Bull. Aust. Math. Soc. **91** (2015), 286–302
doi:10.1017/S0004972714001105

MAXIMAL HARDY SPACES ASSOCIATED TO NONNEGATIVE SELF-ADJOINT OPERATORS

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(Received 20 July 2014; accepted 22 November 2014; first published online 23 December 2014)

Abstract

Let (X, d, μ) be a metric measure space satisfying the doubling, reverse doubling and noncollapsing conditions. Let $\mathscr{L}$ be a nonnegative self-adjoint operator on $L^2(X, d\mu)$ satisfying a pointwise Gaussian upper bound estimate and Hölder continuity for its heat kernel. In this paper, we introduce the Hardy spaces $H^p_{\mathscr{L}}(X)$, $0 < p \le 1$, associated to $\mathscr{L}$ in terms of grand maximal functions and show that these spaces are equivalently characterised by radial and nontangential maximal functions.

2010 *Mathematics subject classification*: primary 42B30; secondary 42B35, 35K08.

Keywords and phrases: Hardy spaces, metric measure spaces, self-adjoint operators, Gaussian heat kernel bounds, maximal functions.

1. Introduction and main result

The theory of real variable Hardy spaces $H^p(\mathbb{R}^n)$ was highly developed in the 1960s and 1970s (see, especially, the classical papers [11, 18]). Recall that, for $0 < p < \infty$, $H^p(\mathbb{R}^n)$ is defined as the space of all bounded tempered distributions f such that the Poisson maximal function

$$M_P f(x) := \sup_{t>0} |e^{-t\sqrt{-\Delta}} f(x)|$$

belongs to $L^p(\mathbb{R}^n)$. Here, $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$ is the classical Laplacian and thus $e^{-t\sqrt{-\Delta}}$ is the Poisson semigroup.

It is well known that the spaces $H^p(\mathbb{R}^n)$ are characterised by some other maximal functions. To state these characterisations, we need to introduce some notation. For $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the radial and nontangential maximal functions of f with respect to φ are defined as

$$M^0_\varphi f(x) := \sup_{t>0} |f * \varphi_t(x)| \quad \text{and} \quad M_\varphi f(x) := \sup_{t>0} \sup_{|y-x|<t} |f * \varphi_t(y)|,$$

This work is supported by the Program of Leading Graduate Schools, MEXT, Japan.

respectively, where $\varphi_t(x) := t^{-n}\varphi(t^{-1}x)$. For a fixed positive integer N, we set

$$\mathcal{F}_N(\mathbb{R}^n) := \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \mathfrak{N}_N(\varphi) \le 1\},$$

where

$$\mathfrak{N}_N(\varphi) := \int_{\mathbb{R}^n} (1+|x|)^N \sum_{|\beta|\le N+1} |\partial^\beta \varphi(x)|\,dx.$$

The grand maximal function of f (with respect to N) is defined as

$$M_N f(x) := \sup_{\varphi\in\mathcal{F}_N(\mathbb{R}^n)} M_\varphi f(x).$$

The following characterisations of $H^p(\mathbb{R}^n)$ are given in [11]: if $0 < p < \infty$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)\,dx \neq 0$ and $N \in \mathbb{N}$ with $N \ge \lfloor n/p \rfloor + 1$, then, for all $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$f \in H^p(\mathbb{R}^n) \iff M^0_\varphi f \in L^p(\mathbb{R}^n) \iff M_\varphi f \in L^p(\mathbb{R}^n) \iff M_N f \in L^p(\mathbb{R}^n).$$

On the other hand, the spaces $H^p(\mathbb{R}^n)$ are also characterised by various kinds of square functions. For example, an L^1 function f belongs to $H^1(\mathbb{R}^n)$ if and only if the Lusin (area integral) function

$$Sf(x) := \bigg(\iint_{\Gamma(x)} \bigg|\frac{\partial}{\partial t} e^{-t\sqrt{-\Delta}} f(y)\bigg|^2 t^{1-n}\,dy\,dt\bigg)^{1/2} \tag{1.1}$$

belongs to $L^1(\mathbb{R}^n)$; see [11, 17]. In 2004, Auscher *et al.* [1] introduced a class of Hardy spaces $H^1_{\mathscr{L}}(\mathbb{R}^n)$ associated to an operator $\mathscr{L}$ by means of the square function in (1.1) with the Poisson semigroup $e^{-t\sqrt{-\Delta}}$ replaced by the semigroup $e^{-t\mathscr{L}}$, under the assumption that $\mathscr{L}$ admits a heat kernel satisfying a pointwise Poisson upper bound. Later, Duong and Yan [5] introduced the BMO space (the space of all functions of bounded mean oscillation) associated to such an $\mathscr{L}$ and they proved in [6] that the BMO space associated to the adjoint operator $\mathscr{L}^*$ is the dual space of the space $H^1_{\mathscr{L}}(\mathbb{R}^n)$. Recently, Auscher *et al.* [2] studied the Hardy space H^1 associated to the Hodge Laplacian on a Riemannian manifold. Meanwhile, Hofmann and Mayboroda [14] investigated Hardy spaces associated to a second order divergence form elliptic operator $\mathscr{L}$ on $\mathbb{R}^n$ with complex coefficients. The theory of the Hardy spaces $H^p_{\mathscr{L}}(X)$, $1 \le p < \infty$, on a metric space X associated to a nonnegative self-adjoint operator $\mathscr{L}$ satisfying Davies–Gaffney estimates was developed in [13]. In all of these developments, the Hardy spaces $H^p_{\mathscr{L}}$ were introduced by means of the Lusin (area integral) function associated to the semigroups $e^{-t\mathscr{L}}$ or $e^{-t\sqrt{\mathscr{L}}}$.

In the case that $\mathscr{L} = -\Delta + V$ is a Schrödinger operator with a locally integrable nonnegative potential V, the H^p and BMO spaces associated to $\mathscr{L}$ were earlier investigated by Dziubański *et al.* (see [7, 9, 10] and the references therein). In these works, the spaces $H^p_{\mathscr{L}}(\mathbb{R}^n)$ were introduced by means of the radial maximal function associated to the semigroup $e^{-t\mathscr{L}}$, instead of using square functions. Note that the operator $\mathscr{L} = -\Delta + V$ satisfies the Davies–Gaffney estimates and it was proved in

[13, 15] that for such a special operator $\mathscr{L}$ the Hardy spaces defined via square functions are equivalent to those defined via maximal functions. Hence, the general theory developed in [13] applies to this Schrödinger setting. However, the spaces $H^p_{\mathscr{L}}(\mathbb{R}^n)$ associated to $\mathscr{L} = -\Delta + V$ enjoy some interesting properties which may not be satisfied by Hardy spaces associated to general operators satisfying Davies–Gaffney estimates. For instance, if the potential V satisfies certain additional assumptions (for example, the reverse Hölder inequality), the space $H^1_{\mathscr{L}}(\mathbb{R}^n)$ associated to $\mathscr{L} = -\Delta + V$ is characterised by the (generalised) Riesz transform $\nabla(-\Delta + V)^{-1/2}$ (see [8] for more details).

In the present paper, we focus on maximal Hardy spaces associated to operators. We shall introduce Hardy spaces associated to nonnegative self-adjoint operators in terms of 'grand' maximal functions and show that such Hardy spaces are equivalently characterised by the radial and nontangential maximal functions.

Now let us describe our result more precisely. We refer to Section 2 for all unfamiliar notation and definitions. Let (X, d, μ) be a metric measure space satisfying the doubling, reverse doubling and noncollapsing conditions. Let $\mathscr{L}$ be a nonnegative self-adjoint operator on $L^2(X, d\mu)$ whose heat kernel satisfies the Gaussian upper bound and Hölder continuity. We introduce the radial, nontangential and grand maximal functions associated to $\mathscr{L}$ as follows.

