LOWER RADICALS IN NONASSOCIATIVE RINGS

R. TANGEMAN and D. KREILING

(Received 2nd April 1970)

Communicated by G. E. Wall

Let W be a universal class of (not necessarily associative) rings and let $A \subseteq W$. Kurosh has given in [6] a construction for LA, the lower radical class determined by A in W. Using this construction, Leavitt and Hoffmann have proved in [4] that if A is a hereditary class (if $K \in A$ and I is an ideal of K, then $I \in A$), then LA is also hereditary. In this paper an alternate lower radical construction is given. As applications, a simple proof is given of the theorem of Leavitt and Hoffmann and a result of Yu-Lee Lee for alternative rings is extended to not necessarily associative rings.

Let $A \subseteq W$ be any class of rings. Define $R_1(A)$ to be the homomorphic closure of A. Proceeding inductively, let β be an ordinal exceeding one and suppose the classes $R_\alpha(A)$ have been defined for all $\alpha < \beta$. If β is not a limit ordinal, define

$$R_\beta(A) = \{K \in W \,|\, I,\ K/I \in R_{\beta-1}(A) \text{ for some } I < K\}.$$

If β is a limit ordinal, define

$$R_\beta(A) = \{K \in W \,|\, K \text{ contains a chain } \{I_\gamma\} \text{ of ideals such that each } I_\gamma \in \bigcup_{\alpha<\beta} R_\alpha(A), \text{ and } K = \cup I_\gamma\}.$$

Finally define $R(A) = \cup R_\alpha(A)$, where the union is taken over all ordinals α.

The following characterization of radical classes is found in [2]. Using this characterization, we prove that $R(A) = L(A)$.

THEOREM 1. *Let W be a universal class and let $A \subseteq W$. Then A is a radical class in W if, and only if, the following conditions are satisfied:*

i) *A is homomorphically closed*

ii) *If I, $K/I \in A$, then $K \in A$*

iii) *The union of a chain of A-ideals of a W-ring K is again an A-ideal of K.*

The following lemma is obvious.

LEMMA 1. *If α and β are ordinals with $\alpha \leqq \beta$, then $R_\alpha(A) \subseteq R_\beta(A)$.*

https://doi.org/10.1017/S1446788700011046 Published online by Cambridge University Press

LEMMA 2. *For every ordinal* $\alpha \geqq 1$, $R_\alpha(A)$ *is homomorphically closed. Hence* $R(A)$ *is homomorphically closed.*

PROOF. $R_1(A)$ is homomorphically closed. Let $\beta > 1$ be an ordinal, and suppose $R_\alpha(A)$ is homomorphically closed for all $\alpha < \beta$. Let $K \in R_\beta(A)$ and let $I < K$. If β is a limit ordinal, there is a chain $\{I_\gamma\}$ of ideals of K such that I_γ belongs to one of the classes $R_\alpha(A)$ with $\alpha < \beta$ and such that $K = \cup I_\gamma$. But $\{(I + I_\gamma)/I\}$ is a chain of ideals of K/I, and K/I is its union. Since

$$(I + I\gamma))/I \cong I\gamma/(I \cap I_\gamma),$$

each of these ideals is a homomorphism of some I_γ, and thus by the induction hypothesis each $(I + I\gamma)/I$ belongs to some $R_\alpha(A)$ with $\alpha < \beta$. This means $K/I \in R_\beta(A)$.

Now suppose $\beta - 1$ exists. Then K contains an ideal J so that J, $K/J \in R_{\beta-1}(A)$. By the induction hypothesis, $(J + I)/I$ and $K/(I + J)$ both belong to $R_{\beta-1}(A)$, since the former is a homomorphic image of J and the latter of K/J. Since

$$[R/I]/[(J + I)/I] \cong R/(J + I),$$

$R/I \in R_\beta(A)$. Thus by transfinite induction $R_\beta(A)$ is homomorphically closed for all ordinals β. It follows immediately that $R(A)$ is homomorphically closed.

