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ON THE "ZERO–TWO" LAW FOR POSITIVE CONTRACTIONS*

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0. Introduction

Let (X,Σ,μ) be a measure space (where μ is a positive σ-additive measure) and let $L^p(X,\Sigma,\mu)$, $1\leqq p\leqq +\infty$ be the usual real Banach lattices.

Let E be a real Banach lattice (all the Banach lattices considered in this paper are real). A linear bounded operator $T:E\to E$ is called a positive contraction of E if T is a positive operator (i.e., $x\in E$, $x\geqq 0\Rightarrow Tx\geqq 0$) and if $\|T\|\leqq 1$.

In 1970 Ornstein and Sucheston obtained a result (Theorem 1.1 of [**2**]) which was the first one in a row of several theorems, usually called "zero–two" laws.

Theorem 1.1 of [**2**] is called the "zero–two" law for positive contractions of L^1-spaces. Using its proof one obtains a second form of the "zero–two" law for positive contractions of L^1-spaces:

Theorem A. *Let T be a positive contraction of $L^1(X,\Sigma,\mu)$. If for some $m\in\mathbf{N}\cup\{0\}$ $\|T^{m+1}-T^m\|<2$, then $\lim_{n\to+\infty}\|T^{n+1}-T^n\|=0$.*

In this paper we study a property possessed by L^1 and L^∞-spaces. As an application, we obtain a new proof of Theorem A.

The property in question (which will be discussed in Section 1) can be stated as follows:

Theorem B. *Let E be an L^1 or an L^∞-space, and let $S,T:E\to E$ be two positive contractions of E such that $S\leqq T$ (i.e., $T-S$ is a positive operator). If $\|T-S\|<1$, then $\|T^n-S^n\|<1$ for every $n\in\mathbf{N}$.*

Using the duality of AM and AL-spaces (see Proposition 9.1, p. 121 of [**3**]), it is obvious that in order to prove Theorem B, it is enough to prove it under the assumption that E is an L^1-space. If E is an L^∞-space, Theorem B is proved directly. We

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will also show that Theorem B fails to be true if we assume that E is an L^p-space, $1<p<+\infty$ (the reason behind the construction made in Section 2 of [**4**]).

In Section 2 we will use Theorem B, the linear modulus of regular operators, and a procedure due to Foguel in [**1**], in order to give a new proof of Theorem A.

Unless stated otherwise, in this paper we use the terminology of Schaefer's book [**3**].

1. Positive contractions in L^p-spaces, $1 \leqq p \leqq +\infty$

Theorem 1.1. *Let E be an L^1-space and let $S, T: E \to E$ be two positive contractions of E such that $S \leqq T$. If $\|T-S\|<1$, then for every $n \in \mathbf{N}$ $\|T^n-S^n\|<1$.*

Proof. Let us assume that for some $n \in \mathbf{N}$, $\|T^n-S^n\|=1$ and let m be the first natural number such that $\|T^m-S^m\|=1$. Clearly, $m \geqq 2$.

Since T^m-S^m is a positive operator, there exists a sequence $(x_\ell)_{\ell \in \mathbf{N}}$ such that $x_\ell \in E$, $x_\ell \geqq 0$, $\|x_\ell\|=1$ for every $\ell \in \mathbf{N}$, the sequence $(\|(T^m-S^m)x_\ell\|)_{\ell \in \mathbf{N}}$ converges and $\lim_{\ell \to +\infty}\|(T^m-S^m)x_\ell\|=1$.

Since E is an L^1-space, for every $\ell \in \mathbf{N}$ $\|(T^m-S^m)x_\ell\|=\|T^m x_\ell\|-\|S^m x_\ell\|$. Hence, $\lim_{\ell \to +\infty}\|T^m x_\ell\|=1$ and $\lim_{\ell \to +\infty}\|S^m x_\ell\|=0$.

The sequences $(\|S^{m-1}x_\ell\|)_{\ell \in \mathbf{N}}$ and $(\|TS^{m-1}x_\ell\|)_{\ell \in \mathbf{N}}$ are bounded; accordingly, we may pick a subsequence $(x_{\ell_h})_{h \in \mathbf{N}}$ of $(x_\ell)_{\ell \in \mathbf{N}}$ such that $(\|S^{m-1}x_{\ell_h}\|)_{h \in \mathbf{N}}$ converges and a subsequence $(x_{\ell_{h_k}})_{k \in \mathbf{N}}$ of $(x_{\ell_h})_{h \in \mathbf{N}}$ such that $(\|TS^{m-1}x_{\ell_{h_k}}\|)_{k \in \mathbf{N}}$ converges.