Definition 1.1. For $f \in \mathcal{S}'_{\mathscr{L}}(X)$, $\Phi \in \mathcal{S}(\mathbb{R}_+)$, $N \in \mathbb{N}_0$ and $x \in X$, define

$$M^0_{\mathscr{L},\Phi} f(x) := \sup_{t>0} |\Phi(t^2\mathscr{L}) f(x)|, \quad M_{\mathscr{L},\Phi} f(x) := \sup_{t>0} \sup_{d(y,x)<t} |\Phi(t^2\mathscr{L}) f(y)|$$

and

$$M_{\mathscr{L},N} f(x) := \sup_{\Phi\in\mathcal{F}_N(\mathbb{R}_+)} M_{\mathscr{L},\Phi} f(x).$$

Here, $\mathcal{F}_N(\mathbb{R}_+)$ is defined in (2.11) and $\Phi(t^2\mathscr{L})f$ is defined in (2.15).

We now introduce Hardy spaces associated to $\mathscr{L}$ by means of grand maximal functions.

Definition 1.2. For $p \in (0, 1]$, we define the Hardy space $H^p_{\mathscr{L}}(X)$ associated to $\mathscr{L}$ as

$$H^p_{\mathscr{L}}(X) := \{f \in \mathcal{S}'_{\mathscr{L}}(X) : M_{\mathscr{L},N_p} f \in L^p(X, d\mu)\}$$

with the quasi-norm given by

$$\|f\|_{H^p_{\mathscr{L}}(X)} := \|M_{\mathscr{L},N_p} f\|_{L^p(X,d\mu)},$$

where

$$N_p := \lfloor 2n/p \rfloor + \lfloor 3n/2 \rfloor + 4.$$

Here, the number n is the 'dimension' of the metric measure space X; see Section 2 below.

https://doi.org/10.1017/S0004972714001105 Published online by Cambridge University Press

The following theorem, which says that the spaces $H^p_{\mathscr{L}}(X)$ are equivalently characterised by radial and nontangential maximal functions, is the main result of the present paper.

THEOREM 1.3. *Suppose that* $\Phi \in \mathcal{S}(\mathbb{R}_+)$, $\Phi(0) \neq 0$ *and* $0 < p \leq 1$. *Then, for any* $f \in \mathcal{S}'_{\mathscr{L}}(X)$, *the following conditions are equivalent:*

(i) $f \in H^p_{\mathscr{L}}(X)$;
(ii) $M_{\mathscr{L},\Phi} f \in L^p(X, d\mu)$;
(iii) $M^0_{\mathscr{L},\Phi} f \in L^p(X, d\mu)$.

Moreover, the following (quasi-)norm equivalence is valid:

$$\|M_{\mathscr{L},N_p} f\|_{L^p(X,d\mu)} \sim \|M_{\mathscr{L},\Phi} f\|_{L^p(X,d\mu)} \sim \|M^0_{\mathscr{L},\Phi} f\|_{L^p(X,d\mu)}.$$

The rest of this paper is organised as follows. In Section 2, we review the main properties of doubling and reverse doubling metric measure spaces and the concepts of Schwartz functions and distributions on them, and review some important estimates derived from the Gaussian heat kernel bounds. In Section 3, we give the proof of Theorem 1.3.

Throughout the paper, the symbol $\mathbb{N}_0$ will denote the set of nonnegative integers. For any positive number σ, we denote by $\lfloor \sigma \rfloor$ the largest integer less than or equal to σ. The letter C will denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. By writing $A \lesssim B$, we mean $A \leq CB$. We also use $A \sim B$ to denote $A \lesssim B \lesssim A$. Constants with subscripts will remain unchanged throughout.

Note. After this paper was submitted for publication, we learned that Dekel *et al.* in a recent preprint [4] also treated maximal Hardy spaces associated to nonnegative self-adjoint operators. Although they also give a proof of Theorem 1.3, their method is totally different from ours.

2. Preliminaries

Let X be a locally compact metric space with a distance d and let μ be a positive, locally finite, regular Borel measure on X. Throughout the paper, we assume that $\mu(X) = \infty$.

Denote by $B(x, r)$ the open ball with centre $x \in X$ and radius $r > 0$ and by $V(x, r)$ its measure $\mu(B(x, r))$. The metric measure space (X, d, μ) satisfies the *doubling condition* if there exists a constant $C_* > 1$ such that

$$V(x, 2r) \leq C_* V(x, r)$$

for all $x \in X$ and $r \in (0, \infty)$. Notice that the doubling condition implies that

$$V(x, \lambda r) \leq C_* \lambda^n V(x, r) \tag{2.1}$$

for all $x \in X$, $r \in (0, \infty)$ and $\lambda \in [1, \infty)$, where $n = \log_2 C_* > 0$ is a constant playing the role of a dimension, but one should not confuse it with dimension. From (2.1), the

https://doi.org/10.1017/S0004972714001105 Published online by Cambridge University Press

local finiteness of μ and the infiniteness of $\mu(X)$, it follows that $\operatorname{diam} X = \infty$. Also, since $B(x, r) \subset B(y, d(x, y) + r)$, (2.1) yields

$$V(x, r) \leq C_* \Big(1 + \frac{d(x, y)}{r}\Big)^n V(y, r) \tag{2.2}$$

for all $x, y \in X$ and $r \in (0, \infty)$. The metric measure space (X, d, μ) is said to satisfy the *reverse doubling condition* if there exists a constant $C_\dagger > 1$ such that

$$V(x, 2r) \geq C_\dagger V(x, r)$$

for all $x \in X$ and $r \in (0, \infty)$. A consequence of the reverse doubling condition is that

$$V(x, \lambda r) \geq C_\dagger^{-1} \lambda^{\varsigma} V(x, r) \tag{2.3}$$

for all $x \in X$, $r \in (0, \infty)$ and $\lambda \in [1, \infty)$, where $\varsigma = \log_2 C_\dagger > 0$. We say that (X, d, μ) satisfies the *noncollapsing condition* if there exists a constant $C_\flat > 0$ such that

$$\inf_{x \in X} V(x, 1) \geq C_\flat.$$

The noncollapsing condition along with the doubling condition yields that, for all $r \in (0, 1]$,

$$\inf_{x \in X} V(x, r) \geq C_*^{-1} C_\flat r^n. \tag{2.4}$$

Throughout the paper, we assume that (X, d, μ) satisfies the doubling, reverse doubling and noncollapsing conditions.

Consider a nonnegative self-adjoint operator $\mathscr{L}$ with domain $D(\mathscr{L})$ dense in $L^2(X, d\mu)$. Let $E(\lambda)$ be the spectral resolution of $\mathscr{L}$. For any bounded Borel measurable function $\Phi : [0, \infty) \to \mathbb{C}$, by the spectral theorem we can define the operator

$$\Phi(\mathscr{L}) = \int_0^\infty \Phi(\lambda)\, dE(\lambda).$$

It is well known that the operator $\Phi(\mathscr{L})$ is bounded on $L^2(X, d\mu)$. We assume that the associated semigroup $P_t = e^{-t\mathscr{L}}$ consists of integral operators with real-valued (heat) kernel $p_t(x, y)$. We say that the heat kernel of $\mathscr{L}$ satisfies the *Gaussian upper bound* if there exist two constants $C_\sharp, c_\sharp > 0$ such that

$$|p_t(x, y)| \leq C_\sharp \frac{\exp\Big(-\frac{d(x,y)^2}{c_\sharp t}\Big)}{\sqrt{V(x, \sqrt{t}) V(y, \sqrt{t})}} \tag{2.5}$$

for all $t \in (0, \infty)$ and $x, y \in X$. We say that the heat kernel of $\mathscr{L}$ satisfies *Hölder continuity* if there exists a positive constant α such that

$$|p_t(x, y) - p_t(x, y')| \leq C_\sharp \Big(\frac{d(y, y')}{\sqrt{t}}\Big)^\alpha \frac{\exp\Big(-\frac{d(x,y)^2}{c_\sharp t}\Big)}{\sqrt{V(x, \sqrt{t}) V(y, \sqrt{t})}} \tag{2.6}$$

for all $t \in (0, \infty)$ and $x, y, y' \in X$ satisfying $d(y, y') \leq \sqrt{t}$. Throughout the paper, we assume that the heat kernel of $\mathscr{L}$ satisfies the Gaussian upper bound and Hölder continuity.

