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ESSENTIAL STATE SURFACES FOR KNOTS AND LINKS

MAKOTO OZAWA

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Abstract

We study a canonical spanning surface obtained from a knot or link diagram, depending on a given Kauffman state. We give a sufficient condition for the surface to be essential. By using the essential surface, we can deduce the triviality and splittability of a knot or link from its diagrams. This has been done on the extended knot or link class that includes all semiadequate, homogeneous knots and links, and most algebraic knots and links. In order to prove the main theorem, we extend Gabai's Murasugi sum theorem to the case of nonorientable spanning surfaces.

2010 *Mathematics subject classification*: primary 57M25; secondary 57Q35.

Keywords and phrases: state surface, knot diagram, adequate knot, homogeneous knot, Murasugi sum.

1. Introduction

In 1930, Frankl and Pontrjagin [9] proved the existence of a Seifert surface for any knot. In 1934, Seifert [28] gave an algorithm for constructing a Seifert surface from a knot diagram. Seifert's algorithm allows us to construct a spanning surface from a knot diagram, depending on a given Kauffman state [15]. In this paper, we give a sufficient condition for the spanning surface to be essential. By using the essential surface, we show that a knot or link is trivial or split if and only if the diagram is trivial or split respectively under our sufficient condition.

Throughout this paper we work in the piecewise linear category. For knot theory, graph theory and 3-manifold theory terminology, we refer to [5, 7, 17].

In Section 2 we define and give examples of state surfaces and state our main results. We prove some lemmas in Section 3, one of which extends Gabai's Murasugi sum theorem [11], and prove our main theorems in Section 4. In Section 5 we list some problems that are of interest for further study. Finally, in Section 6, we summarize some of the recent progress that has been made since the first draft of this paper.

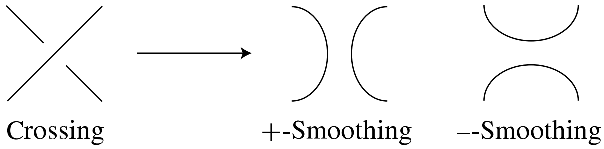

Figure 1. Two smoothings of a crossing.

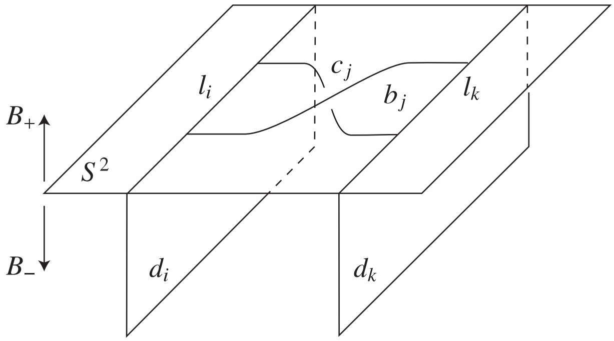

Figure 2. Recovering a crossing by a half twisted band.

2. Definitions, examples and results

Let K be a knot or link in the 3-sphere S^3 and let D be a connected diagram of K on the 2-sphere S^2 that separates S^3 into two 3-balls, say B_+ and B_-. Let $C = \{c_1, \ldots, c_n\}$ be the set of crossings of D. A map $\sigma : C \to \{+, -\}$ is called a *state* for D. For each crossing $c_i \in C$, we take a +-smoothing or −-smoothing of D, according to whether $\sigma(c_i) = +$ or $-$ (see Figure 1).

Historically, in most papers, +-smoothing and −-smoothing are called A-splicing and B-splicing. The terminology +-smoothing and −-smoothing seems reasonable since, if we orient a crossing locally so that it has a ±-sign, then a smoothing along the orientation coincides with a ±-smoothing. After ±-smoothing, we have a collection $l_1, \ldots, l_m$ of loops on S^2 that we call *state loops*. Let $\mathcal{L}_\sigma = \{l_1, \ldots, l_m\}$ be the set of state loops.

Each state loop l_i bounds a unique disk d_i in B_-. We may assume that these disks are mutually disjoint. For each crossing c_j and state loops l_i and l_k whose subarcs replace c_j by $\sigma(c_j)$-smoothing, we attach a half twisted band b_j to d_i and d_k so that it recovers c_j. See Figure 2 for the case where $\sigma(c_j) = +$. In this way, we obtain a spanning surface that consists of disks $d_1, \ldots, d_m$ and half twisted bands $b_1, \ldots, b_n$. We call this surface a *σ-state surface* F_σ.

