0}{}^j(C)$. By Corollary 1 of Theorem 1, it suffices to show that $[x_{N+j} - y_{N+j}, f] \in L_{x_0}{}^{j-1}(C)$ for all $f \in C^*$. But

$$\begin{aligned}
&[x_{N+j} - y_{N+j}, f] \\
&= (f \otimes I)\Delta(x_{N+j} - y_{N+j}) - f(x_0)(x_{N+j} - y_{N+j}) - f(x_{N+j} - y_{N+j})x_0 \\
&= (f \otimes I)(\Delta x_{N+j} - \Delta y_{N+j}) - f(x_0)(x_{N+j} - y_{N+j}) - f(x_{N+j} - y_{N+j})x_0 \\
&= \sum_{i=0}^{N+j} (f(x_i)x_{N+j-i} - f(y_i)y_{N+j-i}) - f(x_0)(x_{N+j} - y_{N+j}) - f(x_{N+j} - y_{N+j})x_0 \\
&= \sum_{i=1}^{N+j-1} (f(x_i)x_{N+j-i} - f(y_i)y_{N+j-i}) \\
&= \sum_{i=1}^{N+j-1} (f(x_{N+j-i})x_i - f(y_{N+j-i})y_i) \\
&= \sum_{i=1}^{j-1} (f(x_{N+j-i})x_i - f(y_{N+j-i})y_i) + \sum_{i=j}^{N+j-1} f(x_{N+j-i})(x_i - y_i).
\end{aligned}$$

Thus by our induction hypothesis, we see that

$$[x_{N+j} - y_{N+j}, f] \in L_{x_0}{}^{j-1}(C) \text{ for all } f \in C^*.$$

PROPOSITION 8. *Let B be a bialgebra over a field K of characteristic zero. Let $\alpha \in G(B), x_1, \ldots, x_j, \in L_1{}^1(B)$ and*

$$d_j = \sum_{(q_1,\ldots,q_j)\in S(j)} \alpha x_1{}^{q_1} \ldots x_j{}^{q_j}/(q_1!) \ldots (q_j!)$$

for $j = 0, 1, 2, \ldots$, where

$$S(j) = \{(q_1, \ldots, q_j) \in \mathbf{N}^j | q_1 + 2q_2 + \ldots + jq_j = j\}.$$

Then $(d_0, d_1, \ldots, d_j, \ldots)$ is a sequence of divided powers in B.

Proof. First note that for any $x \in L_1{}^1(B)$, the sequence

$$(1, x, x^2/2!, \ldots, x^j/j!, \ldots)$$

is a sequence of divided powers in B (the proof is simply by induction). Using this and the fact that

$$\Delta d_j = \sum_{(q_1,\ldots,q_j)} \Delta\alpha\Delta(x_1{}^{q_1}/q_1!) \ldots \Delta(x_j{}^{q_j}/q_j!),$$

one can show that $\Delta d_j = \sum_{i=0}^{j} d_i \otimes d_{j-i}$ by computing.

THEOREM 2. *Let B be a bialgebra over a field K of characteristic zero. Then a sequence $(1 = d_0, d_1, \ldots, d_j, \ldots)$ in B is a sequence of divided powers in B if and only if*

$$d_j = \sum_{(q_1,\ldots,q_j)\in S(j)} x_1^{q_1} \ldots x_j^{q_j}/(q_1!) \ldots (q_j!)$$

(with $j = 0, 1, 2, \ldots$) for some $x_1, \ldots, x_j$ in $L_1^1(B)$, where

$$S(j) = \{(q_1, \ldots, q_j) \in \mathbf{N}^j | q_1 + 2q_2 + \ldots + jq_j = j\}.$$

Proof. By Proposition 8, it suffices to show that if $(1 = d_0, d_1, \ldots, d_j, \ldots)$ is a sequence of divided powers in B, then d_j is of the above form. Since $d_1 \in L_1^1(B)$ by Proposition 2, this is true for $j = 1$. So assume

$$d_i = \sum_{(q_1,\ldots,q_i)\in S(i)} x_1^{q_1} \ldots x_i^{q_i}/(q_1!) \ldots (q_i!)$$

for $i = 0, 1, \ldots, j - 1$. We now show d_j is of the above form. Let $y_j \in L_1^1(B)$ and

$$d_j' = \sum_{(q_1,\ldots,q_j)\in S(j)} x_1^{q_1} \ldots x_{j-1}^{q_{j-1}} y_j^{q_j}/(q_1!) \ldots (q_j!).$$