Examples of settings in which our theory applies include uniformly elliptic divergence form operators, Riemannian manifolds with nonnegative Ricci curvature and Lie groups of polynomial growth. For more examples which satisfy all the above assumptions, we refer the reader to [3, 16].

We use the following notation borrowed from [16]: for $t, \sigma > 0$ and $x, y \in X$, set

$$D_{t,\sigma}(x, y) = [V(x, t)V(y, t)]^{-1/2}\Big(1 + \frac{d(x, y)}{t}\Big)^{-\sigma}.$$

By [16, Lemma 2.1], for any $\sigma > n$ there exists a positive constant C (depending on σ) such that

$$\int_X \Big(1 + \frac{d(x, y)}{t}\Big)^{-\sigma} d\mu(y) \leq CV(x, t) \tag{2.7}$$

for all $t \in (0, \infty)$ and $x \in X$. This together with (2.2) yields that for any $\sigma > 3n/2$,

$$\|D_{t,\sigma}(x, \cdot)\|_{L^1(X,d\mu)} \leq C \tag{2.8}$$

uniformly for all $t \in (0, \infty)$ and $x \in X$.

The following estimate, proved by Kerkyacharian and Petrushev [16], is important to us.

Lemma 2.1 [16, Theorem 3.4]. *Suppose that $m \in \mathbb{N}_0$, $m \geq n + 1$, $r \geq m + n + 1$, $\Phi \in C^m(\mathbb{R}_+)$ and there exists a constant $A > 0$ such that*

$$|\Phi^{(\nu)}(\lambda)| \leq A(1 + \lambda)^{-r}$$

for all $\lambda \in \mathbb{R}_+$ and $\nu \in \{0, 1, \dots, m\}$. Suppose further that

$$\Phi^{2\nu+1}(0) = 0$$

for all $\nu \in \mathbb{N}_0$ such that $2\nu + 1 \leq m$. Then, for any $t > 0$, $\Phi(t\sqrt{\mathscr{L}})$ is an integral operator with a kernel $K_{\Phi(t\sqrt{\mathscr{L}})}(x, y)$; moreover, there exists a constant $C > 0$ (depending on m) such that

$$|K_{\Phi(t\sqrt{\mathscr{L}})}(x, y)| \leq CAD_{t,m}(x, y) \tag{2.9}$$

for all $t \in (0, \infty)$ and $x, y \in X$ and such that

$$|K_{\Phi(t\sqrt{\mathscr{L}})}(x, y) - K_{\Phi(t\sqrt{\mathscr{L}})}(x, y')| \leq CA\Big(\frac{d(y, y')}{t}\Big)^{\alpha} D_{t,m}(x, y) \tag{2.10}$$

for all $t \in (0, \infty)$ and $x, y, y' \in X$ satisfying $d(y, y') \leq t$.

Remark 2.2. In [16], the heat kernel of the operator is assumed to satisfy the 'local' Gaussian upper bound, namely, (2.5) and (2.6) hold only for $t \in (0, 1]$, so the estimates for the kernel $K_{\Phi(t\sqrt{L})}(x, y)$ in [16] are valid only for $t \in (0, 1]$. However, in the current paper we assume that (2.5) and (2.6) hold for all $t \in (0, \infty)$, from which the estimates (2.9) and (2.10) are valid for all $t \in (0, \infty)$.

Now we recall from [16] the notions of Schwartz functions and tempered distributions on X associated to $\mathscr{L}$. The Schwartz class $\mathcal{S}_{\mathscr{L}}(X)$ is defined to be the class of all functions $\phi \in \bigcap_{k\in\mathbb{N}_0} D(\mathscr{L}^k)$ such that

$$\mathcal{P}_{k,m}(\phi) := \operatorname*{ess\,sup}_{x\in X}(1+\rho(x,x_0))^m|\mathscr{L}^k\phi(x)| < \infty$$

for all $k, m \in \mathbb{N}_0$, where $x_0 \in X$ is selected arbitrarily and fixed once and for all. Clearly, the particular selection of x_0 in the above definition is not important, since if $\mathcal{P}_{k,m}(\phi) < \infty$ for one $x_0 \in X$, $\mathcal{P}_{k,m}(\phi) < \infty$ for any other selection of $x_0 \in X$. It is often more convenient to have a directed family of seminorms, so we define, for $k, m \in \mathbb{N}_0$ and $\phi \in \mathcal{S}_{\mathscr{L}}(X)$,

$$\mathcal{P}^*_{k,m}(\phi) := \sum_{\substack{0\le j\le k\\ 0\le \ell\le m}} \mathcal{P}_{j,\ell}(\phi).$$

It was shown in [16] that $\mathcal{S}_{\mathscr{L}}(X)$ is a Fréchet space. The space $\mathcal{S}'_{\mathscr{L}}(X)$ of distributions on X is defined as the set of all continuous linear functionals on $\mathcal{S}_{\mathscr{L}}(X)$. The evaluation of $f \in \mathcal{S}'_{\mathscr{L}}(X)$ on $\phi \in \mathcal{S}_{\mathscr{L}}(X)$ will be denoted by $(f, \phi) := f(\phi)$.

Let $\mathbb{R}_+ := [0, \infty)$ and define

$$\mathcal{S}(\mathbb{R}_+) := \Big\{\Phi \in C^\infty(\mathbb{R}_+) : \text{for all } \nu \in \mathbb{N}_0, \Phi^{(\nu)} \text{ decays rapidly at infinity and } \lim_{\lambda\to 0^+}\Phi^{(\nu)}(\lambda) \text{ exists}\Big\}.$$

Then Borel's theorem concerning the existence of smooth functions with arbitrary Maclaurin series implies that $\mathcal{S}(\mathbb{R}_+) = \mathcal{S}(\mathbb{R})|_{\mathbb{R}_+}$. Throughout the paper, we use the following notation: for any $N \in \mathbb{N}_0$ and any $\Phi \in \mathcal{S}(\mathbb{R}_+)$, we put

$$\|\Phi\|_{(N)} := \sup_{\lambda\in\mathbb{R}_+,\ 0\le\nu\le N} (1+\lambda)^{N+n+1}|\Phi^{(\nu)}(\lambda)|,$$

where $\Phi^{(\nu)}$ is the νth-order derivative of Φ. Then we set

$$\mathcal{F}_N(\mathbb{R}_+) := \{\Phi \in \mathcal{S}(\mathbb{R}_+) : \|\Phi\|_{(N)} \le 1\}. \tag{2.11}$$

Observe that if $\Phi \in \mathcal{S}(\mathbb{R}_+)$, then the function $\Psi : \mathbb{R}_+ \to \mathbb{C}$ defined by $\Psi(\lambda) := \Phi(\lambda^2)$ also lies in $\mathcal{S}(\mathbb{R}_+)$ and, moreover, $\Psi^{2\nu+1}(0) = 0$ for all $\nu \in \mathbb{N}_0$. Also note that for any $m \in \mathbb{N}_0$, there exists a constant $C > 0$, which depends on m but is independent of Φ, such that $\|\Psi\|_{(m)} \le C\|\Phi\|_{(m)}$. By these facts, we can reformulate Lemma 2.1 as follows.

