DACEY GRAPHS

DAVID P. SUMNER

(Received 13 March 1973; revised 9 August 1973)

Communicated by G. Szekeres

1. Introduction

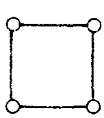
In this paper our graphs will be finite, undirected, and without loops or multiple edges. We will denote the set of vertices of a graph G by $V(G)$. If G is a graph and $u, v \in V(G)$, then we will write $u \sim v$ to denote that u and v are adjacent and $u \nsim v$ otherwise. If $A \subseteq V(G)$, then we let $N(A) = \{u \in V(G) \mid u \sim a$ for each $a \in A\}$. However we write $N(v)$ instead of $N(\{v\})$. When there is no chance of confusion, we will not distinguish between a subset $A \subseteq V(G)$ of vertices of G and the subgraph that it induces. We will denote the cardinality of a set A by $|A|$. The degree of a vertex v is $\delta(v) = |N(v)|$. Any undefined terminology in this paper will generally conform with Behzad and Chartrand [1].

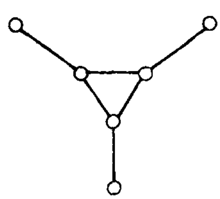
In their work on empirical logic, Foulis and Randall have defined the concept of the logic of a graph (see Foulis [4] and [5] and also Jeffcott [7]).

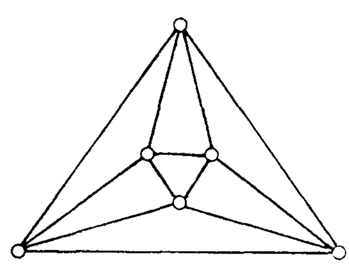
In this context, a graph is defined to be a *Dacey graph* if and only if its logic is an orthomodular poset. It is convenient that a characterization of Dacey graphs in purely graph-theoretic terms is available. We will take this characterization as our definition of a Dacey graph. By a *clique* of a graph G we mean a maximal subset A of the vertices of G such that any two elements of A are adjacent.

DEFINITION. *Let G be a graph. Then G is a Dacey graph if and only if for every clique E of G and every pair of distinct vertices u and v we have $E \subseteq N(u) \cup N(v) \Rightarrow u \sim v$.*

We will hereafter abbreviate Dacey graph to D-graph. As examples of D-graphs we have







The only nontrivial trees that are D-graphs are the stars $K_{1,n}$ for $n \geqq 1$.

It is our intention in this paper to investigate D-graphs from a graph-theoretic point of view. Also we develop some sufficient conditions for a graph to be a D-graph, and several classes of D-graphs are determined. The properties of point closed and point determining are characterized for D-graphs in terms of their clique structure. We obtain several characterizations of the complete graphs as special types of D-graphs. We study the hereditary Dacey graphs (HD-graphs) and strengthen the previously known results (see [3]). Our development here is more constructive than the earlier one. Finally, we consider some interesting connectivity properties of HD-graphs.

REMARK. It is helpful to observe that if E is a clique of some graph G and $E \subseteq N(u) \cup N(v)$ with $u \nsim v$, then $\{u, v\} \cap E = \emptyset$.

2. Point determining and point closed D-graphs

DEFINITION. (1) G is *point determining* if and only if for $u, v \in V(G)$ with $u \neq v$, we have $N(u) \neq N(v)$.

(2) G is *point closed* if and only if for each $v \in V(G)$, $N(N(v)) = \{v\}$.

Note that if a graph is point closed, then it is also point determining. We will be interested in D-graphs that are point closed (or at least point determining). For additional results concerning these latter two properties, see Sumner [8] and [9].

THEOREM 1. *Let G be a D-graph. Then G is point determining if and only if G has at most one isolated point and for each integer $k \geqq 1$, every complete subgraph of order k is contained in at most one clique of order $k+1$.*

PROOF. Let G be a point determining D-graph. Suppose we can find a complete subgraph A of some order $k \geqq 1$ such that $A \subseteq E_1$ and $A \subseteq E_2$ for some two distinct cliques of order $k+1$. Thus $E_1 = A \cup \{v\}$ and $E_2 = A \cup \{u\}$ for some $u, v \in V(G)$ with $u \neq v$ and $u \nsim v$. Suppose $w \in N(v)$. Then if $w \in A$, certainly $w \in N(u)$, while if $w \notin A$, then $E_1 = A \cup \{v\} \subseteq N(u) \cup N(w) \Rightarrow u \sim w$ so that $w \in N(u)$. Hence $N(v) \subseteq N(u)$. Similarly, we have $N(u) \subseteq N(v)$, and thus $N(u) = N(v)$, but this is a contradiction.