Then by Proposition 8, $(1, d_1, \ldots, d_{j-1}, d_j')$ is a sequence of divided powers in B. But by Proposition 7, $d_j - d_j' \in L_1^1(B)$. Thus $d_j = d_j' + z_j$ for some $z_j \in L_1^1(B)$. Let $x_j = y_j + z_j$. Then $x_j \in L_1^1(B)$ and we have

$$d_j = \sum_{(q_1,\ldots,q_j)\in S(j)} x_1^{q_1} \ldots x_j^{q_j}/(q_1!) \ldots (q_j!).$$

COROLLARY. *Assume further that $G(B)$ is a group. Then a sequence $(d_0, d_1, \ldots, d_j, \ldots)$ in B is a sequence of divided powers in B if and only if $d_0 \in G(B)$ and*

$$d_j = \sum_{(q_1,\ldots,q_j)\in S(j)} d_0 x_1^{q_1} \ldots x_j^{q_j}/(q_1!) \ldots (q_j!)$$

(with $j = 0, 1, 2, \ldots$) for some $x_1, \ldots, x_j, \ldots$ in $L_1^1(B)$, where

$$S(j) = \{(q_1, \ldots, q_j) \in \mathbf{N}^j | q_1 + 2q_2 + \ldots + jq_j = j\}.$$

Proof. Again by Proposition 8, we only have to prove the result in one direction. So suppose $(d_0, d_1, \ldots, d_j, \ldots)$ is a sequence of divided powers in B. Then by Proposition 2, $d_0 \in G(B)$ and $d_j \in L_{d_0}{}^j(B)$ for $j = 1, 2, \ldots$. Since $G(B)$ is a group, $L_{d_0}{}^j(B) = d_0 L_1^j(B)$ by Theorem 4 below. Therefore for each j, there exists $y_j \in L_1^j(B)$ such that $d_j = d_0 y_j$. So on one hand, we have

$$\Delta d_j = \Delta d_0 \Delta y_j = (d_0 \otimes d_0)\Delta y_j.$$

But on the other hand,

$$\Delta d_j = \sum_{i=0}^{j} d_i \otimes d_{j-i} = \sum_{i=0}^{j} d_0 y_i \otimes d_0 y_{j-i} = (d_0 \otimes d_0) \sum_{i=0}^{j} y_i \otimes y_{j-i}.$$

Thus

$$(d_0 \otimes d_0)\Delta y_i = (d_0 \otimes d_0) \sum_{i=0}^{j} y_i \otimes y_{j-i}.$$

Since d_0^{-1} exists, we get $\Delta y_i = \sum_{i=0}^{j} y_i \otimes y_{j-i}$, i.e., $(1, y_1, \ldots, y_j, \ldots)$ is a sequence of divided powers in B. Then applying Theorem 2, we are done.

Remarks. (1) In particular, the sequences of divided powers in a Hopf algebra H of characteristic zero are completely determined by the elements of $L_1^1(H)$ and $G(H)$.

(2) The results of this section are for higher differentiations also (cf. Proposition 5).

3. The structure of coalgebras. We now show that high order Lie-like elements are useful in studying the structure of coalgebras.

For $\alpha \in G(C)$, $x \in C$ and $f \in C^*$, we set

$$\begin{aligned}[f, x]_\alpha &= f > x - f(\alpha)x - f(x)\alpha \\ &= (I \otimes f)\Delta x - f(\alpha)x - f(x)\alpha.\end{aligned}$$

We drop the subscript α if it is clear from the context. We shall show (cf. Corollary 1 of Theorem 1)

$$L_\alpha^n(C) = \{x \in C | [f, x] \in L_\alpha^{n-1}(C) \text{ for all } f \in C^*\}.$$

Also, if we put $P_\alpha^1(C) = L_\alpha^1(C)$, and for any integer $n > 1$ set

$$P_\alpha^n(C) = \{x \in C | \Delta x = \alpha \otimes x + x \otimes \alpha + y; y \in \textstyle\sum_{i=1}^{n-1} L_\alpha^i(C) \otimes L_\alpha^{n-i}(C)\},$$

then we shall show $L_\alpha^n(C) = P_\alpha^n(C)$.

Let (C, Δ, ϵ) be a pointed irreducible coalgebra over K and α the group-like element of C. Then $K\alpha$ is the coradical of C and we have

$$K\alpha = C_0 \subset C_1 \subset C_2 \subset \ldots \subset C_i \subset \ldots,$$

the coradical filtration on C, where $C_i = \Lambda^{i+1}K\alpha$ and $C = \bigcup C_i$. Let $C_n^+ = C_n \cap \operatorname{Ker}(\epsilon)$ for every positive integer n. Then it is easy to see that $C_n^+ = \{x - \epsilon(x)\alpha | x \in C_n\}$ and $C_n = K\alpha \oplus C_n^+$ (for details, see [7]).