Remark 2.1. The following historical remarks were suggested by Przytycki.

First, the state surfaces corresponding to the *positive state* σ_+ (that is, $\sigma_+(c_j) = +$ for all j) and the *negative state* σ_- (that is, $\sigma_-(c_j) = -$ for all j) were considered for

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alternating links in the nineteenth century by Tait. They were originally called Tait surfaces, but nowadays are called checkerboard surfaces.

Next, the state surface corresponding to the *Seifert state* $\vec{\sigma}$ (that is, the state determined by an orientation of the knot) that gives the Seifert surface was introduced by Seifert in [28].

Finally, independently, Przytycki has already considered using a surface for any Kauffman state. See [26, Footnote 2].

We may assume that F_σ intersects $N(K)$ in its collar $N(\partial F_\sigma; F_\sigma)$ and write F_σ instead of $F_\sigma \cap E(K)$, where $N(K)$ denotes the regular neighbourhood of K in S^3 and $E(K)$ denotes the exterior of K. We take a (twisted) I-bundle $F_\sigma \tilde{\times} I$ over F_σ in $E(K)$ and call the associated ∂I-bundle $F_\sigma \tilde{\times} \partial I$ over F_σ the *interpolating surface* obtained from F_σ. We denote this surface by $(F_\sigma)^{\sim}$ since it is a double cover of F_σ. Note that any interpolating surface $(F_\sigma)^{\sim}$ is orientable, and it is connected if and only if F_σ is nonorientable.

We use F_σ to construct a graph G_σ, with signs on its edges, by regarding a disk d_i as a vertex v_i and a band b_j as an edge e_j that has the same sign as $\sigma(c_j)$. We call the graph G_σ a *σ-state graph*. In general, a graph is called a *block* if it is connected and has no cut vertex. It is well known that any graph has a unique decomposition into maximal blocks.

Following [4, 16], we say that a diagram D is *σ-adequate* if G_σ has no loops, and that D is *σ-homogeneous* if, in each block of G_σ, all edges have the same sign. We remark that any diagram of any link is σ-adequate for some state σ (for example, the Seifert state), and σ'-homogeneous for some state σ' (for example, the positive state), where the states σ and σ' do not generally coincide.

REMARK 2.2. As was pointed out in [8], the definition of 'adequate' given here seems to differ slightly from the original definition. See Example 2.6 for a discussion of its consistency with the original definition.

EXAMPLE 2.3. Let D be a diagram of the figure-of-eight knot, which has four crossings, c_1, c_2, c_3 and c_4, as shown at the top of Figure 3. To make a σ-state surface, let $\sigma(c_1) = \sigma(c_2) = -$ and $\sigma(c_3) = \sigma(c_4) = +$, for example. Since the σ-state graph G_σ has no loop and all edges in each block have the same sign as in the second part of Figure 3, D is σ-adequate and σ-homogeneous. Moreover, the block decomposition of G_σ corresponds to a Murasugi decomposition of F_σ (see the bottom part of Figure 3).

EXAMPLE 2.4. A diagram D with an orientation is said to be *positive* if all crossings have a positive sign. For any positive diagram D, there exists a state σ such that D is σ-adequate and σ-homogeneous. Indeed, we can take σ so that $\sigma(c_j) = +$ for all c_j, namely, the positive state σ_+. Also, we can take σ so that it yields a canonical Seifert surface F_σ, namely, the Seifert state $\vec{\sigma}$. Note that the states σ_+ and $\vec{\sigma}$ coincide only on a positive diagram.

EXAMPLE 2.5. For any alternating diagram D without nugatory crossings, there exist two states σ_1 and σ_2 such that D is σ_i-adequate and σ_i-homogeneous for $i = 1, 2$. Indeed, we can take $\sigma_1 = \sigma_+$ (or $\sigma_1 = \sigma_-$) and $\sigma_2 = \vec{\sigma}$.