Lemma 2.3. *For any $\Phi \in \mathcal{S}(\mathbb{R}_+)$ and $t > 0$, $\Phi(t^2\mathscr{L})$ is an integral operator with a kernel $K_{\Phi(t^2\mathscr{L})}(x, y)$; moreover, for any $m \in \mathbb{N}_0$ with $m \ge n + 1$, there is a constant $C > 0$, which depends on m but is independent of Φ, such that*

$$|K_{\Phi(t^2\mathscr{L})}(x, y)| \le C\|\Phi\|_{(m)}D_{t,m}(x, y) \tag{2.12}$$

for all $t \in (0, \infty)$ and $x, y \in X$ and such that

$$|K_{\Phi(t^2\mathscr{L})}(x, y) - K_{\Phi(t^2\mathscr{L})}(x, y')| \le C\|\Phi\|_{(m)}\Big(\frac{d(y, y')}{t}\Big)^\alpha D_{t,m}(x, y) \tag{2.13}$$

for all $t \in (0, \infty)$ and $x, y, y' \in X$ satisfying $d(y, y') \le t$.

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A consequence of Lemma 2.3 is the following result.

Corollary 2.4. *Suppose that $\Phi \in \mathcal{S}(\mathbb{R}_+)$. Then, for any $t > 0$, the kernel $K_{\Phi(t^2\mathscr{L})}(x, y)$ belongs to $\mathcal{S}_{\mathscr{L}}(X)$ as a function of x and as a function of y.*

Proof. Fix $t > 0$. Let $k, m \in \mathbb{N}_0$ be such that $m \geq n + 1$. From [16, (5.14)], we see that for any fixed $x \in X$,

$$\mathscr{L}^k[K_{\Phi(t^2\mathscr{L})}(x, \cdot)] = K_{\mathscr{L}^k\Phi(t^2\mathscr{L})}(x, \cdot) = t^{-2k}K_{(t^2\mathscr{L})^k\Phi(t^2\mathscr{L})}(x, \cdot).$$

Hence, by (2.12), we have that for all $y \in X$,

$$\begin{aligned} |\mathscr{L}^k[K_{\Phi(t^2\mathscr{L})}(x, \cdot)](y)| &= t^{-2k}|K_{(t^2\mathscr{L})^k\Phi(t^2\mathscr{L})}(x, y)| \\ &\leq Ct^{-2k}\|\lambda \mapsto \lambda^k\Phi(\lambda)\|_{(m)}D_{t,m}(x, y) \\ &\leq Ct^{-2k}\|\Phi\|_{(k+m)}D_{t,m}(x, y). \end{aligned} \tag{2.14}$$

This implies that $K_{\Phi(t^2\mathscr{L})}(x, \cdot) \in \mathcal{S}_{\mathscr{L}}(X)$ with x fixed. Since $K_{\Phi(t^2\mathscr{L})}(\cdot, y) = \overline{K_{\Phi(t^2\mathscr{L})}(y, \cdot)}$, we also have $K_{\Phi(t^2\mathscr{L})}(\cdot, y) \in \mathcal{S}_{\mathscr{L}}(X)$ with y fixed. □

Thanks to Corollary 2.4, it is now natural to define, for any $f \in \mathcal{S}'_{\mathscr{L}}(X)$ and $\Phi \in \mathcal{S}(\mathbb{R}_+)$,

$$\Phi(\mathscr{L})f(x) := (f, K_{\Phi(\mathscr{L})}(x, \cdot)), \quad x \in X. \tag{2.15}$$

This extends the domain of $\Phi(\mathscr{L})$ from $L^2(X, d\mu)$ to $\mathcal{S}'_{\mathscr{L}}(X)$.

3. Proof of Theorem 1.3

For the proof of Theorem 1.3, we need a sequence of lemmas.

Lemma 3.1. *Suppose that $\Phi \in \mathcal{S}(\mathbb{R}_+)$ and $\Phi(0) = 1$. Then, for any $\Psi \in \mathcal{S}(\mathbb{R}_+)$ and $N \in \mathbb{N}_0$, there exist a family $\{\Theta_{(s)}\}_{0\leq s\leq 1}$ of functions in $\mathcal{S}(\mathbb{R}_+)$ and a constant $C > 0$ such that:*

(i) $\Psi(\lambda) = \int_0^1 \Theta_{(s)}(\lambda)\Phi(s^2\lambda)\, ds$ *for all* $\lambda \in \mathbb{R}_+$;

(ii) $\int_X (1 + (d(x, y))/t)^N |K_{\Theta_{(s)}(t^2\mathscr{L})}(x, y)|\, d\mu(y) \leq Cs^N\|\Psi\|_{(2N+\lfloor 3n/2\rfloor+3)}$ *for all* $t > 0$ *and* $x \in X$.

Proof. We follow [12, Theorem 4.9]. Fix $N \in \mathbb{N}_0$. Let $\{\Omega_{(s)}\}_{0\leq s\leq 1}$ be the unique family of functions in $\mathcal{S}(\mathbb{R}_+)$ such that

$$\partial_s^{N+1}[\Phi(s^2\lambda)^{N+2}] = \Phi(s^2\lambda)\Omega_{(s)}(\lambda) \quad \text{for all } s \in [0, 1], \text{ for all } \lambda \in \mathbb{R}_+. \tag{3.1}$$

Notice that $\Omega_{(s)}$ has the expression

$$\Omega_{(s)}(\lambda) = \sum_{j_1+\cdots+j_{N+1}=N+1} C_{j_1,\ldots,j_k}\partial_s^{j_1}[\Phi(s^2\lambda)]\cdots\partial_s^{j_k}[\Phi(s^2\lambda)], \tag{3.2}$$

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where each $C_{j_1,\dots,j_k}$ is a nonnegative integer. Choose $\Xi \in C^\infty([0,1])$ such that

$$\Xi(s) = s^N/N! \quad \text{for all } s \in [0, 1/2],$$
$$0 \le \Xi(s) \le s^N/N! \quad \text{for all } s \in [1/2, 1],$$
$$\partial_s^j \Xi(1) = 0 \quad \text{for all } j \in \{0, 1, \dots, N+1\}.$$

Then we set

$$\Theta_{(s)}(\lambda) = (-1)^{N+1}\Xi(s)\Omega_{(s)}(\lambda)\Psi(\lambda) - [\partial_s^{N+1}\Xi(s)]\Phi(s^2\lambda)^{N+1}\Psi(\lambda), \quad \lambda \in \mathbb{R}_+. \tag{3.3}$$

Clearly, $\Theta_{(s)} \in \mathcal{S}(\mathbb{R}_+)$ for every $s \in [0,1]$. We claim that (i) and (ii) hold for this choice of $\Theta_{(s)}$.