Set $y_k = x_{\ell_{h_k}}$ for every $k \in \mathbf{N}$.

Let

$$\alpha = \lim_{k \to +\infty} \|S^{m-1}y_k\| \quad \text{and}$$

$$\beta = \lim_{k \to +\infty} \|TS^{m-1}y_k\|.$$

Since $\|T^{m-1}-S^{m-1}\|<1$ and $\lim_{k \to +\infty}\|T^{m-1}y_k\|=1$, it follows that $\alpha>0$.

If $\alpha>0$, then it is obvious that we may choose $(y_k)_{k \in \mathbf{N}}$ such that $S^{m-1}y_k \neq 0$ for every $k \in \mathbf{N}$.

We note that for every $k \in \mathbf{N}$

$$\begin{aligned}\|TS^{m-1}y_k\| &= \|T^m y_k\| - \|T^m y_k - TS^{m-1}y_k\| \geqq \|T^m y_k\| - \|T^{m-1}y_k - S^{m-1}y_k\| \\ &= \|S^{m-1}y_k\| + \|T^m y_k\| - \|T^{m-1}y_k\|.\end{aligned}$$

Since $\lim_{k \to +\infty}(\|T^m y_k\| - \|T^{m-1}y_k\|)=0$, we obtain that

$$\lim_{k \to +\infty}\|TS^{m-1}y_k\| \geqq \lim_{k \to +\infty}\|S^{m-1}y_k\|,$$

that is, $\beta \geqq \alpha$.

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Clearly, $\beta \leqq \alpha$, since for every $k \in \mathbf{N}$ $\|TS^{m-1}y_k\| \leqq \|S^{m-1}y_k\|$. Hence, $\alpha = \beta$.
For every $k \in \mathbf{N}$ let

$$z_k = \frac{S^{m-1}y_k}{\|S^{m-1}y_k\|}.$$

It follows that $\lim_{k\to+\infty} \|Tz_k\| = 1$ and $\lim_{k\to+\infty} \|Sz_k\| = 0$. Hence, $\lim_{k\to+\infty} \|(T-S)z_k\| = 1$. Since for every $k \in \mathbf{N}$ $\|z_k\| = 1$, we obtain that $\|T-S\| = 1$, that is, a contradiction. □

Using the duality of AM and AL-spaces (see, for example, Proposition 9.1, p. 121 of [3]), one can readily see that Theorem 1.1 remains true if, instead of an L^1-space, one considers E to be an L^∞-space.

Under the assumption that E is an L^∞-space, we can prove Theorem 1.1 directly (without using the duality of AM and AL-spaces). Since we think that the direct proof is of interest in itself we will give it here.

Let E be an L^∞-space (hence, E is an AM-space with unit). By a classical result due to S. Kakutani, M. Krein and S. Krein (see, for example, Corollary 1, p. 104 of [3]) there exists a Hausdorff compact topological space K and an isometric lattice isomorphism of E onto $C(K)$ (where, as usual, we note by $C(K)$ the Banach lattice of all continuous functions $f: K \to R$, the norm on $C(K)$ being defined by $\|f\| = \sup_{t\in K} |f(t)|$). Accordingly, in order to prove Theorem 1.1 under the assumption that E is an L^∞-space, it is enough to prove that given a Hausdorff compact topological space K and two positive contractions S, T of $C(K)$ such that $S \leqq T$ and $\|T-S\| \leqq 1$, one has that $\|T^n - S^n\| < 1$ for every $n \in \mathbf{N}$.

Let 1_K be the constant one function (i.e., 1_K is the unit of $C(K)$).

Clearly, $T1_K \leqq 1_K$, since T is a positive contraction.

We will distinguish two cases:

(i) $T1_K = 1_K$ and

(ii) $T1_K \neq 1_K$.

(i) Let $\alpha > 0$ be such that $\|T-S\| = 1-\alpha$. Since $T-S$ is a positive contraction of $C(K)$, we obtain that $\|T-S\| = \|(T-S)1_K\|$. Using the fact that 1_K is the largest element of the unit ball of $C(K)$, we deduce that $(T-S)1_K \leqq (1-\alpha)1_K$. Our assumption $T1_K = 1_K$ implies that $S1_K \geqq \alpha 1_K$; therefore, $S^n 1_K \geqq \alpha^n 1_K$ for every $n \in \mathbf{N}$. Consequently, $\|T^n - S^n\| = \|(T^n - S^n)1_K\| \leqq 1 - \alpha^n$ for every $n \in \mathbf{N}$.