We now show that $R(A)$ satisfies conditions (ii) and (iii) of Theorem 1.

LEMMA 3. *Let* $K \in W$ *and let* $\{I_\alpha\}$ *be a chain of* $R(A)$*-ideals of* K. *Then* $\cup I_\alpha$ *is an* $R(A)$*-ideal of* K.

PROOF. Since K is a set, there is by Lemma 1 an ordinal β with the property that $I_\alpha \in R_\beta(A)$ for each α. Let δ be a limit ordinal exceeding β, then $\cup I_\alpha \in R_\delta(A)$.

LEMMA 4. *Let* $K \in W$, *and suppose* K *contains an ideal* $I \in R(A)$ *such that* $K/I \in R(A)$. *Then* $K \in R(A)$.

PROOF. By Lemma 1, there is an ordinal β such that I, $K/I \in R_\beta(A)$. This means that $K \in R_{\beta+1}(A)$.

THEOREM 2. $R(A) = L(A)$.

PROOF. By Theorem 1 and Lemmas 2, 3, and 4, $R(A)$ is a radical class in W. By the minimality of $L(A)$ among radical classes in W which contain A, it is enough to show $R(A) \subseteq L(A)$. This is accomplished by proving $R_\alpha(A) \subseteq L(A)$ for every ordinal α.

Clearly $R_1(A) \subseteq L(A)$. Let β be an ordinal exceeding one, and assume $R_\alpha(A) \subseteq L(A)$ for all ordinals $\alpha < \beta$. Let $K \in R_\beta(A)$. If β is a limit ordinal, K is the union of a chain of ideals from the classes $R_\alpha(A)$, where $\alpha < \beta$. Thus by the induction hypothesis K is the union of $L(A)$-ideals, so $K \in L(A)$ by Theorem 1.

If β is not a limit ordinal, there is an ideal I of K such that I and K/I both belong to $R_{\beta-1}(A) \subseteq L(A)$. Again, $K \in L(A)$ by Theorem 1. Thus $R_\beta(A) \subseteq L(A)$ for all ordinals $\beta \geqq 1$.

The referee has provided an alternate proof that $L(A) \subseteq R(A)$, independent of Lemma 2 as follows.

Let A_α be the Kurosh classes (see [1]), then $A_1 = R_1(A) \subseteq R(A)$. Let β be an ordinal and suppose $A_\alpha \subseteq R(A)$ for all $\alpha < \beta$. Let $K \in A_\beta$ and let S be the set of all $R(A)$-ideals of K. By Lemma 3 S, is closed under taking unions of chains, so by Zorn's Lemma S contains a maximal element I. If $I = K$ we are done, but if $0 \neq K/I$ there exists

$$0 \neq J/1 < K/I \text{ with } J/I \in A_\alpha \subseteq R(A).$$

By Lemma 4 we have $J \in R(A)$ contradicting the maximality of I. Hence $I = K \in R(A)$ so $A_\alpha \subseteq R(A)$ for each ordinal α. Therefore $LA = \cup A_\alpha \subseteq R(A)$.

We now give a simple proof of the following theorem which appears in [4]. Other results of the form "A has property P implies LA has property P" may, perhaps, be provable in a similar way.

THEOREM 3. [4] *Let $A \subseteq W$ where W is some universal class. Then if A is hereditary, so is $L(A)$.*

PROOF. We prove that $R_\beta(A)$ is hereditary for each $\beta \geqq 1$. This is easily seen to be true if $\beta = 1$. Thus, assume $\beta > 1$, and suppose $R_\alpha(A)$ is a hereditary class for each $\alpha < \beta$. Let $K \in R_\beta(A)$, and suppose I is an ideal of K. If β is a limit ordinal, $K = \cup I_\gamma$ where $\{I_\gamma\}$ is a chain of ideals each belonging to one of the (hereditary) classes $R_\alpha(A)$, $\alpha < \beta$. But then $I = \cup (I_\gamma \cap I)$ so $I \in R_\beta(A)$.