LEMMA 1. *Let (C, Δ, ϵ) be a pointed irreducible coalgebra over K and α the group-like element. Then $P_\alpha^n(C) = L_\alpha^n(C) = C_n^+$ for every positive integer n; hence $C_n = K\alpha \oplus P_\alpha^n(C) = K\alpha \oplus L_\alpha^n(C)$ and $C = K\alpha \oplus L_\alpha^\infty(C)$.*

Proof. We show $P_\alpha^n(C) \subset L_\alpha^n(C) \subset C_n^+ \subset P_\alpha^n(C)$ for any positive integer n. Using the fact that $L_\alpha^n(C) = \{x \in C | [x, f] \in L_\alpha^{n-1}(C) \text{ for all } f \in C^*\}$, one sees easily that $P_\alpha^n(C) \subset L_\alpha^n(C)$. To prove the second inclusion, let x be any element of $L_\alpha^n(C)$. Then

$$\Delta_n x = \textstyle\sum_{s=1}^{n} (-1)^{s-1} \sum_{i_1 < \ldots < i_s} T_{i_1 \ldots i_s}(\otimes^s \alpha \otimes \Delta_{n-s} x).$$

On the other hand

$$C_n = \Lambda^{n+1}K\alpha = \operatorname{Ker}(C \xrightarrow{\Delta_n} \otimes^{n+1}C \to \otimes^{n+1}C/K\alpha).$$

Thus it is clear that $x \in C_n$. Also we know that $\epsilon(x) = 0$ (Proposition 1). Thus $x \in C_n \cap \operatorname{Ker}(\epsilon) = C_n^+$. Since x is an arbitrary element of $L_\alpha^n(C)$, we have $L_\alpha^n(C) \subset C_n^+$. Finally we prove the third inclusion by induction on n.

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If $n = 1$, it is known that $C_1 = K\alpha \oplus L_\alpha^1(C)$ [**7**, Proposition 10.0.1]. Thus $C_1^+ = L_\alpha^1(C) = P_\alpha^1(C)$. So suppose $C_i^+ \subset P_\alpha^i(C)$ for $i = 1, \ldots, n-1$. For any element $z \in C_n^+$, $\Delta z = \alpha \otimes z + z \otimes \alpha + y$, where $y \in \sum_{i=1}^{n-1} C_i^+ \otimes C_{n-i}^+$ [**7**, Proposition 10.0.2]. By our induction hypothesis, we have

$$y \in \sum_{i=1}^{n-1} L_\alpha^{\,i}(C) \otimes L_\alpha^{n-i}(C).$$

Thus $z \in P_\alpha^n(C)$. Since z is any element of C_n^+, we conclude $C_n^+ \subset P_\alpha^n(C)$.

LEMMA 2. *Let (C, Δ, ϵ) be any coalgebra over K, α a group-like element of C and C^α the irreducible component of C containing α. Let Λ_α be the restriction of Λ to C^α, $C_i = \Lambda^{i+1}K\alpha$ and $C_i^\alpha = \Lambda_\alpha^{i+1}K\alpha$ for $i = 0, 1, 2, \ldots$. Then $C_i = C_i^\alpha$ for all i.*

Proof. Since $K\alpha$ is a subcoalgebra of C, each C_i is a subcoalgebra of C and $C_0 \subset C_1 \subset C_2 \subset \ldots$ [**7**, Proposition 9.0.0.i]. Furthermore,

$$\Delta(C_n) \subset \sum_{i=0}^n C_i \otimes C_{n-i} \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

[**7**, Theorem 9.1.6]. Therefore $\bigcup C_i$ is a filtered coalgebra. Since $C_0 = K\alpha$ contains every simple subcoalgebra of $\bigcup C_i$ [**7**, Proposition 11.1.1], $\bigcup C_i$ is pointed irreducible and $\bigcup C_i \subset C^\alpha$. Since $C_i \subset C^\alpha$, $C_i = \Lambda^{i+1}K\alpha = \Lambda_\alpha^{i+1}K\alpha = C_i^\alpha$ for each i and $C^\alpha = \bigcup C_i^\alpha = \bigcup C_i$.