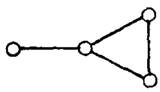
Conversely, suppose G is a D-graph and for each $k \geqq 1$, every complete subgraph of order k is contained in at most one clique of order $k+1$. Let $u, v \in V(G)$ with $u \neq v$ and suppose that $N(u) = N(v)$. Let E be a maximal, complete subgraph of $N(v) = N(u)$. Since not both of u and v are isolated, $|E| \geqq 1$. Thus $E \cup \{v\}$ and $E \cup \{u\}$ are both cliques containing E, but that is a contradiction.

COROLLARY 1. *If G is a point determining D-graph and if E is a clique in G with maximum order, then for any $v \notin E$, $|E - N(v)| \geqq 2$.*

PROOF. Since $v \notin E$, $E - N(v) \neq \varnothing$. So if $|E - N(v)| < 2$, we must have $E - N(v) = \{u\}$ for some $u \in V(G)$. Hence $F = (E - \{u\}) \cup \{v\}$ is a complete subgraph of G with $|F| = |E|$. Thus by the maximality of E, F is a clique in G. But then $E - \{u\}$ is a complete subgraph of order $|E| - 1$ contained in two distinct cliques of order $|E|$.

As a consequence, we obtain the following characterization of complete graphs in terms of the D-graph property.

COROLLARY 2. *A graph G is complete if and only if G is a connected, point determining D-graph which does not contain an induced subgraph of the form*

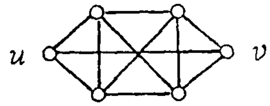


PROOF. Clearly every complete graph satisfies the given conditions. Suppose G is a point determining, connected D-graph that is not complete. We will show that G must contain an induced subgraph of the given form. By the previous corollary, we can find a clique E of G such that for every $v \notin E$, $|E - N(v)| \geqq 2$. Since G is not complete, $V(G) \neq E$, and so since G is connected, there exist $v \notin E$ and $u \in E$ with $v \sim u$. Let $w_1, w_2 \in E$ such that $v \nsim w_1$ and $v \nsim w_2$. Then $\{u, v, w_1, w_2\}$ induces a subgraph of the indicated form.

COROLLARY 3. *If G is a point determining D-graph with largest clique of order k and if E is a clique in G of order $k - 1$, then there exists at most one $v \in E$ such that $E - \{v\}$ is contained in a clique different from E.*

PROOF. Suppose that $u, v \in E$ with $u \neq v$, $E - \{u\} \subseteq A$, and $E - \{v\} \subseteq B$ where A and B are distinct cliques different from E. Then since a complete subgraph of order $k - 2$ can be contained in at most one clique of order $k - 1$, we have $|A| > k - 1$ and $|B| > k - 1$. Thus $|A| = |B| = k$. Hence $A = (E - \{u\}) \cup \{a, b\}$ and $B = (E - \{v\}) \cup \{c, d\}$ for some $a, b, c, d \in V(G) - E$. We note that $\{a, b\} \cap \{c, d\} = \varnothing$; for if $a = c$, for example, then since $u \in B$, $u \sim c$, and so $u \sim a$. So since $E - \{u\} \subseteq A$, we have $E \subseteq N(a)$, contrary to E being a clique. Let $F = (E - \{u, v\}) \cup \{a, b, c, d\}$. Then since $E \subseteq N(a) \cup N(c)$, we have $a \sim c$. Similarly, $a \sim d$, $b \sim c$, and $b \sim d$ (of course, $c \sim d$ and $a \sim b$ since A and B are complete). Thus F is complete, but $|F| = k + 1$, but this is a contradiction.