First we verify (i). Consider the integral

$$I(\lambda) = (-1)^{N+1}\int_0^1 \Xi(s)\{\partial_s^{N+1}[\Phi(s^2\lambda)^{N+2}]\}\Psi(\lambda)\,ds, \quad \lambda \in \mathbb{R}_+. \tag{3.4}$$

Integrating by parts $N+1$ times and noting that the boundary terms in the first N integrations by parts vanish,

$$\begin{aligned} I(\lambda) &= -[\partial_s^N\Xi(s)]\Phi(s^2\lambda)^{N+2}\Psi(\lambda)|_{s=0}^1 + \int_0^1 [\partial_s^{N+1}\Xi(s)]\Phi(s^2\lambda)^{N+2}\Psi(\lambda)\,ds \\ &= \Psi(\lambda) + \int_0^1 [\partial_s^{N+1}\Xi(s)]\Phi(s^2\lambda)^{N+2}\Psi(\lambda)\,ds, \end{aligned}$$

where we used $\Phi(0) = 1$. Hence, by (3.4), (3.1) and (3.3),

$$\Psi(\lambda) = I(\lambda) - \int_0^1 [\partial_s^{N+1}\Xi(s)]\Phi(s^2\lambda)^{N+2}\Psi(\lambda)\,ds = \int_0^1 \Theta_{(s)}(\lambda)\Phi(s^2\lambda)\,ds.$$

Next we verify (ii). Since $\Xi(s)$ is constant for $s \in [0, \frac{1}{2}]$, we have $|\partial_s^{N+1}\Xi(s)| \le Cs^N$ for all $s \in [0,1]$. From this fact, (3.2) and (3.3), it is not difficult to see that for every $m \in \mathbb{N}_0$,

$$\|\Theta_{(s)}\|_{(m)} \le Cs^N\|\Psi\|_{(m+N+1)}, \tag{3.5}$$

where the constant C depends on Φ and m, but is independent of $s \in [0,1]$ and Ψ. Take $m = N + \lfloor 3n/2 \rfloor + 2$ $(\ge n+1)$. Then it follows from (2.12), (3.5) and (2.8) that

$$\begin{aligned} \int_X \Big(1 + \frac{d(x,y)}{t}\Big)^N |K_{\Theta_{(s)}(t^2\mathscr{L})}(x,y)|\,d\mu(y) &\le C\|\Theta_{(s)}\|_{(m)} \int_X \Big(1 + \frac{d(x,y)}{t}\Big)^N D_{t,m}(x,y)\,d\mu(y) \\ &= \|\Theta_{(s)}\|_{(m)} \int_X D_{t,m-N}(x,y)\,d\mu(y) \\ &\le Cs^N\|\Psi\|_{(m+N+1)} = Cs^N\|\Psi\|_{(2N+\lfloor 3n/2\rfloor+3)}. \end{aligned}$$

This verifies (ii) and completes the proof. □

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Lemma 3.2. *Suppose that $\Phi \in \mathcal{S}(\mathbb{R}_+)$ with $\Phi(0) = 1$. Then, for any $N \in \mathbb{N}_0$, there exists a constant $C > 0$ such that for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$ and $x \in X$,*

$$M_{\mathscr{L},2N+\lfloor 3n/2\rfloor+3} f(x) \le C T^N_{\mathscr{L},\Phi} f(x), \tag{3.6}$$

where

$$T^N_{\mathscr{L},\Phi} f(x) = \sup_{y\in X, t>0} |\Phi(t^2\mathscr{L})f(y)|\Big(1 + \frac{d(x,y)}{t}\Big)^{-N}. \tag{3.7}$$

Proof. For any given $\Psi \in \mathcal{S}(\mathbb{R}_+)$, write $\Psi(\cdot) = \int_0^1 \Theta_{(s)}(\cdot)\Phi(s^2\cdot)\,ds$ as in Lemma 3.1. Then, for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$, $t \in (0,\infty)$ and $y \in X$,

$$\begin{aligned}\Psi(t^2\mathscr{L})f(y) &= \int_0^1 \Theta_{(s)}(t^2\mathscr{L})\Phi(s^2t^2\mathscr{L})f(y)\,ds\\ &= \int_0^1\int_X \Phi(s^2t^2\mathscr{L})f(z)K_{\Theta_{(s)}(t^2\mathscr{L})}(y,z)\,d\mu(z)\,ds.\end{aligned}$$

It follows that

$$\begin{aligned}|\Psi(t^2\mathscr{L})f(y)| &\le \int_0^1\int_X |\Phi(s^2t^2\mathscr{L})f(z)||K_{\Theta_{(s)}(t^2\mathscr{L})}(y,z)|\,d\mu(z)\,ds\\ &\le T^N_{\mathscr{L},\Phi}f(x)\int_0^1\int_X\Big(1+\frac{d(x,z)}{st}\Big)^N|K_{\Theta_{(s)}(t^2\mathscr{L})}(y,z)|\,d\mu(z)\,ds\\ &\le T^N_{\mathscr{L},\phi}f(x)\int_0^1\int_X s^{-N}\Big(1+\frac{d(x,y)+d(y,z)}{t}\Big)^N|K_{\Theta_{(s)}(t^2\mathscr{L})}(y,z)|\,d\mu(z)\,ds.\end{aligned}$$

Note that if $y \in B(x,t)$, then $1 + (d(x,y) + d(y,z))/t < 2(1 + (d(y,z))/t)$. Hence, by Lemma 3.1(ii),

$$\begin{aligned}M_{\mathscr{L},\Psi}f(x) &\le 2^N T^N_{\mathscr{L},\Phi}f(x)\int_0^1\int_X s^{-N}\Big(1+\frac{d(y,z)}{t}\Big)^N|K_{\Theta_{(s)}(t^2\mathscr{L})}(y,z)|\,d\mu(z)\,ds\\ &\le C\|\Psi\|_{(2N+\lfloor 3n/2\rfloor+3)}T^N_{\mathscr{L},\Phi}f(x),\end{aligned}$$

which yields the desired inequality (3.6). □

Lemma 3.3. *For any $\Phi \in \mathcal{S}(\mathbb{R}_+)$, $p \in (0,1]$ and $N \in \mathbb{N}_0$ with $N > n/p$, there exists a constant $C > 1$ such that for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$,*

$$C^{-1}\|M_{\mathscr{L},\Phi}f\|_{L^p(X,d\mu)} \le \|T^N_{\mathscr{L},\Phi}f\|_{L^p(X,d\mu)} \le C\|M_{\mathscr{L},\Phi}f\|_{L^p(X,d\mu)},$$

where $T^N_{\mathscr{L},\Phi}f$ is defined by (3.7).

Proof. Obviously, $M_{\mathscr{L},\Phi}f(x) \le 2^N T^N_{\mathscr{L},\Phi}f(x)$ for every $x \in X$, so the first inequality holds as long as $C > 2^N$. To see the second inequality, set $q = n/N$, so that $q < p$. Observe that

$$|\Phi(t^2\mathscr{L})f(y)| \le M_{\mathscr{L},\Phi}f(z) \quad \text{whenever } z \in B(y,t).$$

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From this and (2.1),

$$\begin{aligned}|\Phi(t^2\mathscr{L})f(y)|^q &\le \frac{1}{V(y,t)}\int_{B(y,t)}[M_{\mathscr{L},\Phi}f(z)]^q\,d\mu(z)\\ &\le \frac{V(x,t+d(x,y))}{V(y,t)}\frac{1}{V(x,t+d(x,y))}\int_{B(x,t+d(x,y))}[M_{\mathscr{L},\Phi}f(z)]^q\,d\mu(z)\\ &\lesssim \Big(1+\frac{d(x,y)}{t}\Big)^n \mathcal{M}_{\mathrm{HL}}([M_{\mathscr{L},\Phi}f(\cdot)]^q)(x),\end{aligned}$$

where $\mathcal{M}_{\mathrm{HL}}$ is the Hardy–Littlewood maximal operator. Since $N = n/q$, this says that for all $x \in X$,

$$[T^N_{\mathscr{L},\Phi}f(x)]^q \lesssim \mathcal{M}_{\mathrm{HL}}([M_{\mathscr{L},\Phi}f(\cdot)]^q)(x).$$

Then, since $p/q > 1$, the Hardy–Littlewood maximal theorem yields

$$\int_X [T^N_{\mathscr{L},\Phi}f(x)]^p\,d\mu(x) \lesssim \int_X \{\mathcal{M}_{\mathrm{HL}}([M_{\mathscr{L},\Phi}f(\cdot)]^q)(x)\}^{p/q}\,d\mu(x) \lesssim \int_X [M_{\mathscr{L},\Phi}f(x)]^p\,d\mu(x).$$