THEOREM 3. *With the same assumptions as in Lemma 2, the following sets are equal for every integer $n > 1$.*

(1) $L_\alpha^n(C^\alpha)$,
(2) $P_\alpha^n(C^\alpha)$,
(3) $(C_n^\alpha)^+ = C_n^\alpha \cap \operatorname{Ker}(\epsilon)$,
(4) $C_n^+ = C_n \cap \operatorname{Ker}(\epsilon)$,
(5) $L_\alpha^n(C)$,
(6) $\{x \in C \mid [x, f] \in L_\alpha^{n-1}(C) \text{ for all } f \in C^*\}$,
(7) $P_\alpha^n(C)$,
(8) $\{x \in C \mid \Delta x = \alpha \otimes x + x \otimes \alpha + y; y \in L_\alpha^{n-1}(C) \otimes L_\alpha^{n-1}(C)\}$, *and*
(9) $\{x \in C \mid [f, x] \in L_\alpha^{n-1}(C) \text{ for all } f \in C^*\}$.

Proof. (1) = (2) = (3) by Lemma 1 and (3) = (4) by Lemma 2. To see that (4) = (5), we show (1) $\subset$ (5) $\subset$ (4). But $L_\alpha^n(C^\alpha) \subset L_\alpha^n(C)$ and $L_\alpha^n(C) \subset C_n$ by definitions. $L_\alpha^n(C) \subset \operatorname{Ker}(\epsilon)$ is known (Proposition 1). (5) = (6) is also known (Corollary 1 of Theorem 1). (6) = (7) since one sees easily that (2) $\subset$ (7) $\subset$ (6). Since (7) $\subset$ (8) $\subset$ (9) is easily seen, to complete the proof it suffices to show (9) $\subset$ (4). Let x be any element of (9) and assume

$$\Delta x = \alpha \otimes x + x \otimes \alpha + \sum_{i=1}^m y_i \otimes z_i,$$

where $\{z_i\}_{i=1}^m$ is a linearly independent set. Since

$$[f, x] = (I \otimes f)\Delta x - f(\alpha)x - f(x)\alpha \in L_\alpha^{n-1}(C)$$

for all $f \in C^*$, we see $y_i \in L_\alpha^{n-1}(C)$ for all i (by picking proper f's). Thus

$\Delta x \in C \otimes K\alpha + C_{n-1} \otimes C$; and (by [7, Proposition 9.0.0(a)]),

$$x \in \Delta^{-1}(C \otimes K\alpha + C_{n-1} \otimes C) = C_{n-1}\Lambda K\alpha = \Lambda^{n+1}K\alpha = C_n,$$

Finally we show $x \in \mathrm{Ker}(\epsilon)$.

$$x = (\epsilon \otimes I)\Delta x = (\epsilon \otimes I)(\alpha \otimes x + x \otimes \alpha + \sum_{i=1}^{m} y_i \otimes z_i)$$
$$= \epsilon(\alpha)x + \epsilon(x)\alpha + \sum_{i=1}^{m} \epsilon(y_i)z_i = x + \epsilon(x)\alpha.$$

This implies $\epsilon(x)\alpha = 0$, hence $\epsilon(x) = 0$. Thus $x \in C_n \cap \mathrm{Ker}(\epsilon)$. Since x is any element of (9), we have (9) $\subset$ (4).

Remark. For any coalgebra C, we can write down its pointed irreducible components as $C^\alpha = K\alpha \oplus L_\alpha^\infty(C)$, where $\alpha \in G(C)$. By virtue of Theorem 1, we know pretty well the structure of C^α. If C is the direct sum of its pointed irreducible components (e.g., C is pointed and cocommutative), we have

$$C = \sum_{\alpha \in G(C)} \oplus (K\alpha \oplus L_\alpha^\infty(C)).$$

THEOREM 4. *Let B be a bialgebra and $\alpha, \beta \in G(B)$. Then $\alpha L_\alpha^q(B) \subset L_{\alpha\beta}^q(B)$, $L_\beta^q(B)\alpha \subset L_{\beta\alpha}^q(B)$ (equalities if α^{-1} exists in $G(B)$) and $L_\alpha^r(B)L_\beta^s(B) \subset L_{\alpha\beta}^{r+s}(B)$ for any positive integers q, r, and s.*

Proof. For any $x \in L_\beta^q(B)$,

$$\begin{aligned}\Delta_q\alpha x &= \Delta_q\alpha\Delta_q x\\ &= (\otimes^{q+1}\alpha)(\textstyle\sum_{s=1}^{q} (-1)^{s-1} \sum_{i_1<\ldots<i_s} T_{i_1\ldots i_s}(\otimes^s\beta \otimes \Delta_{q-s}x)\\ &= \textstyle\sum_{s=1}^{q} (-1)^{s-1} \sum_{i_1<\ldots<i_s} T_{i_1\ldots i_s}(\otimes^s\alpha\beta \otimes \Delta_{q-s}\alpha x).\end{aligned}$$