(ii) If we assume that $T1_K \neq 1_K$, then $g = 1_K - T1_K$ is a positive element of $C(K)$. Define a positive contraction R of $C(K)$ by $Rf = gf$ for every $f \in C(K)$.

Clearly, $T+R$ and $S+R$ are positive operators. Moreover, $T+R$ and $S+R$ are positive contractions since $(S+R)1_K \leqq (T+R)1_K = 1_K$.

Set $U = S+R$ and $V = T+R$.

Clearly, $\|V-U\| = \|T-S\| < 1$ and the positive contractions U, V are in the case (i); accordingly, $\|V^n - U^n\| < 1$ for every $n \in \mathbf{N}$.

For every $n \in \mathbf{N}$ $0 \leqq T^n - S^n \leqq V^n - U^n$ since

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$$V^n=(T+R)^n=\sum_{\substack{i_1,i_2,\ldots,i_n,j_1,j_2,\ldots,j_n\in\{0,1\}\\ i_k+j_k=1,k=1,2,\ldots,n}} T^{i_1}R^{j_1}T^{i_2}R^{j_2}\ldots T^{i_n}R^{j_n}$$

$$U^n=(S+R)^n=\sum_{\substack{i_1,i_2,\ldots,i_n,j_1,j_2,\ldots,j_n\in\{0,1\}\\ i_k+j_k=1,k=1,2,\ldots,n}} S^{i_1}R^{j_1}S^{i_2}R^{j_2}\ldots S^{i_n}R^{j_n}$$

and

$$S^{i_1}R^{j_1}S^{i_2}R^{j_2}\ldots S^{i_n}R^{j_n}\leqq T^{i_1}R^{j_1}T^{i_2}R^{j_2}\ldots T^{i_n}R^{j_n}$$

for every

$$i_1,i_2,\ldots,i_n,j_1,j_2,\ldots,j_n\in\{0,1\}, i_k+j_k=1, k=1,2,\ldots,n.$$

Accordingly, $\|T^n-S^n\|<1$.

We have therefore proved directly that Theorem 1.1 remains true if we replace the L^1-space E by an L^∞-space.

Unfortunately, Theorem 1.1 does not remain true if one replaces the L^1-space E by an L^p-space, $1<p<+\infty$.

Indeed, let $L^p(X,\Sigma,\mu)$, $1\leqq p<+\infty$ be the 2-dimensional L^p-space defined as follows: $X=\{1,2\}$, $\Sigma=\mathscr{P}(\{1,2\})$ and the measure μ is generated by $\mu(\{1\})=\mu(\{2\})=1/2$. Accordingly, we may think of $L^p(X,\Sigma,\mu)$, $1\leqq p<+\infty$ as the Banach lattice $\mathbf{R}^2$ endowed with the norm

$$\|(a_1,a_2)\|_p=\left(\frac{|a_1|^p}{2}+\frac{|a_2|^p}{2}\right)^{1/p}$$

for every $(a_1,a_2)\in\mathbf{R}^2$.

Let $S,T:L^p(X,\Sigma,\mu)\to L^p(X,\Sigma,\mu)$ be two linear bounded operators defined as follows:

$$T(a_1,a_2)=(a_1,a_2)\begin{bmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{bmatrix}=\left(\frac{a_1+a_2}{2},\frac{a_1+a_2}{2}\right)$$

and

$$S(a_1,a_2)=(a_1,a_2)\begin{bmatrix}0 & 1/2\\ 0 & 0\end{bmatrix}=\left(0,\frac{a_1}{2}\right)$$

for every $(a_1,a_2)\in L^p(X,\Sigma,\mu)$, $1\leqq p<+\infty$.

Obviously, S, T are positive operators and $S\leqq T$.

The operator T is a contraction since for every $(a_1,a_2)\in\mathbf{R}^2$, $a_1,a_2\geqq 0$

$$\|T(a_1,a_2)\|_p=\frac{a_1+a_2}{2}\leqq\|(a_1,a_2)\|_p$$

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for every $1 \leqq p < +\infty$. Hence, S is a contraction as well.

An easy computation shows that

$$\sup\{a_1+a_2 : a_1, a_2 \in \mathbb{R};\ a_1, a_2 \geqq 0;\ \tfrac{1}{2}(a_1^p + a_2^p) = 1\} = 2$$

for every $1 \leqq p < +\infty$.