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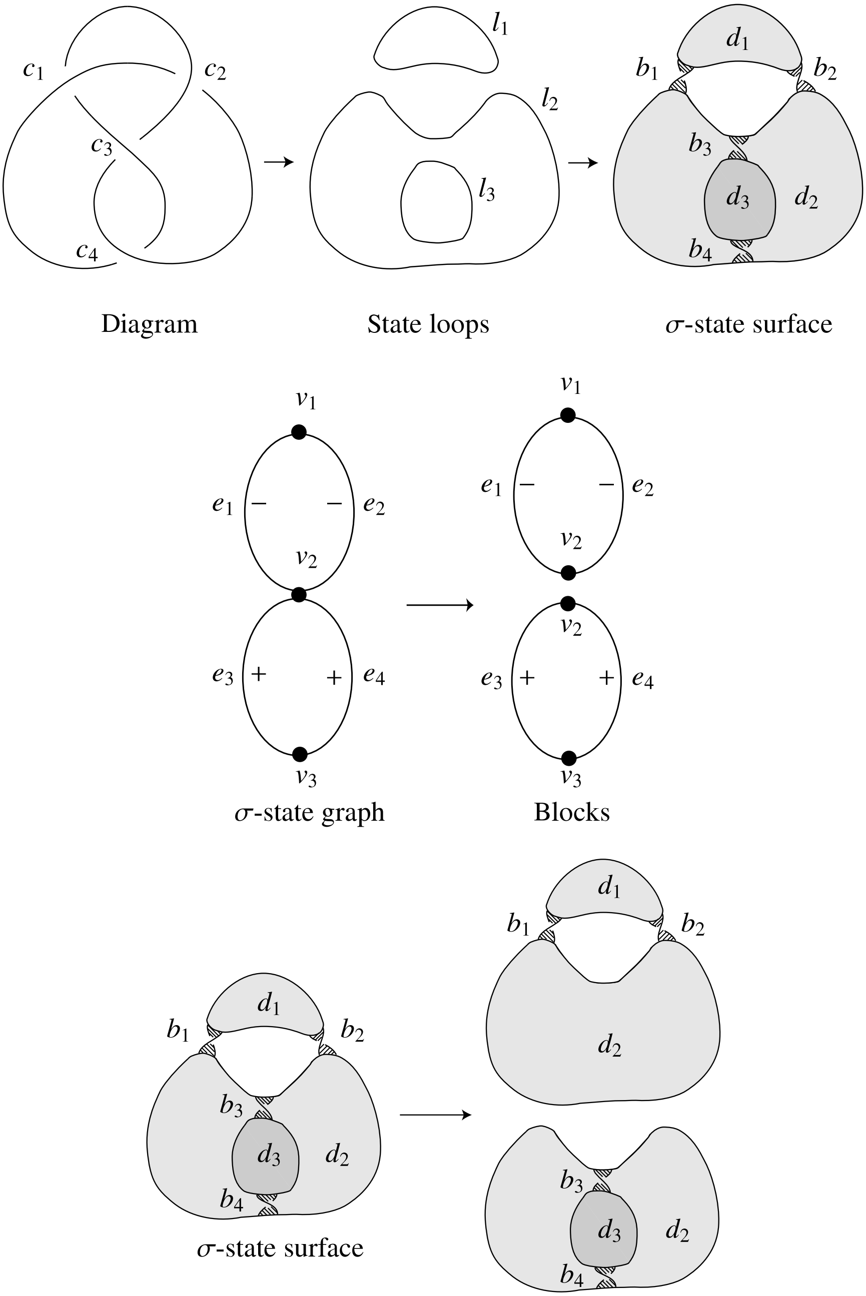

FIGURE 3. A state surface, σ-state graph and block decomposition, and Murasugi decomposition.

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Example 2.6. We say that a diagram D is *homogeneous* [4] if D is $\vec{\sigma}$-homogeneous for the Seifert state $\vec{\sigma}$. Note that D is automatically $\vec{\sigma}$-adequate since the $\vec{\sigma}$-state surface $F_{\vec{\sigma}}$ is orientable and thus $G_{\vec{\sigma}}$ has no loop.

We say that a diagram D is *semiadequate* [16] if D is σ-adequate for the positive state σ_+ or the negative state σ_-. Note that D is automatically $\sigma_\pm$-homogeneous since $\sigma_\pm(c_j) = \pm$ for all j.

We say that a diagram D is *adequate* [29] if D is σ-adequate for both the positive state σ_+ and the negative state σ_-. Note also that D is automatically $\sigma_\pm$-homogeneous since $\sigma_\pm(c_j) = \pm$ for all j.

Example 2.7. We say that an arborescent link L is *strictly arborescent* if the absolute value of each weight is greater than 1. Note that there exist a diagram D of L and a state σ such that D is σ-adequate and σ-homogeneous. Indeed, a strictly arborescent link L is the boundary of a σ-state surface that is a Murasugi sum of twisted annuli or Möbius bands with one or more full twists. See [3] or [12] for the definition and construction of surfaces for arborescent links.