COROLLARY 4. *If G is a point determining D-graph, then every clique of order two either constitutes an endline (i.e., one of its vertices is an endpoint) or is the edge uv in an induced subgraph of the form*



https://doi.org/10.1017/S1446788700029232 Published online by Cambridge University Press

PROOF. Suppose the edge uv forms a clique of order two and neither u nor v is an endpoint. Then there exist cliques A and B different from $\{u, v\}$ with $u \in A$ and $v \in B$. But by the theorem, each of u and v is contained in at most one clique of order two, so $|A| \geqq 3$ and $|B| \geqq 3$. Let $x, y \in A - \{u\}$ with $x \neq y$ and r, $s \in B - \{v\}$ with $r \neq s$. Note that $x \sim y$ and $r \sim s$. Since $\{u, v\}$ forms a clique, $N(u) \cap N(v) = \emptyset$, so that $\{x, y, r, s\}$ is a set of four distinct vertices, and since G is a D-graph, it follows that $\{x, y, r, s\}$ is complete. Thus $\{u, v, x, y, r, s\}$ induces a subgraph of the indicated form.

DEFINITION. Two endpoints u and v of a graph G are *coincident* if and only if $N(u) = N(v)$.

Among those graphs that have no cliques of order larger than three, our next result characterizes those that are point determining D-graphs.

THEOREM 2. *If G is a connected graph with no cliques of order larger than three, then G is a point determining D-graph if and only if every edge of G either lies in exactly one triangle or is an endline adjacent to no other endline.*

PROOF. Suppose G is a point determining D-graph. Let $e = uv$ be an edge of G. If e does not lie in any triangle, then $\{u, v\}$ forms a clique and so, since G has no complete subgraphs of order four, it follows from Corollary 4 that e is an endline. Since G is point determining, e cannot be adjacent to any other endline. As a consequence of Theorem 1 with $k = 2$, e lies in at most one triangle.

Conversely, suppose G satisfies the given conditions. We first observe that G is point determining. For it $N(u) = N(v)$ for distinct vertices u and v, then we may choose $w \in N(u) = N(v)$. However, not both of uw and vw can be endlines since they form adjacent edges. Hence we may assume that uw lies in a triangle. Thus there is some $x \in G$ with $x \sim u$ and $x \sim w$. But then $x \sim v$ so that the edge xw lies in the two triangles xwv and xwu.

Finally, suppose that G is not a D-graph. Let E be a clique in G with $E \subseteq N(x) \cup N(y)$ and $x \nsim y$. Then there exist $a, b \in E$ with $a \neq b$, $a \sim x$, and $b \sim y$. Thus ab is not an endline and hence lies in a unique triangle abc. But then E must be $\{a, b, c\}$. Without loss of generality, $c \sim x$. But then the triangles cax and abc both contain the edge ac.

Our next theorem characterizes those D-graphs that are point closed.

THEOREM 3. *If G is a D-graph, then G is point closed if and only if for every clique E of G and $u \notin E$, there exist v_1, $v_2 \in E$ with $v_1 \neq v_2$, $u \nsim v_1$, and $u \nsim v_2$.*

PROOF. Suppose G is a point closed D-graph and E is a clique in G. Let $u \notin E$. Then there exists $v_1 \in E$ with $u \nsim v_1$. Suppose $u \in N(E - \{v_1\})$. Then since $N(N(v_1)) = \{v_1\}$, we have $u \notin N(N(v_1))$ so there exists $w \in G$ with $w \sim v_1$ and

https://doi.org/10.1017/S1446788700029232 Published online by Cambridge University Press

$w \nsim u$. But then $E \subseteq N(u) \cup N(w)$ with $w \nsim u$, but that is impossible in a D-graph.

Now suppose G is such that for every clique E and $u \notin E$, there exist $v_1, v_2 \in E$ with $u \nsim v_1$ and $u \nsim v_2$. Suppose $N(N(u)) \neq \{u\}$. Let E be a maximal complete subgraph of $N(u)$. Then $E \cup \{u\}$ is a clique of G and if $v \in N(N(w)) - \{u\}$, then v is adjacent to all but one element of $E \cup \{u\}$, but that is impossible.

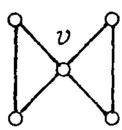
COROLLARY 5. *In a connected, point closed D-graph with at least three vertices, there are no cliques of order two, and every clique of order three meets every other clique in at most one vertex.*

COROLLARY 6. *Let G be a point closed D-graph and* $v \in V(G)$. *Then one of the following holds:*

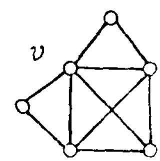
(i) *v lies in exactly one clique;*

(ii) *v is the point v in an induced subgraph of the form* (a) *below, or*

(iii) *v is the point v in an induced subgraph of the form in* (b).