If β is not a limit ordinal, there is an ideal J of K so that J, $K/J \in R_{\beta-1}(A)$. Since $R_{\beta-1}(A)$ is hereditary, $I \cap J$ and

$$(J + I)/J \cong I/(I \cap J)$$

both belong to $R_{\beta-1}(A)$. This implies $I \in R_\beta(A)$.

The proof of Theorem 4 requires the following lemma.

LEMMA 5. *If P is a radical class in W and for some $K' \in W$ a subring $K \subseteq K'$ is the set-theoretic union of P-ideals of K', then $K \in P$.*

PROOF. If $K = \cup I_\alpha \notin P$, then $K/I \in SP = \{H \in W \mid H$ has no nonzero P-ideals$\}$ for some $I \neq K$. Then for some α we have $I_\alpha \nsubseteq I$, so $(I_\alpha + I)I \cong I_\alpha/(I \cap I_\alpha)$ is a nonzero P-ideal of K/I. This contradiction proves that $K \in P$.

The following theorem is proved for alternative rings in [7] by Yu-Lee Lee.

THEOREM 4. *If A_1 and A_2 are homomorphically closed, hereditary classes of W-rings, then $L(A_1 \cap A_2) = LA \cap LA_2$.*

PROOF. Trivially $L(A_1 \cap A_2) \subseteq LA_1 \cap LA_2$. Since $K \in LA_1 \cap LA_2$ if and only if $K \in R_\gamma(A_1) \cap R_\gamma(A_2)$ for some ordinal number γ. It suffices to prove

$$R_\gamma(A_1) \cap R_\gamma(A_2) \subseteq LA_1 \cap A_2),$$

for each ordinal $\gamma \geqq 1$. This is clear for $\gamma = 1$. Let β be an ordinal number greater than 1 and suppose

$$R_\alpha(A_1) \cap R_\alpha(A_2) \subseteq L(A_1 \cap A_2)$$

for each ordinal $\alpha < \beta$. Let $K \in R_\beta(A_1) \cap R_\beta(A_2)$.

If β is a limit ordinal, K is the union of a chain $\{I_\gamma\}_{\gamma \in C}$ of ideals each belonging to one of the classes $R_\alpha(A_1)$ for $\alpha < \beta$. Also K is the union of a chain $\{J_\delta\}_{\delta \in D}$ of ideals each belonging to one of the classes $R_\alpha(A_2)$ for $\alpha < \beta$. If $x \in K$, $x \in J\delta$ for some $\delta \in D$ and $x \in I_\gamma$ for some $\gamma \in C$, so $x \in J_\delta \cap I_\gamma$ for some $(\delta,\gamma) \in D \times C$. Since $J_\delta \in R_\alpha(A_2)$ for some $\alpha < \beta$, and since $R_\alpha(A_2)$ is hereditary (see proof of Theorem 3), $J_\delta \cap I_\lambda \in R_\alpha(A_2)$. Similarly $J_\delta \cap I_\gamma \in R_\eta(A_1)$ for some $\eta < \beta$. Thus

$$J_\delta \cap I_\gamma \in R_\mu(A_1) \cap R_\mu(A_2),$$

where $\mu = \max[\eta, \alpha]$. Since $\mu < \beta$, the induction hypothesis implies $J_\delta \cap I_\gamma \in L(A_1 \cap A_2)$ so that K is the set-theoretic union of $L(A_1 \cap A_2)$-ideals. Thus, by Lemma 5, $K \in L(A_1 \cap A_2)$.