We now review the definition of essential surfaces. Let M be an orientable compact 3-manifold and let F be a compact surface, but not a 2-sphere, properly embedded in M, possibly with boundary. Let i denote the inclusion map $F \to M$. We say that F is π_1*-injective* if the induced map

$$i_* : \pi_1(F) \to \pi_1(M)$$

is injective. We say that F is ∂-π_1*-injective* if the induced map

$$i_* : \pi_1(F, \partial F) \to \pi_1(M, \partial M)$$

is injective for every choice of two base points in ∂F. A surface F in M is π_1*-essential* if F is π_1-injective, ∂-π_1-injective and not ∂-parallel in M.

A disk D, embedded in M, is a *compressing disk* for F if $D \cap F = \partial D$ and ∂D is an essential loop in F. A disk D, embedded in M, is a ∂*-compressing disk* for F if $D \cap F \subset \partial D$ is an essential arc in F and

$$D \cap \partial M = \partial D - \mathrm{int(D \cap F)}.$$

We say that F is *incompressible* or ∂*-incompressible* if there exists no compressing disk or ∂-compressing disk respectively for F. A surface F in M is *essential* if F is incompressible, ∂-incompressible and not ∂-parallel in M. We remark that a σ-state surface F_σ is π_1-essential in $E(K)$ if and only if the interpolating surface $(F_\sigma)^{\tilde{}}$ obtained from F_σ is essential in $E(K)$.

The following theorem gives a sufficient condition for the state surface to be π_1-essential.

Theorem 2.8. *If a diagram is both σ-adequate and σ-homogeneous for some state σ, then the σ-state surface is π_1-essential.*

If F_σ is nonorientable and π_1-essential, then the interpolating surface $(F_\sigma)^{\tilde{}}$ is connected and essential. Therefore, the knot satisfies the Neuwirth conjecture [21],

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which states that, for any nontrivial knot K, there exists a closed surface S that contains K and is such that $S \cap E(K)$ is connected and essential in $E(K)$.

COROLLARY 2.9. *If a diagram is σ-adequate and σ-homogeneous for a state $\sigma \neq \vec{\sigma}$, then the knot satisfies the Neuwirth conjecture. In particular, all adequate knots satisfy the Neuwirth conjecture.*

REMARK 2.10. It can be confirmed that every 10-crossing knot diagram in the Rolfsen knot table [27], except for 8_{19}, 10_{124}, 10_{128}, 10_{134}, 10_{139} and 10_{142}, is σ-adequate and σ-homogeneous for a positive or negative state σ distinct from the Seifert state $\vec{\sigma}$. Also, every 11-crossing knot diagram in the Hoste–Thistlethwaite knot table [14], except for $K11_n93$, $K11_n95$, $K11_n118$, $K11_n126$, $K11_n136$, $K11_n169$, $K11_n171$, $K11_n180$ and $K11_n181$, is σ-adequate and σ-homogeneous for a positive or negative state σ distinct from the Seifert state $\vec{\sigma}$. Furthermore, it can be checked that 10_{134}, 10_{142}, $K11_n93$, $K11_n95$, $K11_n136$, $K11_n169$, $K11_n171$, $K11_n180$ and $K11_n181$ bound π_1-essential, nonorientable checkerboard surfaces. (It may be necessary to deform the diagram by the Reidemeister move of type III.)

REMARK 2.11. Futer *et al.* [10] estimated the hyperbolic volume of adequate knots by using the guts of state surfaces.

REMARK 2.12. We can construct a spanning surface other than F_σ from a state σ by letting the loop l_i bound a disk d'_i in B_+. Theorem 2.8 holds for all state surfaces obtained by such a method. Moreover, we can construct a branched surface as in [13] that consists of disks $d_1, \ldots, d_m$ in B_- and disks $d'_1, \ldots, d'_m$ in B_+ bounded by $l_1, \ldots, l_m$ respectively, and half twisted bands $b_1, \ldots, b_n$.

REMARK 2.13. Suppose that a diagram D is σ-adequate and σ-homogeneous for a state σ. If F_σ is orientable, then it is a minimal genus Seifert surface by [11, Theorem 2] or [4, Corollary 4.1].

On the other hand, as pointed out by M. Hirasawa, a similar phenomenon need not occur in the nonorientable case. Indeed, there exist 2-bridge links with two continued fractions $-3\ 2\ -2\ 3$ and $2\ 3\ 2$, where the notation follows Adams [1].