(a)



(b)

PROOF. Suppose v lies in at least two cliques. Then there exist $a, b \in G$ with $v \sim a$, $v \sim b$, and $a \nsim b$. Let E be a clique containing $\{a, v\}$. Then $b \notin E$, so there exists $c \in E - \{a\}$ with $c \nsim b$. Let F be a clique containing $\{b, v\}$. Then $a \notin F$, so there exists $d \in F - \{b\}$ with $d \nsim a$. If $d \nsim c$, then $\{a, b, c, d, v\}$ induces a subgraph of the form in (a). If $d \sim c$, then let D be a clique containing $\{v, d, c\}$. Then since $a \nsim b$, $D \nsubseteq N(a) \cup N(b)$ so there exists $y \in D$ with $y \nsim a$ and $y \nsim b$. Thus $\{a, b, c, d, v, y\}$ induces a subgraph of the form in (b).

COROLLARY 7. *A graph G is complete if and only if it is a connected, point closed D-graph that does not contain an induced subgraph of the form* (a) *or* (b) *of Corollary* 6.

The following result is proved in Sumner [8].

THEOREM 4. *If G is a point determining, connected graph that is not complete, then there exists an edge e of G such that G − e is also point determining.*

We note that every complete graph is a point closed D-graph and also that the removal of any edge of a complete graph results in a D-graph. It is curious that these properties, in fact, characterize complete graphs.

THEOREM 5. *A graph G is complete if and only if G is a connected, point closed D-graph in which the removal of any edge again results in a D-graph.*

https://doi.org/10.1017/S1446788700029232 Published online by Cambridge University Press

PROOF. Suppose G satisfies the given conditions but is not complete. Then since G is point closed, it is also point determining and hence by the previous theorem, there exists an edge e of G such that $G - e$ is also point determining. Let $e = uv$. Let E be a clique of G which contains u and v. Then $F = E - \{u\}$ and $D = E - \{v\}$ are complete in $G - e$ and from Theorem 3, denoting the neighborhood sets in $G - e$ by $N_0, N_0(E - \{u\}) = N_0(E - \{v\}) = \varnothing$ since G is point closed. Thus F and D are cliques in $G - e$. Hence $E - \{u, v\}$ is a complete subgraph of order $|E| - 2$ which is contained in the two cliques F and D of $G - e$ both having order $|E| - 1$. But this is impossible since by Theorem 1 we would have $G - e$ not point determining.

DEFINITION. *Let G be a graph. We will say that the large cliques are sparsely scattered if and only if there do not exist cliques A, B, and C, all of order at least four such that $|A \cap B| \geqq 2$ and $|B \cap C| \geqq 2$.*

COROLLARY 8. *Let G be a graph such that the large cliques are sparsely scattered. Then G is a point closed D-graph if and only if for every clique E and $u \notin E$, there exist $v_1, v_2 \in E$ with $v_1 \neq v_2$, $v_1 \nsim u$, and $v_2 \nsim u$.*

PROOF. As a consequence of Theorem 3, it is enough to show that under the assumption that the large cliques of G are sparsely scattered, the given condition implies that G is a D-graph. Suppose G is not a D-graph. Let B be a clique of G and let u and v be distinct vertices of G with $B \subseteq N(u) \cup N(v)$ and $u \nsim v$. Since $u \notin B$, there exist $a, b \in B$ with $\{a, b\} \cap N(u) = \varnothing$. Hence $a, b \in N(v)$. Similarly, there exist $c, d \in N(x) \cap B$ such that $\{c, d\} \cap N(v) = \varnothing$. Let A and C be cliques containing $\{a, b, v\}$ and $\{c, d, u\}$, respectively. Since c must be nonadjacent to at least two elements of A and a is nonadjacent to at least two elements of B, we have $|A| \geqq 4$ and $|C| \geqq 4$. But clearly $|B| \geqq 4$, $|A \cap B| \geqq 2$, and $|B \cap C| \geqq 2$ contrary to the assumption that the large cliques are sparsely scattered.

