RINGS CHARACTERIZED BY THEIR WEAKLY-INJECTIVE MODULES

by SERGIO R. LÓPEZ-PERMOUTH

(Received 14 June, 1991)

1. Introduction. The notation in this paper will be standard and it may be found in [**2**] or [**8**]. Throughout the paper, the notation $A \subset' B$ will mean that A is an essential submodule of the module B. Given an arbitrary ring R and R-modules M and N, we say that M is weakly N-injective if and only if every map $\varphi: N \to E(M)$ from N into the injective hull $E(M)$ of M may be written as a composition $\sigma \circ \hat{\varphi}$, where $\hat{\varphi}: N \to M$ and $\sigma: M \to E(M)$ is a monomorphism. This is equivalent to saying that for every map $\varphi: N \to E(M)$, there exists a submodule X of $E(M)$, isomorphic to M, such that $\varphi(N)$ is contained in X. In particular, M is weakly R-injective if and only if, for every $x \in E(M)$, there exists $X \subset E(M)$ such that $x \in X \cong M$. We say that M is weakly-injective if and only if it is weakly N-injective for every finitely generated module N. Clearly, M is weakly-injective if and only if, for every finitely generated submodule N of $E(M)$, there exists $X \subset E(M)$ such that $N \subset X \cong M$.

Any weakly N-injective module M satisfies the closely related property that, for every submodule K of N, if N/K embeds in $E(M)$ then N/K embeds in M. Following [**10**], we refer to any such module as being N-tight. If M is N-tight for every finitely generated module N, we say that M is tight.

Weakly-injective (tight) modules are closed under finite sums and under essential extensions. However, they remarkably fail to be closed under direct summands [**11**]. A natural question would be to ask for what rings is it true that summands of weakly-injective (tight) modules are weakly-injective (tight). We show that these rings are precisely the weakly-semisimple rings. Following [**11**], a ring R such that every right R-module is weakly-injective will be referred to as a right weakly-semisimple ring. It is not hard to see, following the arguments in [**10**] and [**11**], that a ring R is weakly-semisimple if and only if every right R-module is tight. Weakly-semisimple rings are right QI-rings. That is, if R is a weakly-semisimple ring then every quasi-injective right R-module is injective (see [**3**], [**4**], [**5**], [**8**], [**9**] for background on right QI-rings). If R is hereditary and noetherian then R is a right weakly-semisimple ring if and only if it is a right QI-ring [**11**].

The opposite extreme to weakly-semisimple rings consists of those rings over which the only weakly-injective modules are the trivial ones (i.e. the injective modules). We show that this happens if and only if the ring is semisimple artinian.

A ring R is said to be a right q.f.d. ring if and only if every cyclic right R-module has finite Goldie dimension. This is equivalent to the requirement for every cyclic right R-module to have finitely generated (possibly zero) socle [**12**]. Examples of right q.f.d. rings include rings with right Krull dimension. In particular, right noetherian rings are also right q.f.d. rings. R is a right q.f.d. ring if and only if every finitely generated right R-module has finite Goldie dimension [**6**]. It is easy to see that if R is a right q.f.d. ring then every right R-module contains, as an essential submodule, a direct sum of uniform submodules. Right q.f.d. rings have also been studied in [**1**], and in [**13**]. Arbitrary sums of (weakly-)injective right modules over a ring R are weakly-injective if and only if R is a right q.f.d. ring [**1**].

A ring R is said to be right semi-artinian if every (cyclic) right module has non-zero (and thus essential) socle (see [7], for example). Right semi-artinian rings are called right socular in [8].

2. Direct summands of weakly-injective modules.

PROPOSITION 2.1. *Every completely reducible module over an arbitrary ring R is a direct summand of a weakly-injective R-module.*

Proof. Let M be a completely reducible right R-module. Let us write $M = \bigoplus_{i \in I} [S_i]$, where $[S_i]$ represents the homogeneous component of M corresponding to the simple submodule $S_i \subset M$. It follows that, for every $i \in I$, there exists a cardinal $\aleph_i$ such that $[S_i] \cong S_i^{(\aleph_i)}$. Let $\aleph$ be an infinite cardinal greater than both the cardinality of R and the number of summands of M. In particular, for every $i \in I$, $\aleph > \aleph_i$. Notice that for every finitely generated right R-module N, if $\bigoplus_{\alpha \in \Gamma} U_\alpha$ is an internal direct sum of nonzero submodules of N then the cardinality of Γ is less than $\aleph$. Let $V = M \oplus E(M^{(\aleph)})$. We claim that V is weakly-injective. Notice, first of all, that $E(V) \cong E(M^{(\aleph)})$ and $\operatorname{Soc} V = \operatorname{Soc} E(V) \cong \bigoplus_{i \in I} [S_i]^{(\aleph)} \cong \bigoplus_{i \in I} (S_i^{(\aleph_i)})^{(\aleph)} \cong \bigoplus_{i \in I} S_i^{(\aleph)}$. Let N be a finitely generated submodule of $E(V)$. Then the number of simple summands in any decomposition of $\operatorname{Soc} N$ is less than $\aleph$. Let us say that $\operatorname{Soc} N = \bigoplus_{i \in I} [[S_i]]$, where $[[S_i]]$ denotes the (possibly zero) homogeneous component of $\operatorname{Soc} N$ corresponding to S_i. Since, for every $i \in I$, the number of simple summands in $[[S_i]]$ is less than $\aleph$, we conclude that the homogeneous component of $\operatorname{Soc} E(V)$ corresponding to S_i equals $[[S_i]] \oplus K_i$ for some $K_i \cong (S_i)^{(\aleph)}$. Hence, we get $\operatorname{Soc} V = \operatorname{Soc} N \oplus T$ for some $T \cong \operatorname{Soc} V$. Therefore, $E(\operatorname{Soc} V) = E(V) = E(N) \oplus E(T)$, and $E(T) \cong E(V)$. Let Y be a submodule of $E(T)$ isomorphic to V and define $X = E(N) \oplus Y$. Then