REMARK 2.14. The converse of Theorem 2.8 does not hold in general. It is true that if a σ-state surface F_σ is π_1-essential, then the diagram D is σ-adequate. However, it is not true in general that if a σ-state surface F_σ is π_1-essential, then the diagram D is σ-homogeneous.

By using a π_1-essential state surface, we can prove the next theorem, which tells us that we can deduce the triviality and splittability of a knot or link from its diagram. In this paper, we say that a diagram D is *nontrivial* if it contains at least one crossing and that D is *nonsplit* if it is connected.

THEOREM 2.15. *Let K be a knot or link that admits a σ-adequate and σ-homogeneous diagram D without nugatory crossings for some state σ. Then the following hold.*

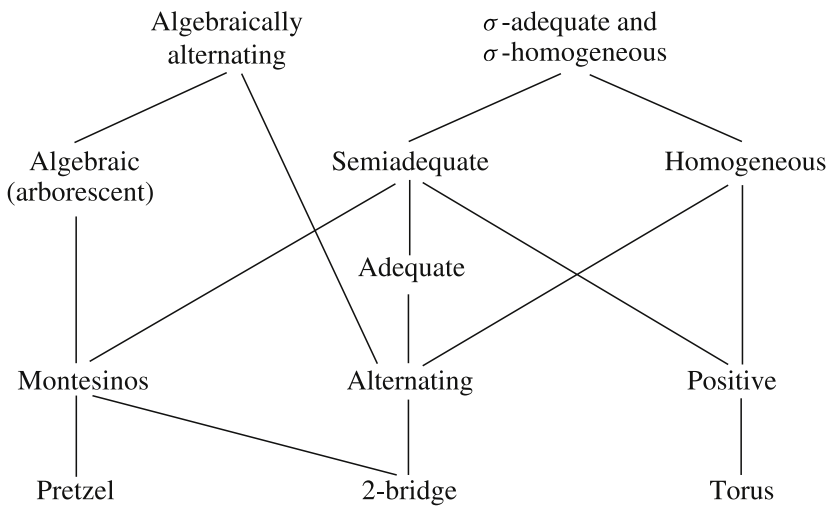

FIGURE 4. The Hasse diagram for the set of knot diagrams partially ordered by inclusion.

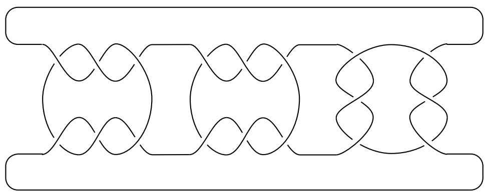

FIGURE 5. An algebraic link that is neither σ-adequate nor σ-homogeneous for any state σ.

(1) *D is nontrivial if and only if K is nontrivial.*
(2) *D is nonsplit if and only if K is nonsplit.*

The determining problem for the triviality and splittability of a knot or link has been solved for the following classes. For triviality, the determining problem has been solved for alternating knots [19], homogeneous links [4], semiadequate links [29] and Montesinos knots [16]. For splittability, the determining problem has been solved for alternating links [18], homogeneous links [4], semiadequate links [29] and positive links [22].

Figure 4 shows the Hasse diagram of the various classes of knots and links. Here, almost all algebraic links have σ-adequate and σ-homogeneous diagrams for some state σ (see Example 2.7), but some algebraic links seem to be neither σ-adequate nor σ-homogeneous for any state σ (see Figure 5). Algebraically alternating knots and links are defined in [24] so that these classes include both alternating and algebraic knots and links. Some results on closed, incompressible surfaces are obtained in [24].

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3. Lemmas

The next lemma is stated for knots, but it also holds for links K, provided that $E(K) - F$ is irreducible. Note that, for a connected diagram D and a state surface F_σ, $E(K) - F_\sigma$ will be a handlebody and hence irreducible.

Lemma 3.1 [23, Lemma 2]. *Let K be a knot in S^3 and let F be an incompressible, orientable surface properly embedded in $E(K)$. If F is ∂-compressible in $E(K)$, then F is a ∂-parallel annulus.*

Similarly, we have the following lemmas.