Now suppose $\beta - 1$ exists, and let $K \in R_\beta(A_1) \cap R_\beta(A_2)$. Then there exist ideals I and J such that I, $K/I \in R_{\beta-1}(A_1)$ and J, $K/J \in R_{\beta-1}(A_2)$. Since $R_{\beta-1}(A_1)$ and $R_{\beta-1}(A_2)$ are hereditary,

$$I \cap J \in R_{\beta-1}(A_1) \cap R_{\beta-1}(A_2)$$

so $I \cap J \in L(A_1 \cap A_2)$. Since $R_{\beta-1}(A_1)$ is homomorphically closed (Lemma 2),

$$I/(I \cap J) \cong (I+J)/J \in R_{\beta-1}(A_1).$$

Since $R_{\beta-1}(A_2)$ is hereditary, $(I+J)/J$, as an ideal of K/J is a member of $R_{\beta-1}(A_2)$. Thus

$$I/(I \cap J) \cong (I+J)/J \in R_{\beta-1}(A_1) \cap R_{\beta-1}(A_2) \subseteq L(A_1 \cap A_2).$$

Thus $I \cap J$ and $I/(I \cap J)$ belong to $R_{\beta-1}(A_1) \cap R_{\beta-1}(A_2) \subseteq L(A_1 \cap A_2)$.

Thus since $I \cap J$ and $I/(I \cap J)$ belong to $L(A_1 \cap A_2)$, $I \in L(A_1 \cap A_2)$. Similarly, $J \in L(A_1 \cap A_2)$ so that $I + J$ is an $L(A_1 \cap A_2)$-ideal of K. Also $K/(I+J)$ belongs to

$$R_{\beta-1}(A_1) \cap R_{\beta-1}(A_2) \subseteq L(A_1 \cap A_2)$$

since it is the homomorphic image of both K/J and K/I. Thus, since $I+J$ and $K/(I+J)$ belong to $L(A_1 \cap A_2)$, we have that $K \in L(A_1 \cap A_2)$.

We have shown that $R_\beta(A_1) \cap R(A_2) \subseteq L(A_1 \cap A_2)$ which proves the theorem.

COROLLARY. *If* A_i, $i = 1, 2, \cdots, n$, *are homomorphically closed, hereditary classes of W-rings, then* $L(\bigcap_{i=1}^n A_i) = \bigcap_{i=1}^n LA_i$.

PROOF. By induction.

It is shown in [1] that the Kurosh-Amitsur construction terminates at ω, the first infinite ordinal in case W is an associative universal class. If A is hereditary in an associative class, then $LA = A_3$, the third step in the Kurosh-Amitsur construction (see [3]). To see that similar properties do not hold for the construction of Theorem 2, let W be the class of all associative rings and $Z \subseteq W$ the class of rings having zero multiplication. Then the classes $R_n(Z)$, n finite, are all distinct. Jacobson has given in [5] an example of an LZ-ring K which is not the sum (and thus not the union) of its nilpotent ideals. Therefore $K \notin R_\omega(Z)$ so that $LZ \neq R_\omega(Z)$.

References

[1] T. Anderson, N. Divinsky, and A. Sulinsky, 'Lower, Radical Properties for Associative and Alternative Rings'; *J. London Math. Soc.* 41 (1966), 417–424.

[2] S. A. Amitsur, 'Radicals in Rings and Bicategories', *Amer. J. Math.* 76 (1954), 100–125.

[3] E. P. Armendariz and W. G. Leavitt, 'The Hereditary Property in the Lower Radical Construction', *Canad. J. Math.* 20 (1968), 474–476.

[4] A. E. Hoffman and W. G. Leavitt, 'Properties Inherited by the Lower Radical.' *Protugaliae Mathematica* 27 (1968), 63–66.

[5] N. Jacobson, *Structure of Rings* (Amer. Math. Soc. Coll. Publ. 37, Providence, 1956).

[6] A. Kurosh, 'Radicals in Rings and Algebras', *Math. Sb.* 33, (1953), 13–26.

[7] Yu-Lee Lee, 'On Intersections and Unions of Radical Classes', *J. Aust. Math. Soc.* (To appear).

University of Florida
Western Illinois University
Macomb, Illinois 61455
U.S.A.