As an immediate consequence of this we obtain a result originally due to Greechie and Miller [6].

COROLLARY 9. *Let G be a graph such that every clique has order at least three and no two cliques meet in more than one vertex. Then G is a point closed D-graph.*

We may generalize this result in another direction by:

THEOREM 6. *Let G be a graph and let $k \geqq 0$ be an integer such that for every two cliques E_1 and E_2, $|E_1 \cap E_2| \leqq k$. Then if for every clique E with $|E| \leqq 2k$ there is some $r \geqq 0$ such that E contains $2r + 1$ vertices no $r + 1$ of which are in any other clique, then G is a D-graph.*

https://doi.org/10.1017/S1446788700029232 Published online by Cambridge University Press

PROOF. Suppose G is not a D-graph and let E be a clique with $E \subseteq N(a) \cup N(b)$ and $a \nsim b$. Thus there exist $x, y \in E$ with $a \sim x$ and $b \sim y$. Let F and D be cliques of G such that $\{a\} \cup (N(a) \cap E) \subseteq F$ and $\{b\} \cap (N(b) \cap E) \subseteq D$. Then $E \subseteq (F \cap E) \cup (D \cap E)$ so that $|E| \leqq |F \cap E| + |D \cap E| \leqq 2k$. But for every $r \geqq 0$ and any $2r+1$ vertices in E, there are $r+1$ of them in F or $r+1$ of them in D, both cliques different from E. Thus this is a contradiction and G must be a D-graph.

However, a graph satisfying the conditions of the previous theorem need not be point closed (nor even point determining) as may be seen by considering K_4 with one edge deleted.

DEFINITION. *If G is a graph, then by the line graph of G we mean the graph $L(G)$ whose vertices are the edges of G; two vertices of $L(G)$ are adjacent if and only if they are adjacent edges in G.*

The next theorem characterizes those line graphs which are also D-graphs. The proof is straightforward but tedious and is omitted. The proof may be found in Sumner [9].

THEOREM 7. *Let G be a connected graph of order at least five. Then the line graph $L(G)$ is a D-graph if and only if every triangle in G contains two vertices* of degree two and for each $v \in V(G)$,

(i) *If $\delta(v) = 2$, then v either lies in a triangle or is adjacent to an endpoint.*
(ii) *If $\delta(v) = 3$, then $N(v)$ is an independent set.*
(iii) *If $\delta(v) = 4$, then the graph induced by $N(v)$ contains an isolated vertex.*

COROLLARY 10. *If G is a connected graph with $|G| \geqq 5$ and $\delta(G) \geqq 3$, then $L(G)$ is a D-graph if and only if G has no triangles; and in this case, $L(G)$ is also point closed.*

We will denote the diameter of a graph G by $d(G)$ and the distance between two vertices x and y by $d(x, y)$. We have the following bound on the diameter of a D-graph.

THEOREM 8. *Let G be a connected D-graph of order p and let $\varepsilon(G)$ be the order of a largest clique in G. Then $d(G) \leqq [(1/2)(p - \varepsilon(G) + 4)]$.*

PROOF. Let $d(G) = d$. Fix $x, y \in G$ with $d(x\ y) = d$, and let P be a path $x = p_0 p_1 \cdots p_d = y$ from x to y of length d. The theorem is trivially true if $d \leqq 2$, so we will suppose $d \geqq 3$. Since P is a shortest path between x and y, we have $p_i \sim p_j$ for $p_i, p_j \in P$ if and only if $|i - j| = 1$. Thus for $i = 1, 2, \cdots, d-2$, let E_i be a clique containing $\{p_i, p_{i+1}\}$. Then $p_{i-1} \nsim p_{i+2}$, so $E_i \nsubseteq N(p_{i+1}) \cup N(p_{i+2})$; hence there exists $x_i \in E_i$ with $x_i \nsim p_{i-1}$ and $x_i \nsim p_{i+2}$. Therefore since P is a shortest path between x and y, $N(x_i) \cap P = \{p_i, p_{i+1}\}$. Thus $Q = \{x_1, x_2, \cdots, x_{d-2}\}$ is a set of $d-2$ distinct points and $Q \cap P = \varnothing$.

Let E be a clique in G of order $\varepsilon(G)$. We claim that $|E \cap (P \cup Q)| \leqq 3$. Clearly $|E \cap P| \leqq 2$.