This completes the proof. □

For our purpose we introduce two auxiliary maximal type functions: for $f \in \mathcal{S}'_{\mathscr{L}}(X)$, $\Phi \in \mathcal{S}(\mathbb{R}_+)$, $K \in \mathbb{N}_0$, $N \in \mathbb{N}_0$ and $\varepsilon \in (0,1]$, we set

$$M^{\varepsilon K}_{\mathscr{L},\Phi}f(x) = \sup_{0<t<1/\varepsilon}\ \sup_{y\in B(x,t)} |\Phi(t^2\mathscr{L})f(y)|\Big(\frac{t}{t+\varepsilon}\Big)^K(1+\varepsilon d(y,x_0))^{-K},$$

$$T^{\varepsilon NK}_{\mathscr{L},\Phi}f(x) = \sup_{0<t<1/\varepsilon}\ \sup_{y\in X}|\Phi(t^2\mathscr{L})f(y)|\Big(1+\frac{d(x,y)}{t}\Big)^{-N}\Big(\frac{t}{t+\varepsilon}\Big)^K(1+\varepsilon d(y,x_0))^{-K}.$$

Lemma 3.4. *For any $\Phi \in \mathcal{S}(\mathbb{R}_+)$, $p \in (0,1]$ and $N \in \mathbb{N}_0$ with $N > n/p$, there exists $C > 0$ such that for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$, $\varepsilon \in (0,1]$ and $K \in \mathbb{N}_0$,*

$$\|T^{\varepsilon NK}_{\mathscr{L},\Phi}f\|_{L^p(X,d\mu)} \le C\|M^{\varepsilon K}_{\mathscr{L},\Phi}f\|_{L^p(X,d\mu)}.$$

Proof. The proof is the same as that of Lemma 3.3 and is omitted. □

Lemma 3.5. *For any $\Phi \in \mathcal{S}_{\mathscr{L}}(X)$, $p \in (0,1]$ and $f \in \mathcal{S}'_{\mathscr{L}}(X)$, there exists $K \in \mathbb{N}_0$ such that $M^{\varepsilon K}_{\mathscr{L},\Phi}f \in L^p(X,d\mu)\cap L^\infty(X,d\mu)$ for $0 < \varepsilon \le 1$.*

Proof. By the definition of $\mathcal{S}'_{\mathscr{L}}(X)$, there exist $k_0, m_0 \in \mathbb{N}_0$ such that

$$|\Phi(t^2\mathscr{L})f(y)| = |(f, K_{\Phi(t^2\mathscr{L})}(y,\cdot))| \le C\mathcal{P}^*_{k_0,m_0}(K_{\Phi(t^2\mathscr{L})}(y,\cdot)). \tag{3.8}$$

Let $M \in \mathbb{N}_0$ be such that $M \ge \max\{m_0 + n/2, n+1\}$. Then, by (2.14) and (2.2),

$$
\begin{aligned}
\mathcal{P}^*_{k_0,m_0}(K_{\Phi(t^2\mathscr{L})}(y,\cdot)) &= \sum_{\substack{0\le k\le k_0\\ 0\le m\le m_0}} \sup_{z\in X}(1+d(z,x_0))^m|\mathscr{L}^k[K_{\Phi(t^2\mathscr{L})}(y,\cdot)](z)| \\
&\le C\sum_{0\le k\le k_0}\sup_{z\in X}(1+d(z,x_0))^{m_0}t^{-2k}\|\Phi\|_{(k+M)}D_{t,M}(y,z) \\
&\le C\sum_{0\le k\le k_0}\sup_{z\in X}\frac{t^{-2k}(1+d(z,x_0))^{m_0}}{V(z,t)}\Big(1+\frac{d(y,z)}{t}\Big)^{-M+n/2} \\
&\le C\sum_{0\le k\le k_0}\sup_{z\in X}\frac{t^{-2k}(1+d(z,x_0))^{m_0}}{V(z,t)}\Big(1+\frac{d(y,z)}{t}\Big)^{-m_0}.
\end{aligned} \tag{3.9}
$$

Note that if $t\in(0,1]$, then by (2.4) and the triangle inequality for the distance d,

$$
\begin{aligned}
&\sum_{0\le k\le k_0}\sup_{z\in X}\frac{t^{-2k}(1+d(z,x_0))^{m_0}}{V(z,t)}\Big(1+\frac{d(y,z)}{t}\Big)^{-m_0} \\
&\quad\le C\sup_{z\in X}t^{-(2k_0+n)}\Big(1+\frac{d(z,x_0)}{t}\Big)^{m_0}\Big(1+\frac{d(y,z)}{t}\Big)^{-m_0} \\
&\quad\le Ct^{-(2k_0+n)}\Big(1+\frac{d(y,x_0)}{t}\Big)^{m_0}\le Ct^{-(2k_0+n+m_0)}(1+d(y,x_0))^{m_0}.
\end{aligned} \tag{3.10}
$$

If $t\in(1,1/\varepsilon]$, then from (2.3) and the triangle inequality for the distance d, it follows that

$$
\begin{aligned}
&\sum_{0\le k\le k_0}\sup_{z\in X}\frac{t^{-2k}(1+d(z,x_0))^{m_0}}{V(z,t)}\Big(1+\frac{d(y,z)}{t}\Big)^{-m_0} \\
&\quad\le Ct^{-\varsigma}(1+d(z,x_0))^{m_0}\Big(1+\frac{d(y,z)}{t}\Big)^{-m_0} \\
&\quad\le Ct^{m_0-\varsigma}\Big(1+\frac{d(z,x_0)}{t}\Big)^{m_0}\Big(1+\frac{d(y,z)}{t}\Big)^{-m_0} \\
&\quad\le Ct^{m_0-\varsigma}\Big(1+\frac{d(y,x_0)}{t}\Big)^{m_0}\le Ct^{m_0-\varsigma}(1+d(y,x_0))^{m_0}.
\end{aligned} \tag{3.11}
$$

Also note that if $t\in(0,1]$ and $K\ge 2k_0+n+m_0$, then

$$
\Big(\frac{t}{t+\varepsilon}\Big)^K t^{-(2k_0+n+m_0)}\le\Big(\frac{1}{t+\varepsilon}\Big)^{2k_0+n+m_0}\le\varepsilon^{-(2k_0+n+m_0)}, \tag{3.12}
$$

while if $t\in(1,1/\varepsilon)$, then, for any $K\in\mathbb{N}_0$,

$$
\Big(\frac{t}{t+\varepsilon}\Big)^K t^{m_0-\varsigma}\le t^{|m_0-\varsigma|}\le\varepsilon^{-|m_0-\varsigma|}. \tag{3.13}
$$

We now choose $K\in\mathbb{N}_0$ such that $K\ge\max\{2k_0+n+m_0,\ m_0+n/p\}$. Then from (3.8) to (3.13) it follows that for any fixed $\varepsilon\in(0,1]$ and for all $t\in(0,1/\varepsilon]$,

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$$|\Phi(t^2\mathscr{L})f(y)|\Big(\frac{t}{t+\varepsilon}\Big)^K(1+\varepsilon d(y,x_0))^{-K} \le |\Phi(t^2\mathscr{L})f(y)|\Big(\frac{t}{t+\varepsilon}\Big)^K\varepsilon^{-K}(1+d(y,x_0))^{-K}$$
$$\le C(1+d(y,x_0))^{-K+m_0},$$

where the constant C depends on ε. Hence,

$$\begin{aligned} M^{\varepsilon K}_{\Phi,\mathscr{L}}f(x) &\le C \sup_{0<t<1/\varepsilon}\sup_{y\in B(x,t)}(1+d(y,x_0))^{-K+m_0} \\ &\le C \sup_{0<t<1/\varepsilon}\sup_{y\in B(x,t)}(1+d(x,x_0))^{-K+m_0}(1+d(x,y))^{K-m_0} \\ &\le C(1+d(x,x_0))^{-(K-m_0)}, \end{aligned}$$

where the constant C depends on ε. Since $p(K-m_0)>n$, it follows by (2.7) that $M^{\varepsilon K}_{\mathscr{L},\Phi}f\in L^p(X,d\mu)\cap L^\infty(X,d\mu)$. □

