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DECOMPOSABILITY OF FINITE RINGS

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Abstract

Let p be a prime, let R be a finite p-ring with identity and suppose that the radical of R has p^m elements. If R is indecomposable as a ring then there are at most $m+1$ minimal ideals in $R/\mathrm{Rad}\, R$.

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Our main result is the following theorem.

THEOREM. *Let p be a prime, let R be a finite p-ring with identity and suppose* Rad R *has p^m elements. If R is indecomposable as a ring then there are at most $m+1$ minimal ideals in $R/\mathrm{Rad}\, R$.*

Stewart (1972), Theorem 3.3, has proved a related result for the case $p = 2$ and the above theorem improves his result, obtained via Lemma 3.4, that $\mu(J) \leqslant 2m^2$ to $\mu(J) \leqslant m+1$. Other results bounding the size of rings with a given radical are Theorem 3.8 of Hall (1940) and Theorem 6.2 of Flanigan (1973).

We also prove the following result which improves the bound given in Theorem 3.5 of Stewart (1972).

PROPOSITION. *Let R be a finite indecomposable 2-ring with identity and with group of units G. If $R/\mathrm{Rad}\, R$ has t simple summands distinct from $\mathbf{Z}_2$ then*

$$|R| \leqslant 2^{1-t}|G|^2.$$

The results of this paper appear in Mainwaring (1978) and are one of the major tools used there to describe completely all finite rings whose unit groups are dihedral.

Proof of the theorem. Suppose that R is indecomposable and that

$$\bar{R} = R/\operatorname{Rad} R = Y_1 \oplus \ldots \oplus Y_s$$

where each Y_i is a matrix ring over a finite field of characteristic p: we must prove that $s \leqslant m+1$. It is well known (see, for example, Proposition 5, page 54 of Jacobson (1964)) that there are orthogonal idempotents $f_1, \ldots, f_s$ in R whose sum is 1 such that $v(f_i) = \delta_i$ where $\delta_1, \ldots, \delta_s$ are the obvious s central orthogonal idempotents in $\bar{R}$ whose sum is 1 and $v: R \to \bar{R}$ is the natural map. Notice that R is an internal direct sum (as an abelian group) of the $f_i R f_j$ $(1 \leqslant i,j \leqslant s)$ and that if $i \neq j$,

$$v(f_i R f_j) = \delta_i \bar{R} \delta_j = 0$$

so that $f_i R f_j \subseteq \operatorname{Rad} R$. Thus

$$\prod_{i \neq j} |f_i R f_j| \leqslant |\operatorname{Rad} R| = p^m$$

which means that the number k of nonzero $f_i R f_j$ with $i \neq j$ is at most m.

Notice that if A_1 and A_2 are nonempty disjoint sets whose union is $\{1, 2, \ldots, s\}$ then there exist $i \in A_1$ and $j \in A_2$ such that either $f_i R f_j \neq 0$ or $f_j R f_i \neq 0$, since otherwise $R = R_1 \oplus R_2$ where each

$$R_l = \sum_{i,j \in A_l} f_i R f_j$$

is an ideal of R since the f's are orthogonal. If we take A_1 successively as $\{1\}, \{1, i_1\}, \ldots, \{1, i_1, \ldots, i_{s-2}\}$ we see that there exist $i_1, \ldots, i_{s-1}$ all distinct from 1 and $j_1, \ldots, j_{s-2}$ with $j_l \in \{1, i_1, \ldots, i_l\}$ such that each of

$$f_1 R f_{i_1} \cup f_{i_1} R f_1, \quad f_{j_1} R f_{i_2} \cup f_{i_2} R f_{j_1}, \quad \ldots, \quad f_{j_{s-2}} R f_{i_{s-1}} \cup f_{i_{s-1}} R f_{j_{s-2}}$$

is nonzero. This leads to at least $s-1$ different nonzero $f_i R f_j$ with $i \neq j$ and hence $s - 1 \leqslant k \leqslant m$, as required.

Proof of the Proposition. Let $J = \operatorname{Rad} R$, put $|J| = 2^m$ and let $R/J \simeq Y \oplus w\mathbf{Z}_2$ where Y is a direct sum of t matrix rings over fields of characteristic 2 and includes no $\mathbf{Z}_2$ summands. By the theorem above, $t + w \leqslant m+1$. We denote by W^* the group of units of any ring W. Since $R^*/(1+J) \simeq Y^*$ (McDonald (1974), Theorem XX1.5) and since $|Y| \leqslant |Y^*|^2$ by Lemma 2.3 of Stewart (1972),

$$|R| = |J|\,|Y|\,2^w \leqslant |J|^2\,|Y^*|^2\,2^{w-m} = |R^*|^2\,2^{w-m} \leqslant 2^{1-t}\,|G|^2.$$

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