Thus $\alpha x \in L_{\alpha\beta}^q(B)$ and we conclude $\alpha L_\beta^q(B) \subset L_{\alpha\beta}^q(B)$. Similarly we can prove $L_\beta^q(B)\alpha \subset L_{\beta\alpha}^q(B)$. If α^{-1} exists in $G(B)$, then any element $x \in L_{\alpha\beta}^q(B)$ can be written as $\alpha(\alpha^{-1}x) \in \alpha L_\beta^q(B)$ and any element $y \in L_{\beta\alpha}^q(B)$ can be written as $(y\alpha^{-1})\alpha \in L_\beta^q(B)\alpha$. Hence the two inclusions become equalities.

We prove the third inclusion by induction on $n = r + s$. For any $x \in L_\alpha^1(B)$ and $y \in L_\beta^1(B)$, we have $\Delta x = \alpha \otimes x + x \otimes \alpha$ and $\Delta y = \beta \otimes y + y \otimes \beta$. Thus

$$\Delta xy = \Delta x\Delta y = (\alpha \otimes x + x \otimes \alpha)(\beta \otimes y + y \otimes \beta)$$
$$= \alpha\beta \otimes xy + xy \otimes \alpha\beta + \alpha y \otimes x\beta + x\beta \otimes \alpha y.$$

Since αy and $x\beta$ are in $L_{\alpha\beta}^1(B)$ by the first two inclusions, we see that $xy \in L_{\alpha\beta}^2(B)$ using Theorem 3. Hence $L_\alpha^1(B)L_\beta^1(B) \subset L_{\alpha\beta}^2(B)$. So suppose $L_\alpha^p(B)L_\beta^q(B) \subset L_{\alpha\beta}^{p+q}(B)$ whenever $p + q < r + s$. Let $x \in L_\alpha^r(B)$ and $y \in L_\beta^s(B)$. Using Theorem 3, we can assume

$$\Delta x = \alpha \otimes x + x \otimes \alpha + u, \qquad u \in L_\alpha^{r-1}(B) \otimes L_\alpha^{r-1}(B),$$
$$\Delta y = \beta \otimes y + y \otimes \beta + v, \qquad v \in L_\beta^{s-1}(B) \otimes L_\beta^{s-1}(B).$$

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By the first two inclusions and our induction hypothesis, it is easy to see that

$$\Delta xy = \alpha\beta \otimes xy + xy \otimes \alpha\beta + w, \qquad w\, L_{\alpha\beta}{}^{r+s-1}(B) \otimes L_{\alpha\beta}{}^{r+s-1}(B).$$

Thus by Theorem 3 we see $xy \in L_{\alpha\beta}{}^{r+s}(B)$. This completes the proof.

The following is a consequence of Theorems 3 and 4.

COROLLARY. *Let B be a bialgebra and B^α, B^β two pointed irreducible components of B, where $\alpha, \beta \in G(B)$. Suppose that*

$$K\alpha = B_0{}^\alpha \subset B_1{}^\alpha \subset \ldots \subset B_i{}^\alpha \subset \ldots$$

and

$$K\beta = B_0{}^\beta \subset B_1{}^\beta \subset \ldots \subset B_j{}^\beta \subset \ldots$$

are the coradical filtrations. Then $B_i{}^\alpha B_j{}^\beta \subset B_{i+j}{}^{\alpha\beta}$ for any non-negative integers i and j. If α^{-1} exists in $G(B)$, then $\alpha B_i{}^\beta = B_i{}^{\alpha\beta}$ and $B_i{}^\beta\alpha = B_i{}^{\beta\alpha}$ for any non-negative integer i.

REFERENCES

1. S. T. Chang, *Higher order derivations and high order Lie-like elements*, Ph.D. thesis, Queen's University, 1971.
2. L. A. Grünenfelder, *Hopf-Algebren und Coradikal*, Math. Z. *116* (1970), 166–182.
3. R. G. Heyneman and M. E. Sweedler, *Affine Hopf algebras. I*, J. of Algebra *13* (1969), 192–241.
4. G. Hochschild, *Algebraic groups and Hopf algebras*, Illinois J. Math. *14* (1970), 52–65.
5. R. G. Larson, *Cocommutative Hopf algebras*, Can. J. Math. *19* (1967), 350–360.
6. Y. Nakai, *High order derivations. I*, Osaka J. Math. *7* (1970), 1–27.
7. M. E. Sweedler, *Hopf algebras* (Benjamin Inc., New York, 1969).

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