We also need the following auxiliary function: if $f\in\mathcal{S}'_{\mathscr{L}}(X)$, $\Phi\in\mathcal{S}(\mathbb{R}_+)$, $K\in\mathbb{N}_0$, $N\in\mathbb{N}_0$ and $0<\varepsilon\le 1$, we set

$$\begin{aligned} \widetilde{M}^{\varepsilon K}_{\mathscr{L},\Phi}f(x) = &\sup_{0<t<1/\varepsilon}\sup_{y\in B(x,t)}\Big(\sup_{z\in B(y,t)}\frac{t^\alpha|\Phi(t^2\mathscr{L})f(z)-\Phi(t^2\mathscr{L})f(y)|}{d(z,y)^\alpha}\Big) \\ &\times\Big(\frac{t}{t+\varepsilon}\Big)^K(1+\varepsilon d(y,x_0))^{-K}, \end{aligned}$$

where $\alpha>0$ is the same constant as in (2.6).

Lemma 3.6. *Suppose that $\Phi\in\mathcal{S}(\mathbb{R}_+)$ with $\Phi(0)=1$. Then, for any $N\in\mathbb{N}_0$ and $K\in\mathbb{N}_0$, there exists $C>0$ such that for all $f\in\mathcal{S}'_{\mathscr{L}}(X)$, $\varepsilon\in(0,1]$ and $x\in X$,*

$$\widetilde{M}^{\varepsilon K}_{\mathscr{L},\Phi}f(x)\le CT^{\varepsilon NK}_{\mathscr{L},\Phi}f(x).$$

Proof. Fix $K,N\in\mathbb{N}_0$. By Lemma 3.1 and its proof, we can write

$$\Phi(\cdot)=\int_0^1\Theta_{(s)}(\cdot)\Phi(s^2\cdot)f\,ds, \tag{3.14}$$

where $\{\Theta_{(s)}\}_{0\le s\le 1}$ is a family of functions in $\mathcal{S}(\mathbb{R}_+)$ with the following property: for any $m\in\mathbb{N}_0$, there exists a constant C (depending on Φ,m,N and K) such that

$$\|\Theta_{(s)}\|_{(m)}\le Cs^{N+K}\quad\text{for all } s\in[0,1]. \tag{3.15}$$

From (3.14), it follows that for all $f\in\mathcal{S}'_{\mathscr{L}}(X)$ and $t\in(0,\infty)$,

$$\Phi(t^2\mathscr{L})f=\int_0^1\Theta_{(s)}(t^2\mathscr{L})\Phi(s^2t^2\mathscr{L})f\,ds, \tag{3.16}$$

which holds pointwise and also in the sense of distributions in $\mathcal{S}'_{\mathscr{L}}(X)$. We fix $m\in\mathbb{N}_0$ such that $m\ge(3/2)n+N+K+1$ and fix arbitrary $x\in X$. Let $t\in(0,1/\varepsilon)$, $y\in B(x,t)$ and $z\in B(y,t)$. By (3.15) and (2.13),

$$|K_{\Theta_{(s)}(t^2\mathscr{L})}(z,w)-K_{\Theta_{(s)}(t^2\mathscr{L})}(y,w)|\le Cs^{N+K}\Big(\frac{d(z,y)}{t}\Big)^\alpha D_{t,m}(y,w).$$

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By this kernel estimate, (3.16) and (2.8), we can estimate as follows:

$$\begin{aligned}
&\frac{t^\alpha|\Phi(t^2\mathscr{L})f(z)-\Phi(t^2\mathscr{L})f(y)|}{d(z,y)^\alpha}\\
&\quad=\frac{t^\alpha}{d(z,y)^\alpha}\bigg|\int_X \Phi(s^2t^2\mathscr{L})f(w)K_{\Theta_{(s)}(t^2\mathscr{L})}(z,w)\,d\mu(w)\\
&\qquad-\int_X \Phi(s^2t^2\mathscr{L})f(w)K_{\Theta_{(s)}(t^2\mathscr{L})}(y,w)\,d\mu(w)\bigg|\\
&\quad\le\int_0^1\int_X\frac{t^\alpha\,|\Phi(s^2t^2\mathscr{L})f(w)|\,|K_{\Theta_{(s)}(t^2\mathscr{L})}(z,w)-K_{\Theta_{(s)}(t^2\mathscr{L})}(y,w)|}{d(z,y)^\alpha}\,d\mu(w)\,ds\\
&\quad\lesssim\int_0^1\int_X|\Phi(s^2t^2\mathscr{L})f(w)|\,s^{N+K}D_{t,m}(y,w)\,d\mu(w)\,ds\\
&\quad\lesssim T^{\varepsilon NK}_{\mathscr{L},\Phi}f(x)\int_0^1\int_X s^{N+K}\Big(1+\frac{d(x,w)}{st}\Big)^N\Big(\frac{st}{st+\varepsilon}\Big)^{-K}\\
&\qquad\times(1+\varepsilon\,d(w,x_0))^K D_{t,m}(y,w)\,d\mu(w)\,ds\\
&\quad\lesssim T^{\varepsilon NK}_{\mathscr{L},\Phi}f(x)\Big(\frac{t}{t+\varepsilon}\Big)^{-K}\int_X\Big(1+\frac{d(x,w)}{t}\Big)^N(1+\varepsilon\,d(w,x_0))^K D_{t,m}(y,w)\,d\mu(w)\\
&\quad\lesssim T^{\varepsilon NK}_{\mathscr{L},\Phi}f(x)\Big(\frac{t}{t+\varepsilon}\Big)^{-K}\int_X\Big(1+\frac{d(x,y)}{t}\Big)^N\Big(1+\frac{d(y,w)}{t}\Big)^N\\
&\qquad\times(1+\varepsilon\,d(w,x_0))^K D_{t,m}(y,w)\,d\mu(w)\\
&\quad\lesssim T^{\varepsilon NK}_{\mathscr{L},\Phi}f(x)\Big(\frac{t}{t+\varepsilon}\Big)^{-K}\int_X(1+\varepsilon\,d(y,x_0))^K(1+\varepsilon\,d(y,w))^K D_{t,m-N}(y,w)\,d\mu(w)\\
&\quad\lesssim T^{\varepsilon NK}_{\mathscr{L},\Phi}f(x)\Big(\frac{t}{t+\varepsilon}\Big)^{-K}(1+\varepsilon d(y,x_0))^K\int_X D_{t,m-N-K}(y,w)\,d\mu(w)\\
&\quad\lesssim T^{\varepsilon NK}_{\mathscr{L},\Phi}f(x)\Big(\frac{t}{t+\varepsilon}\Big)^{-K}(1+\varepsilon\,d(y,x_0))^K,
\end{aligned}$$

where for the last inequality we used (2.8) and $m-N-K>3n/2$. From this, the desired inequality follows immediately. □

Now we are ready to give the proof of the main theorem.