If $E \cap P = \{p_i, p_{i+1}\}$, then $E \cap Q$ can contain at most x_i. If $E \cap P = \{p_i\}$, then $E \cap Q$ can contain at most $\{x_{i+1}, x_i\}$. Thus in either of these cases, $|E \cap (P \cup Q)| \leqq 3$.

Suppose that $E \cap P = \varnothing$. Then if $x_{r_1}, x_{r_2}, x_{r_3}$, and x_{r_4} are elements of $E \cap Q$ with $r_1 < r_2 < r_3 < r_4$, the path $p_0 p_1 \cdots p_{r_1} x_{r_1} x_{r_4} p_{r_4+1} \cdots p_d$ has length $d-1$, but that is impossible. Hence $|E \cap Q| \leqq 3$. So here too, $|E \cap (P \cup Q)| \leqq 3$.

Therefore we have

$$p \geqq |P| + |Q| + (|E| - 3) = (d+1) + (d-2) + \varepsilon(G) - 3,$$

so

$$d \leqq \tfrac{1}{2}(p - \varepsilon(G) + 4).$$

3. Hereditary Dacey graphs

DEFINITION. A graph G is an *HD-graph* if and only if every induced subgraph of G is a D-graph.

Our purpose in the remainder of this paper is to develop the previously known results on HD-graphs in a shorter and more constructive manner. Also we will establish some interesting connectivity properties of HD-graphs, the most surprising of which is Theorem 10.

We will henceforth refer to a path of length three as a *hook*.

The next lemma is well known (see Foulis [3]).

LEMMA 1. *A graph G is an HD-graph if and only if it does not contain a hook as an induced subgraph.*

PROOF. Since a hook is not a D-graph, no HD-graph can contain a hook as an induced subgraph

On the other nand, suppose that G contains no hook as an induced subgraph. We first observe that such a graph must be a D-graph. For suppose E is a clique of G and $u, v \in V(G)$ such that $E \subseteq N(v) \subset N(u)$ but $u \nsim v$. Then $v \notin E$ so there exists $x \in E$ with $x \nsim v$ and so $x \sim u$. Similarly there exists $y \in E$ with $y \nsim u$ and $y \sim v$. But then $uxyv$ is a hook in G. Thus any graph without an induced hook is a D-graph. However, if G has no induced subgraph isomorphic to a hook, neither does any induced subgraph of G. Thus by our observation above, every induced subgraph of G must be a D-graph and hence G is an HD-graph.

REMARK. It is evident that every two vertices of a connected HD-graph are a distance at most two apart. In fact, an equivalent condition for a connected graph G to be an HD-graph is that every induced, connected subgraph of G have diameter at most two. It is also worth noting that every induced subgraph of an HD-graph is again an HD-graph.

https://doi.org/10.1017/S1446788700029232 Published online by Cambridge University Press

DEFINITION. *If G is a connected graph and A and B are disjoint subsets of $V(G)$ with $V(G) = A \dot{\cup} B$, then we write $G = A \oplus B$ if and only if $a \in A$, $b \in B \Rightarrow a \sim b$. In this case, we say that A (and B) is a direct summand of G.*

LEMMA 2. *Let G be a connected HD-graph of order p and let $v \in V(G)$ be a cutpoint. Then $\delta(v) = p - 1$.*

PROOF. Suppose v is a cutpoint of G and $u \in G - v$ such that $v \nsim u$. Let A be the component of $G - v$ which contains u. Since G has diameter at most two, there exists a vertex $w \in A$ with $w \sim u$ end $w \sim v$. Let B be any component of $G - v$ other than A. Then since G is connected, there exists $t \in B$ with $t \sim v$. But then $tvwu$ forms a hook. But this is a contradiction.

If G is connected and $A \subseteq V(G)$ such that $G - A$ is not connected, we will refer to A as a *cut set* of G. If no proper subset of A is a cut set, we will say that A is a *minimal cut set.*

THEOREM 9. *If G is a connected HD-graph and $A \subseteq V(G)$ is a minimal cut set of G, then $G = A \oplus (G - A)$, i.e., A is a direct summand of G.*

PROOF. If $|A| = 1$, then $G = A \oplus (G - A)$ by the the previous lemma. Hence we may assume that $|A| \geqq 2$. Let $a \in A$. Then by the minimality of $A, A - \{a\}$ is not a cut set. Thus $G - (A - \{a\}) = (G - A) \cup \{a\}$ is a connected HD-graph having a as a cutpoint. Hence by the previous lemma, a is adjacent to every element of $G - A$ and since this holds for every $a \in A$, the theorem follows.

