Theorems connected with Simson's Line.

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[A digest of Mr Burgess' paper is given below.]

FIGURE 16.

(i) If XYZ is the Simson Line P(ABC); if PM is perpendicular to XYZ and cuts the sides BC, CA, AB in U, V, W:

then $$PU \, . \, PV \, . \, PW = PA \, . \, PB \, . \, PC = 4R^2 . \, PM.$$

(ii) If PX_1, PY_1, PZ_1; PX_2, PY_2, PZ_2 are the two sets of three straight lines which make angle α with the sides of the triangle ABC; so that X_1, Y_1, Z_1 are collinear, and X_2, Y_2, Z_2; and if $X_1Y_1Z_1$, $X_2Y_2Z_2$, intersect in Q:
Q is shown to lie on PM, and the Simson Line P(ABC) is shown to be the following Simson Lines

$$P(QX_1X_2), \text{ etc.}; \; P(AY_1Z_1), \text{ etc.}; \; P(AY_2Z_2), \text{ etc.}$$

(iii) *When* $\alpha = 45°$,

M is the mid-point of PQ; and if O is the orthocentre of ABC, OQ is therefore parallel to XYZ (since XYZ bisects OP). The locus of Q, as P moves on the circle, is given by the equation

$$\rho = 2R[\cos A \cos\theta - \sin\theta \sin\{2\theta - (B - C)\}]$$

with reference to O as pole and OA as initial line. The curve has three loops of different sizes and can easily be traced from the fact that OQ is at right angles to PQ.

If, in particular, ABC is equilateral (so that O is the circumcentre), the locus of Q is given by $\rho = R\cos 3\theta$, a hypotrochoid with three loops each of which is in area one-twelfth of the circle.

FIGURE 17.

If PX_1, PY_1, PZ_1; PX_2, PY_2, PZ_2 make angle α_1 with the sides of ABC;
and PX_3, PY_3, PZ_3; PX_4, PY_4, PZ_4 ,, ,, α_2 ,, ,, ,, ,, ,, ;
if Q_1, Q_2 are the two corresponding positions of Q and if the four lines $X_1Y_1Z_1$, etc., intersect one another besides in T_1, T_2, T_3, T_4 and intersect the Simson Line P(ABC) in L_1, L_2, L_3, L_4:

(i) T_1, T_2, T_3, T_4 lie on a circle which has P as centre and cuts orthogonally the circles $T_1Q_1Q_2$, $T_2Q_1Q_2$, $T_3Q_1Q_2$, $T_4Q_1Q_2$;

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(ii) L_1, L_2, L_3, L_4 are the mid-points of T_1T_3, T_2T_4, T_1T_4, T_2T_3; and T_1T_2, T_3T_4 are parallel to XYZ and equidistant from it;

(iii) the Simson Line P(ABC) is also the following 34 Simson Lines :—

$P(AY_1Z_1)$, etc.; $P(BZ_1X_1)$, etc.; $P(CX_1Y_1)$, etc.;
$P(Q_1X_1X_2)$, $P(Q_2X_3X_4)$, etc., etc.; $P(Q_1T_1T_4)$, $P(Q_1T_2T_3)$,
$P(Q_2T_1T_3)$, $P(Q_2T_2T_4)$; $P(T_1X_1X_3)$, $P(T_2X_2X_4)$, $P(T_3X_1X_4)$,
$P(T_4X_2X_3)$, etc., etc.

If A_1, A_2, A_3, A_4 be four concyclic points; $O_1, \ldots, O_4$ the orthocentres of the four triangles formed by them:
the quadrilaterals $A_1A_2A_3A_4$, $O_1O_2O_3O_4$ are equal in all respects, and if C, F be their circumcentres, the four Simson Lines of A_1, A_2, A_3, A_4 are concurrent at the mid-point P of CF, which is also the mid-point of $A_1O_1, \ldots, A_4O_4$.

FIGURE 18.

If $A_1, \ldots, A_5$ are points on a circle whose centre is C; $O_1, \ldots, O_5$ the orthocentres of the five triangles formed by sets of three consecutive vertices of the pentagon $A_1A_2A_3A_4A_5$; $Q_1, \ldots, Q_5$ the orthocentres of the five triangles formed each by one side and the opposite vertex; if $B_1, \ldots, B_5$, are the mid-points of the sides, and $G_1, \ldots, G_5$ the mid-points of the diagonals; if $F_1, \ldots, F_5$ are the circumcentres of the five cyclic quadrilaterals (see above) formed by orthocentres of triangles whose vertices are chosen from $A_1, \ldots, A_5$; and $P_1, \ldots, P_5$ are the mid-points of $CF_1, \ldots, CF_5$:

It is clear that the pentagon $P_1P_2P_3P_4P_5$ has its sides parallel to and half the length of the sides of the original pentagon; and if D is the circumcentre of this cyclic pentagon, and S the point of trisection such that $CS = 2SD$, the five straight lines $A_1P_1, \ldots, A_5P_5$ are concurrent at S which is a point of trisection of each, as also of the other ten lines $O_1B_1, \ldots, O_5B_5$, $Q_1G_1, \ldots, Q_5G_5$; S being the centre of homology of the two similar and similarly-situated pentagons.

Again the pentagon $F_1F_2F_3F_4F_5$ is equal in all respects to $A_1A_2A_3A_4A_5$ and similarly situated to it; and if E be its circumcentre, D bisects CE and is the centre of homology of $A_1A_2A_3A_4A_5$, $F_1F_2F_3F_4F_5$.

The theory can be extended according to the following table:—

Number of given points A_1, A_2, ...	Specification of polygon.	Ratio of linear dimensions to original polygon.	Circumcentre denoted by	Centre of homology with original polygon.
4	$O_1O_2O_3O_4$	1	F	P
5	$F_1F_2F_3F_4F_5$	1	E	D
	$P_1P_2P_3P_4P_5$	$\frac{1}{2}$	D	S
6	$E_1E_2E_3E_4E_5E_6$	1	H	K
	$D_1D_2D_3D_4D_5D_6$	$\frac{1}{2}$	K	L
	$S_1S_2S_3S_4S_5S_6$	$\frac{1}{3}$	L	M
7	$H_1H_2H_3H_4H_5H_6H_7$	1	etc.	etc.
	K_1 .... K_7	$\frac{1}{2}$		
	L_1 . ... L_7	$\frac{1}{3}$		
	M_1 .... M_7	$\frac{1}{4}$		
		etc.		

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