Now let $p \in \mathbb{R}$ be such that $1 < p < +\infty$. If $(a_1, a_2) \in \mathbb{R}^2$ is such that $a_1, a_2 \geqq 0$,

$$\|(a_1, a_2)\|_p = \left(\frac{a_1^p}{2} + \frac{a_2^p}{2}\right)^{1/p} = 1,$$

then

$$\|(T-S)(a_1, a_2)\|_p^p = \left\|\left(\frac{a_1+a_2}{2}, \frac{a_2}{2}\right)\right\|_p^p$$

$$= \frac{(a_1+a_2)^p}{2^p} \cdot \frac{1}{2} + \frac{a_2^p}{2^p} \cdot \frac{1}{2} \leqq \frac{2^p}{2^p} \cdot \frac{1}{2} + \frac{2}{2^p} \cdot \frac{1}{2} < 1.$$

We have therefore proved that $T-S$ (as a positive contraction of $L^p(X, \Sigma, \mu)$) has the property that $\|T-S\| < 1$.

Clearly, $\|T\| = 1$ since $T(1,1) = (1,1)$. It is also obvious that $T^2 = T$ and $S^2 = 0$. Accordingly, $\|T^2 - S^2\| = 1$.

2. A new approach to the "zero–two" law in L^1-spaces

In this section our goal is to use Theorem 1.1 in order to obtain a new proof of Theorem A of the Introduction.

Let E be an L^1-space and let T be a positive contraction of E.

As in [4, § 3], let $\ell \in \mathbf{N}$, $n \in \mathbf{N} \cup \{0\}$ be given and let $V_\ell^{(1)}$ and Q_ℓ be two positive operators such that

$$T^{\ell(n+1)} = \left(\frac{I+T}{2}\right)^\ell V_\ell^{(1)} + Q_\ell.$$

For every $d \in \mathbf{N}$, $d \geqq 2$ we define the operator $V_\ell^{(d)}$ recursively by the formula

$$V_\ell^{(d)} = T^{\ell(n+1)} V_\ell^{(d-1)} + V_\ell^{(1)} Q_\ell^{d-1}.$$

By induction one proves (see [4, § 3]) that

$$T^{d\ell(n+1)} = \left(\frac{I+T}{2}\right)^\ell V_\ell^{(d)} + Q_\ell^d \tag{1}$$

for every $d \in \mathbf{N}$.

Proposition 2.1. *For every* $d \in \mathbf{N}$ $\|V_\ell^{(d)}\| \leqq 1 + \|V_\ell^{(1)}\|$.

Proof. Since for $d=1$ the proposition is obviously true, we will assume that $d \geqq 2$. Formula (1) shows that

$$\left(\frac{I+T}{2}\right)^\ell V_\ell^{(d-1)}$$

is a (positive) contraction for every $d \in \mathbf{N}$, $d \geqq 2$.

Let $x \in E$ be such that $x \geqq 0$ and $\|x\| \leqq 1$. Since E is an L^1-space we obtain that for every $d \in \mathbf{N}$, $d \geqq 2$

$$1 \geqq \left\|\left(\frac{I+T}{2}\right)^\ell V_\ell^{(d-1)} x\right\| = \frac{\sum_{i=0}^{\ell} \binom{\ell}{i} \|T^i V_\ell^{(d-1)} x\|}{2^\ell}$$

$$\geqq \frac{\sum_{i=0}^{\ell} \binom{\ell}{i}}{2^\ell} \|T^{\ell(n+1)} V_\ell^{(d-1)} x\| = \|T^{\ell(n+1)} V_\ell^{(d-1)} x\|.$$

We have therefore proved that $\|T^{\ell(n+1)} V_\ell^{(d-1)}\| \leqq 1$ for every $d \leqq \mathbf{N}$, $d \geqq 2$.

The operator Q_ℓ is a positive contraction (as a consequence of the way in which Q_ℓ was defined).

Accordingly,

$$\|V_\ell^{(d)}\| \leqq \|T^{\ell(n+1)} V_\ell^{(d-1)}\| + \|V_\ell^{(1)} Q_\ell^{d-1}\| \leqq 1 + \|V_\ell^{(1)}\|$$

for every $d \in \mathbf{N}$, $d \geqq 2$. □

Proposition 3.1 of [4] and the proposition we have just proved are similar. In both propositions we obtain upper bounds for the sequence $(\|V_\ell^{(d)}\|)_{d \in \mathbf{N}}$. The similarity is strengthened by the fact that

$$\lim_{q \to +\infty} ((\ell+1)^{1/q} + \|V_\ell^{(1)}\|) = 1 + \|V_\ell^{(1)}\|$$

for every $\ell \in \mathbf{N}$.