COROLLARY 11. *Let G be a connected HD-graph of order $p \geqq 2$. Then*

(i) *$k(G) + \Delta(G) \geqq p$, where $k(G)$ is the connectivity of G and $\Delta(G)$ is the maximal degree of G.*

(ii) $\Delta(G) \geqq p/2$.

(iii) *If G is regular and $p \geqq 3$, then G is Hamiltonian.*

PROOF. All of (i), (ii), and (iii) are clear for complete graphs, and so we will assume G is not complete for the remainder of this proof.

(i) Let A be a cut set of order $k(G)$. Then $G = A \oplus (G - A)$ and hence for any $a \in A$, $\Delta(G) \geqq \delta(a) \geqq |G - A|$. Thus

$$p = |A| + |G - A| \leqq k(G) + \Delta(G).$$

(ii) Since $\Delta(G) \geqq k(G) \geqq p - \Delta(G)$, it follows that $\Delta(G) \geqq p/2$.

(iii) For $p \geqq 3$, denoting the minimal degree of G by $\delta(G)$, we have for a regular HD-graph $G, \delta(G) = \Delta(G) \geqq p/2$ and hence, by the well-known theorem of Dirac [2], G is Hamiltonian.

The next two corollaries were known previously (see Foulis [3]).

COROLLARY 12. *A nontrivial connected graph G is an HD-graph if and only if there exist subgraphs A and B of G which are HD-graphs and* $G = A \oplus B$.

PROOF. If G is not complete, then for any minimal cut set A, $G = A \oplus (G - A)$. If G is complete, then $G = A \oplus (G - A)$ for any subgraph A of G.

Conversely, if $G = A \oplus B$, then any induced hook of G must lie entirely in either A or B and hence if A and B are both HD-graphs, then so is G.

COROLLARY 13. *If G is a nontrivial HD-graph, then exactly one of G and* $\bar{G}$ *(the complement of G) is connected.*

PROOF. At least one of G and $\bar{G}$ must be connected, so we may assume that G is connected. Thus $G = A \oplus B$ for some subgraphs A and B. But then no vertex of A is adjacent to any vertex of B in $\bar{G}$. Thus $\bar{G}$ is not connected.

COROLLARY 14. *A graph G is a complete bipartite graph if and only if G is a connected D-graph with no triangles.*

PROOF. Clearly every complete bipartite graph is a connected D-graph with no triangles.

Suppose G is a connected D-graph with no triangles. Then G must clearly be an HD-graph and hence $G = A \oplus B$ for some subgraphs A and B. But then if either of A or B contained an edge, G would contain a triangle. Thus each of A and B is an independent set of vertices and G is a complete bipartite graph.

LEMMA 3. *If G is a nontrivial connected HD-graph and S is a maximal independent set in G, then* $N(S) \neq \varnothing$ *and* $N(S)$ *is a direct summand of G.*

PROOF. By Corollary 12, G contains two subgraphs A and B with $G = A \oplus B$. Since S is independent, $S \subseteq A$ or $S \subseteq B$. Without loss of generality, we can assume that $S \subseteq A$ so that $\varnothing \neq B \subseteq N(S)$. Let $v \in N(S)$. If $G - (S \cup N(S)) = \varnothing$, then $G = S \oplus N(S)$ and we are finished. So we suppose there exists $u \in G - (S \cup N(S))$. We claim that $v \sim u$. Suppose not. Then since $u \notin S$, there exists $w \in S$ with $w \sim u$. But $u \notin N(S)$, so there exists $t \in S$ with $t \nsim u$. Since $v \in N(S)$, $v \sim w$, and $v \sim t$, S is independent so that $t \nsim w$. Thus $uwvt$ is a hook, but this is a contradiction. Hence every u, v with $v \in N(S)$, $u \notin N(S)$ are adjacent and thus $N(S)$ is a direct summand of G.