$$X \cong E(N) \oplus M \oplus E(M^{(\aleph)}) = M \oplus E(N \oplus M^{(\aleph)}) = M \oplus E(\operatorname{Soc} N \oplus (\operatorname{Soc} M)^{(\aleph)})$$
$$\cong M \oplus E\Big(\bigoplus_{i \in I} [[S_i]] \oplus \bigoplus_{i \in I} (S_i)^{(\aleph)}\Big) \cong M \oplus E\Big(\bigoplus_{i \in I} (S_i)^{(\aleph)}\Big) \cong M \oplus E(M^{(\aleph)}) = V.$$

Since $N \subset X$, this concludes our proof.

COROLLARY 2.2. *Over a right semi-artinian ring R, every right R-module is a summand of a weakly-injective right module.*

Proof. This follows from the previous proposition since weak-injectivity is preserved by essential extensions.

PROPOSITION 2.3. *Over arbitrary rings, every module is a summand of a tight module. If R is a right q.f.d. ring, every right R-module is a summand of a weakly-injective right module.*

Proof. Let M be a right module over the right q.f.d. ring R and let $\aleph$ be any infinite cardinal. Consider the module $N = M \oplus E(M^{(\aleph)})$. Since $E(N)$ is isomorphic to a submodule of N, N is tight. In light of Theorem 3.1 ahead, if R is a right q.f.d. ring then N is weakly-injective.

THEOREM 2.4. *Let R be a ring. Then*

(1) *direct summands of weakly-injective (tight) right R-modules are weakly-injective (tight) if and only if R is a right weakly-semisimple ring, and*

(2) *every weakly-injective (tight) right module is injective if and only if R is semisimple-artinian.*

Proof. If weakly-injective (tight) right R-modules were closed under direct summands, Proposition 2.1 implies that every completely reducible right R-module would be weakly-injective (tight) and thus injective. This implies that R is right noetherian (see **[8]** or **[12]**). Then, by Proposition 2.3 and the hypothesis, R is right weakly-semisimple. One can argue in the same way to prove that if every weakly-injective module is injective then every right R-module is injective and hence R is semisimple artinian.

3. Weak-injectivity versus tightness.

THEOREM 3.1. *Let R be a right q.f.d. ring. Then every tight right R-module is weakly-injective.*

Proof. Let M be a tight right R-module over the right q.f.d. ring R. Let N be a finitely generated submodule of $E(M)$. Since R is a right q.f.d. ring, N contains as an essential submodule a finite direct sum $\bigoplus_{i\in I_1} U_i$ of uniform submodules. By the tightness of M, there exists an embedding $\varphi: N\to M$. Let K be a complement of $\varphi(N)$ in M, and let $\bigoplus_{j\in I_2} U_j$ be a direct sum of uniform submodules of M which is essential in K. For convenience, assume $I_1\cap I_2=\varnothing$ and set $I=I_1\cup I_2$. Then $E(M)=E(\varphi(N))\oplus E(K)$ contains as an essential submodule the sum

$$\bigoplus_{i\in I_1} E(\varphi(U_i))\oplus\bigoplus_{j\in I_2} E(U_j)\cong\bigoplus_{i\in I} E(U_i).$$