If we assume that $V_\ell^{(1)}$ is a positive contraction, then in the case of an L^1-space (Proposition 2.1) we obtain that $\|V_\ell^{(d)}\| \leqq 2$ for every $d \in \mathbf{N}$, while in the case of an L^p-space, $1 < p < +\infty$ given in Proposition 3.1 of [4], the upper bound for the sequence $(\|V_\ell^{(d)}\|)_{d \in \mathbf{N}}$ depends on ℓ (tends to $+\infty$ as ℓ tends to $+\infty$).

The next proposition is similar to Proposition 4.2 of [4]. As expected, in the case of an L^1-space the statement is stronger.

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Proposition 2.2. *Let E be an L^1-space and let T be a positive contraction of E. If for some $m \in \mathbb{N} \cup \{0\}$ $\|T^{m+1}-T^m\|<2$, then $\|T^{m+1}-(T^{m+1} \wedge T^m)\|<1$.*

Proof. It is well known (see, for example, Theorem 1.5, pp. 232–233 of [3]) that if $|T^{m+1}-T^m|=T^{m+1}+T^m-2(T^{m+1} \wedge T^m)$ is the linear modulus of $T^{m+1}-T^m$, then $\|T^{m+1}-T^m\|=\||T^{m+1}-T^m|\|$.

Let $\eta>0$ be such that $\|T^{m+1}-T^m\|=2(1-\eta)$ and let us assume that $\|T^{m+1}-(T^{m+1} \wedge T^m)\|=1$. It follows that there exists $x \in E$, $x \geqq 0$, $\|x\| \leqq 1$ such that $\|(T^{m+1}-(T^{m+1} \wedge T^m))x\|>1-\eta/4$.

Accordingly, $\|T^{m+1}x\|>1-\eta/4$ and $\|(T^{m+1} \wedge T^m)x\|<\eta/4$. Hence,

$$\||T^{m+1}-T^m|x\|=\|T^{m+1}x\|+\|T^m x\|$$

$$-2\|(T^{m+1} \wedge T^m)x\|>1-\frac{\eta}{4}+1-\frac{\eta}{4}-2\cdot\frac{\eta}{4}$$

$$=2-\eta=2\left(1-\frac{\eta}{2}\right).$$

We have obtained a contradiction, since $\||T^{m+1}-T^m|\|=2(1-\eta)$. □

Proposition 2.3. *Let E be an L^1-space and let T be a positive contraction of E. If for some $m \in \mathbb{N} \cup \{0\}$ $\|T^{m+1}-(T^{m+1} \wedge T^m)\|<1$, then $\lim_{n \to +\infty}\|T^{n+1}-T^n\|=0$.*

Proof It is known (see the proof of Theorem B in Section 4 of [4]) that using Stirling's formula, one can find a positive constant $\gamma>0$ such that for every Banach lattice E, for every positive contraction T of E and for every $\ell \in \mathbb{N}$

$$\left\|\left(\frac{I+T}{2}\right)^{\ell}-T\left(\frac{I+T}{2}\right)^{\ell}\right\| \leqq \gamma/\sqrt{\ell}.$$

Now let E be an L^1-space, let T be a positive contraction of E and let $m \in \mathbb{N} \cup \{0\}$ be such that $\|T^{m+1}-(T^{m+1} \wedge T^m)\|<1$.

Let $\varepsilon>0$ and let $\ell_\varepsilon \in \mathbb{N}$ be such that $\gamma/\sqrt{\ell_\varepsilon}<\varepsilon/4$.

By Theorem 1.1 we obtain that

$$\|T^{\ell_\varepsilon(m+1)}-(T^{m+1} \wedge T^m)^{\ell_\varepsilon}\|<1; \quad \text{therefore,}$$

$$\left\|T^{\ell_\varepsilon(m+1)}-\left(\frac{I+T}{2}\right)^{\ell_\varepsilon}(T^{m+1} \wedge T^m)^{\ell_\varepsilon}\right\|$$

$$\leqq \sum_{i=0}^{\ell_\varepsilon} \frac{\binom{\ell_\varepsilon}{i}}{2^{\ell_\varepsilon}}\|T^{\ell_\varepsilon(m+1)}-T^i(T^{m+1} \wedge T^m)^{\ell_\varepsilon}\|$$

$$\leqq \frac{\|T^{\ell_\varepsilon(m+1)}-(T^{m+1} \wedge T^m)^{\ell_\varepsilon}\|}{2^{\ell_\varepsilon}}+\sum_{i=1}^{\ell_\varepsilon} \frac{\binom{\ell_\varepsilon}{i}}{2^{\ell_\varepsilon}}<1.$$