DEFINITION. Let G be a connected, nontrivial graph. A subset $A \subseteq V(G)$ will be called a *disconnecting* set if and only if $G - A$ is either a disconnected graph or the trivial graph. If no proper subset of A is also a disconnecting set, then we will say that A is a *minimal disconnecting* set.

THEOREM 10. *If G is a nontrivial connected HD-graph, then* $S \subseteq V(G)$ *is a maximal independent set if and only if* $N(S)$ *is a minimal disconnecting set.*

https://doi.org/10.1017/S1446788700029232 Published online by Cambridge University Press

PROOF. Let $S \subseteq V(G)$ be a maximal independent set. Then $G = N(S) \oplus (G - N(S))$. We claim that $G - N(S)$ is not connected or is trivial. If $S = \{v\}$ for some vertex v, then $N(S) = N(v) = G - v$ and in this case, $G - v$ is a minimal disconnecting set. Hence we may assume that S is nontrivial. Since $S \subseteq G - N(S)$, $G - N(S)$ is nontrivial. Let $A = (G - N(S)) - S$. If $A = \varnothing$, then $G - N(S) = S$ is not connected. Hence we may assume $A \neq \varnothing$. Let $a \in A$ such that $|N(a) \cap S|$ is as large as possible. Since $a \notin N(S)$, there exists $s_0 \in S$ with $a \nsim s_0$. Now suppose that $A \cup S = G - N(S)$ is connected and hence a connected HD-graph. Then in $A \cup S$, $d(a, s_0) = 2$, so there exists $b \in A$ such that $a \sim b$ and $b \sim s_0$. Now let $s \in N(a) \cap S$. Then in order that $sabs_0$ not be a hook, we must have $s \sim b$. Thus $s \in N(b) \cap S$. Hence since $s_0 \in N(b) \cap S$ while $s_0 \notin N(a) \cap S$, $|N(a) \cap S| < |N(b) \cap S|$ which is contrary to the choice of a. Thus $G - N(S)$ is not connected. Since $G = N(S) \oplus (G - N(S))$, no proper subset of $N(S)$ can disconnect G. Hence $N(S)$ is a minimal disconnecting set.

Now suppose that A is a minimal disconnecting set. If $G - A = \{v\}$, then by the minimality of A, $G - (A - \{a\}) = \{a, v\}$ is connected for each $a \in A$. Therefore $N(v) = A$ and $S = \{v\}$ is a maximal independent set. Thus we may assume $G - A$ is nontrivial and thus A is a minimal cut set. Hence $G = A \oplus (G - A)$ and $G - A$ is not connected. So if we choose $S_1, S_2, \cdots, S_k$ maximal independent subsets of each component of $G - A$, we obtain $S_1 \cup S_2 \cup \cdots \cup S_k = S$ is a maximal independent subset of G such that $N(S) = A$.

References

[1] M. Behzad and G. Chartrand, *Introduction to the theory of graphs*, (Allyn and Bacon, Inc., Boston (1972).)

[2] G. A. Dirac, 'Some theorems on abstract graphs', *Proc. London Math. Soc.* 2 (1952), 69–81.

[3] D. J. Foulis, *Empirical logic*, xeroxed course notes, (University of Massachusetts, Amherst, Massachusetts (1969–1970).)

[4] C. H. Randall and D. J. Foulis, 'An approach to empirical logic', *Amer. Math. Monthly* 77 (1970), 363–374.

[5] C. H. Randall and D. J. Foulis, 'Operational statistics, I: basic concepts', *J. Mathematical Phys.* 13 (1972), 1667–1675.

[6] R. J. Greechie and F. R. Miller, *On structures related to states on an empirical logic*, I: *weights on finite spaces*, mimeographed notes, (Kansas State University, Manhattan, Kansas (1970).)

[7] B. L. Jeffcott, *Orthologics*, (Ph. D. Dissertation, University of Massachusetts, Amherst, Massachusetts (1971).)

[8] D. P. Sumner, 'Point determination in graphs', *Discrete Math.* 5 (1973), 179–187.

[9] D. P. Sumner, Indecomposable graphs, (Ph.D. Dissertation, University of Massachusetts, Amherst, Massachusetts (1971).)

Department of Mathematics
University of South Carolina
Columbia, South Carolina 29208
U. S. A.