Now, since R is a right q.f.d. ring, this sum is weakly-injective (by **[1]**) and therefore there exists an embedding $\psi:\bigoplus_{i\in I} E(U_i)\to E(M)$ such that $N\subset\bigoplus_{i\in I}\psi(E(U_i))$. Since N is finitely generated, $N\subset\bigoplus_{i\in J}\psi(E(U_i))$ for some finite subset $J\subset I$. Then $E(N)$ is a summand of $\bigoplus_{i\in J}\psi(E(U_i))$. Using the Krull–Schmidt theorem, we may then, without losing generality, assume that $N\subset'\bigoplus_{i\in J}\psi(E(U_i))$. It follows that $|I_1|=|J|$ and, by the Azumaya–Krull–Schmidt theorem, that there exists an isomorphism $\rho:\bigoplus_{i\in I_2} E(U_i)\to\bigoplus_{j\notin J}\psi(E(U_j))$. This isomorphism ρ extends to another isomorphism $\hat{\rho}:E(K)\to E\left(\bigoplus_{j\notin J}\psi(E(U_j))\right)$. The isomorphism $\varphi:N\to\varphi(N)$ extends in turn to an isomorphism $\hat{\varphi}:E(N)=\bigoplus_{j\in J}\psi(E(U_j))\to\bigoplus_{i\in I_1} E(U_i)=E(\varphi(N))$. Let $M_1=M\cap E(\varphi(N))$. Then $\varphi(N)\subset' M_1\subset' E(\varphi(N))$. It follows that $\eta=\hat{\varphi}^{-1}\oplus\rho: M_1\oplus K\to E(M)$ is an embedding satisfying that $N\subset\eta(M_1\oplus K)$. Since $M_1\oplus K\subset' M$, there is an extension $\hat{\eta}:M\to E(M)$ of η such that $N\subset\hat{\eta}(M)$. Therefore M is weakly-injective, as claimed.

The following theorem comes close to being a converse for Theorem 3.1.

THEOREM 3.2. *Let R be a ring such that every tight right R-module is weakly-injective. Then for every cyclic right R-module M, the homogeneous components of* Soc M *are finitely generated.*

Proof. Suppose N is cyclic and Soc N contains an infinitely generated homogeneous component $[S] \cong S^{(\aleph)}$, where $\aleph$ is some infinite cardinal and S is a simple submodule of N. Let K be a complement of $[S]$ in N. Then N/K is a cyclic module with infinitely generated homogeneous and essential socle. So, without loss of generality, let us assume that $\operatorname{Soc} N \subset' N$ and $\operatorname{Soc} N \cong S^{(\aleph)}$ for some simple submodule $S \subset N$ and an infinite cardinal $\aleph$. It follows that $M = S^{(\aleph)} \oplus E(S^{(\aleph)})$ is tight since $E(M) \cong \{0\} \oplus E(S^{(\aleph)}) \subset M$. However, $S^{(\aleph)} \subset' N$ and the embedding of $S^{(\aleph)}$ as an essential submodule of $E(M)$ extends to an embedding φ of N in $E(M)$. If there existed a submodule $X \subset E(M)$ such that $\varphi(N) \subset X \cong M$, by the modular law, $\varphi(N)$ would have a summand isomorphic to $S^{(\aleph)}$. However this is impossible since $\varphi(N)$ is cyclic. Therefore M is tight but not weakly-injective, concluding our proof.

The proof of the above theorem suggests how one can create an example of a tight module which is not weakly-injective.

EXAMPLE 3.3. *Let R be the ring of endomorphisms of an infinite dimensional vector space V over a division ring D. Then there exists a tight R-module M which is not weakly-injective.*

Proof. The socle of R is essential in R and it consists of a direct sum of $\aleph$ pairwise isomorphic minimal right ideals, where $\aleph = \dim_D V$. So, it follows as in the proof of Theorem 3.2 that if S is a minimal right ideal of R, the module $M = S^{(\aleph)} \oplus E(S^{(\aleph)}) \cong \operatorname{Soc} R \oplus R$ is tight but not weakly-injective.

REFERENCES

1. A. H. Al-Huzali, S. K. Jain and S. R. López-Permouth, Rings whose cyclics have finite Goldie dimension, to appear in *J. Algebra.*

2. F. W. Anderson and K. R. Fuller, *Rings and categories of modules* (Springer, 1974).

3. A. K. Boyle, Hereditary QI-rings, *Trans. Amer. Math. Soc.* **192** (1974), 115–120.

4. A. K. Boyle, Injectives containing no proper quasi-injective submodules, *Comm. Algebra* **4** (1976), 775–785.

5. A. K. Boyle and K. R. Goodearl, Rings over which certain modules are injective, *Pacific J. Math.* **58** (1975), 43–53.

6. V. P. Camillo, Modules whose quotients have finite Goldie dimension, *Pacific J.Math.* **69** (1977), 337–338.

7. N. V. Dung and P. F. Smith, On semi-artinian V-modules, *J. Pure Appl. Algebra*, to appear.

8. C. Faith, *Algebra II*, *Ring theory* (Springer, 1976).

9. C. Faith, On hereditary rings and Boyle's conjecture, *Arch. Math.* (*Basel*) **27** (1976), 113–119.

10. J. S. Goland and S. R. López-Permouth, QI-filters and tight modules, *Comm. Algebra* **19** (1991), 2217–2229.

11. S. K. Jain, S. R. López-Permouth and S. Singh, On a class of QI-rings, *Glasgow Math. J.* **34** (1992), 75–81.

12. R. P. Kurshan, Rings whose cyclic modules have finitely generated socle, *J. Algebra* **15** (1970), 376–386.

13. R. C. Shock, Dual generalizations of the Artinian and Noetherian conditions, *Pacific J. Math.* **54** (1974), 227–235.

Ohio University
Athens
Ohio 45701
U.S.A.

https://doi.org/10.1017/S0017089500008934 Published online by Cambridge University Press