Set

$$Q_{\ell_\varepsilon}=T^{\ell_\varepsilon(m+1)}-\left(\frac{I+T}{2}\right)^{\ell_\varepsilon}(T^{m+1}\wedge T^m)^{\ell_\varepsilon}$$

and let us define a sequence $(V_{\ell_\varepsilon}^{(d)})_{d\in\mathbf{N}}$ as follows: $V_{\ell_\varepsilon}^{(1)}=(T^{m+1}\wedge T^m)^{\ell_\varepsilon}$ and for every $d\in\mathbf{N}$, $d\geqq 2$ we define $V_{\ell_\varepsilon}^{(d)}$ using the recursion formula

$$V_{\ell_\varepsilon}^{(d)}=T^{\ell_\varepsilon(m+1)}V_{\ell_\varepsilon}^{(d-1)}+V_{\ell_\varepsilon}^{(1)}Q_{\ell_\varepsilon}^{d-1}.$$

Since $\|Q_{\ell_\varepsilon}\|<1$ we may choose $d_\varepsilon\in\mathbf{N}$ such that $\|Q_{\ell_\varepsilon}^{d_\varepsilon}\|<\varepsilon/4$.

Set $n_\varepsilon=d_\varepsilon\ell_\varepsilon(m+1)$. Using formula (1) as well as Proposition 2.1 we obtain

$$\|T^{n_\varepsilon+1}-T^{n_\varepsilon}\|=\left\|\left(T\left(\frac{I+T}{2}\right)^{\ell_\varepsilon}-\left(\frac{I+T}{2}\right)^{\ell_\varepsilon}\right)V_{\ell_\varepsilon}^{(d_\varepsilon)}+TQ_{\ell_\varepsilon}^{d_\varepsilon}-Q_{\ell_\varepsilon}^{d_\varepsilon}\right\|$$

$$\leqq\left\|T\left(\frac{I+T}{2}\right)^{\ell_\varepsilon}-\left(\frac{I+T}{2}\right)^{\ell_\varepsilon}\right\|\|V_{\ell_\varepsilon}^{(d_\varepsilon)}\|+\|TQ_{\ell_\varepsilon}^{d_\varepsilon}-Q_{\ell_\varepsilon}^{d_\varepsilon}\|$$

$$\leqq(\gamma/\sqrt{\ell_\varepsilon})\cdot 2+2\cdot\frac{\varepsilon}{4}<\varepsilon.$$

We have therefore proved that for every $\varepsilon>0$ there exists $n_\varepsilon\in\mathbf{N}$ such that $\|T^{n_\varepsilon+1}-T^{n_\varepsilon}\|<\varepsilon$. Since the sequence $(\|T^{n+1}-T^n\|)_{n\in\mathbf{N}\cup\{0\}}$ is a decreasing one, we have proved that $\lim_{n\to+\infty}\|T^{n+1}-T^n\|=0$. □

The results obtained in this section enable us to arrive at a new proof of Theorem A. Indeed, let T be a positive contraction of an L^1-space E, and assume that $\|T^{m+1}-T^m\|<2$ for some $m\in\mathbf{N}\cup\{0\}$. Using Proposition 2.2, we obtain that $\|T^{m+1}-(T^{m+1}\wedge T^m)\|<1$; therefore, by Proposition 2.3 $\lim_{n\to+\infty}\|T^{n+1}-T^n\|=0$.

REFERENCES

1. S. R. Foguel, More on the "zero–two" law, *Proc. Amer. Math. Soc.* **61** (1976), 262–264.

2. D. Ornstein and L. Sucheston, An operator theorem on L_1 convergence to zero with applications to Markov kernels, *Ann. Math. Statist.* **41** (1970), 1631–1639.

3. H. H. Schaefer, *Banach Lattices and Positive Operators*, Springer-Verlag, Berlin–Heidelberg–New York, 1974.

4. R. Zaharopol, The modulus of a regular linear operator and the "zero–two" law in L^p-spaces $(1<p<+\infty,\ p\neq 2)$, *J. Funct. Anal.* **68** (1986), 300–312.

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