Proof of Theorem 1.3. Clearly, (i) ⇒ (ii) ⇒ (iii) and $\|M^0_{\mathscr{L},\Phi}f\|_{L^p(X,d\mu)}\le\|M_{\mathscr{L},\Phi}f\|_{L^p(X,d\mu)}$ $\le\|\Phi\|_{(N_p)}\|M_{\mathscr{L},N_p}f\|_{L^p(X,d\mu)}$ for all $f\in\mathcal{S}'_{\mathscr{L}}(X)$. Combining Lemmas 3.2 and 3.3, we see that (ii) ⇒ (i) and $\|M_{\mathscr{L},N_p}f\|_{L^p(X,d\mu)}\lesssim\|M_{\mathscr{L},\Phi}f\|_{L^p(X,d\mu)}$. Hence, it remains to show that (iii) ⇒ (ii) and $\|M_{\mathscr{L},\Phi}f\|_{L^p(X,d\mu)}\lesssim\|M^0_{\mathscr{L},\Phi}f\|_{L^p(X,d\mu)}$.

Suppose now that $f\in\mathcal{S}'_{\mathscr{L}}(X)$ is such that $M^0_{\mathscr{L},\Phi}f\in L^p(X,d\mu)$. By Lemma 3.5, we can choose K so large that $M^{\varepsilon K}_{\mathscr{L},\Phi}f\in L^p(X,d\mu)\cap L^\infty(X,d\mu)$ for $0<\varepsilon\le1$. Then, by Lemmas 3.4 and 3.6, we have $\widetilde{M}^{\varepsilon K}_{\mathscr{L},\Phi}f\in L^p(X,d\mu)$ and $\|\widetilde{M}^{\varepsilon K}_{\mathscr{L},\Phi}f\|_{L^p(X,d\mu)}\le C_1\|M^{\varepsilon K}_{\mathscr{L},\Phi}f\|_{L^p(X,d\mu)}$, where C_1 is independent of $\varepsilon\in(0,1]$. Given $\varepsilon\in(0,1]$, we set

$$\Omega_\varepsilon=\{x\in X:\widetilde{M}^{\varepsilon K}_{\mathscr{L},\Phi}f(x)\le C_2M^{\varepsilon K}_{\mathscr{L},\Phi}f(x)\},$$

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where $C_2 = 2^{1/p}C_1$. Note that

$$\int_X [M^{\varepsilon K}_{\mathscr{L},\Phi} f(x)]^p \, d\mu(x) \le 2 \int_{\Omega_\varepsilon} [M^{\varepsilon K}_{\mathscr{L},\Phi} f(x)]^p \, d\mu(x).$$

Indeed, this follows from

$$\int_{\Omega_\varepsilon^c} [M^{\varepsilon K}_{\mathscr{L},\Phi} f(x)]^p \, d\mu(x) \le C_2^{-p} \int_{\Omega_\varepsilon^c} [\widetilde{M}^{\varepsilon K}_{\mathscr{L},\Phi} f(x)]^p \, d\mu(x) \le (C_1/C_2)^p \int_X [M^{\varepsilon K}_{\mathscr{L},\Phi} f(x)]^p \, d\mu(x)$$

and $(C_1/C_2)^p = \frac{1}{2}$.

We claim that for $0 < r < p$, there exists $C_3 > 0$, independent of ε, such that

$$M^{\varepsilon K}_{\mathscr{L},\Phi} f(x) \le C_3 \{\mathcal{M}_{\mathrm{HL}}([M^0_{\mathscr{L},\Phi} f(\cdot)]^r)(x)\}^{1/r} \quad \text{for all } x \in \Omega_\varepsilon.$$

Once this claim is established, the required inequality $\|M_{\mathscr{L},\Phi} f\|_{L^p(X,d\mu)} \lesssim \|M^0_{\mathscr{L},\Phi} f\|_{L^p(X,d\mu)}$ will follow from the Hardy–Littlewood maximal theorem and the monotone convergence theorem (see, for instance, [17, Ch. 3] and [12, Ch. 4] for details).

Let us now prove the claim. Fix any $x \in \Omega_\varepsilon$. By the definition of $M^{\varepsilon K}_{\mathscr{L},\Phi} f(x)$, there exist $y \in X$ and $t > 0$ such that $d(y, x) < t < 1/\varepsilon$ and

$$|\Phi(t^2\mathscr{L}) f(y)| \Big(\frac{t}{t+\varepsilon}\Big)^K (1 + \varepsilon d(y, x_0))^{-K} \ge \frac{1}{2} M^{\varepsilon K}_{\mathscr{L},\Phi} f(x). \tag{3.17}$$

We fix such y and t. Then, by the definitions of $\widetilde{M}^{\varepsilon K}_{\mathscr{L},\Phi} f$ and Ω_ε,

$$\begin{aligned}
\sup_{z \in B(y,t)} \frac{t^\alpha |\Phi(t^2\mathscr{L}) f(z) - \Phi(t^2\mathscr{L}) f(y)|}{d(z, y)^\alpha} &\le \Big(\frac{t}{t+\varepsilon}\Big)^{-K} (1 + \varepsilon d(y, x_0))^K \widetilde{M}^{\varepsilon K}_{\mathscr{L},\Phi} f(x) \\
&\le C_2 \Big(\frac{t}{t+\varepsilon}\Big)^{-K} (1 + \varepsilon d(y, x_0))^K M^{\varepsilon K}_{\mathscr{L},\Phi} f(x) \\
&\le C_3 |\Phi(t^2\mathscr{L}) f(y)|,
\end{aligned} \tag{3.18}$$

where $C_3 = 2C_2$. Let $C_4 \ge \max(1, (2C_3)^{1/\alpha})$. Then we note that

$$|\Phi(t^2\mathscr{L}) f(z)| \ge \tfrac{1}{2} |\Phi(t^2\mathscr{L}) f(y)| \quad \text{for all } z \in B(y, t/C_4). \tag{3.19}$$

Indeed, since $d(z, y) < t/C_4 < t$, it follows from (3.18) that

$$\begin{aligned}
|\Phi(t^2\mathscr{L}) f(z) - \Phi(t^2\mathscr{L}) f(y)| &\le C_3 \frac{d(z, y)^\alpha}{t^\alpha} |\Phi(t^2\mathscr{L}) f(y)| \le C_3 C_4^{-\alpha} |\Phi(t^2\mathscr{L}) f(y)| \\
&\le \tfrac{1}{2} |\Phi(t^2\mathscr{L}) f(y)|,
\end{aligned}$$

which yields (3.19). Now (3.19) together with (3.17) gives that

$$|\Phi(t^2\mathscr{L}) f(z)| \ge \tfrac{1}{4} M^{\varepsilon K}_{\mathscr{L},\Phi} f(x) \quad \text{for all } z \in B(y, t/C_4).$$

Also, since $C_4 \ge 1$ and $d(y, x) < t$, we have $B(y, t/C_4) \subset B(x, 2t)$. Therefore,

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$$\begin{aligned}\mathcal{M}_{\mathrm{HL}}\Big([M^0_{\mathscr{L},\Phi}f(\cdot)]^r\Big)(x) &\geq \frac{1}{V(x,2t)}\int_{B(x,2t)}[M^0_{\mathscr{L},\Phi}f(z)]^r\,d\mu(z)\\ &\geq \frac{1}{V(x,2t)}\int_{B(x,2t)}|\Phi(t^2\mathscr{L})f(z)|^r\,d\mu(z)\\ &\geq \frac{V(y,t/C_4)}{V(x,2t)}\frac{1}{V(y,t/C_4)}\int_{B(y,t/C_4)}|\Phi(t^2\mathscr{L})f(z)|^r\,d\mu(z)\\ &\gtrsim [M^{\varepsilon K}_{\mathscr{L},\Phi}f(x)]^r.\end{aligned}$$

This establishes the claim and finishes the proof of Theorem 1.3. □

Acknowledgements

I would like to express my gratitude to my advisor, Professor Hitoshi Arai, for his guidance and encouragement. I am also grateful to Professor Gérard Kerkyacharian for explaining many details of his paper [16].

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