Lemma 3.2 [25, Lemma 2.2]. *Let K be a knot in S^3 and F be a π_1-injective non-orientable surface properly embedded in $E(K)$. If F is not ∂-π_1-injective, then F is an unknotted, half-twisted Möbius band and K is trivial.*

Lemma 3.3 [2, Theorem 9.8], [20, Proposition 2.3], [23, Theorem 2, 3]. *Let D be a reduced, prime, alternating diagram. Then the checkerboard surface obtained from D is π_1-essential.*

Let F be a spanning surface for a link K. Suppose that there exists a 2-sphere S that decomposes S^3 into two 3-balls B_1 and B_2 such that $F \cap S$ is a disk. Put $F_i = F \cap B_i$ for $i = 1, 2$. Then we say that F has a *Murasugi decomposition* into F_1 and F_2, which we denote by $F = F_1 * F_2$. Conversely, we say that F is obtained from F_1 and F_2 by a *Murasugi sum* along a disk $F \cap S$.

Put $E = S - \operatorname{int}(F \cap S)$ and let δ be a disk in B_1 such that

$$\delta \cap (F_1 \cup E) = \partial\delta \cap (F_1 \cup E) = \partial\delta$$

and $\partial\delta \cap E$ consists of mutually disjoint arcs $\alpha_1, \ldots, \alpha_n$. Then there exist mutually disjoint arcs $\alpha'_1, \ldots, \alpha'_n$ in $F \cap S$ that form the mutually disjoint loops given by $\alpha_1 \cup \alpha'_1, \ldots, \alpha_n \cup \alpha'_n$ in S. Moreover, there exist mutually disjoint disks $\delta'_1, \ldots, \delta'_n$ in B_2 that are bounded by $\alpha_1 \cup \alpha'_1, \ldots, \alpha_n \cup \alpha'_n$ respectively.

We call the disk $\delta \cup (\delta'_1 \cup \cdots \cup \delta'_n)$ the *extended disk* of δ towards B_2. We remark that the extended disk of δ is uniquely determined by δ and that in general it intersects the interior of F_2.

The following key lemma extends [11, Theorem 1] to nonorientable surfaces.

Lemma 3.4. *If F_1 and F_2 are π_1-essential, then $F = F_1 * F_2$ is also π_1-essential.*

Proof. Suppose that F_1 and F_2 are π_1-essential. We will show that the interpolating surface $(F)\tilde{} = F \tilde{\times} \partial I$ is essential. By [23, Claim 9], $(F)\tilde{}$, $(F_1)\tilde{}$ and $(F_2)\tilde{}$ are incompressible and ∂-incompressible in $F \tilde{\times} I$, $F_1 \tilde{\times} I$ and $F_2 \tilde{\times} I$ respectively.

Suppose that $(F)\tilde{}$ is compressible. Let C be a compressing disk for $(F)\tilde{}$ in the outside of $F \tilde{\times} I$. Put $E = S - \operatorname{int}(F \cap S)$. Without loss of generality, we may assume that C and E are in general position, and that the number of components of $C \cap E$ is minimal over all compressing disks C.

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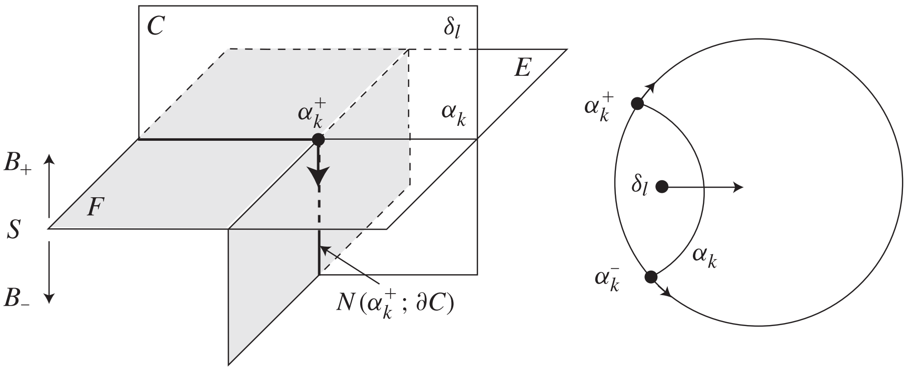

FIGURE 6. Marking $a_k^\pm$ with an arrow and an orientation induced by α_k.

If $C \cap E = \emptyset$, then C is a compressing disk for $(F_1)^{\sim}$ or $(F_2)^{\sim}$. Otherwise, $C \cap E$ consists of arcs, say $\alpha_1, \ldots, \alpha_p$. Let $\delta_1, \ldots, \delta_q$ be subdisks on C, separated by $\alpha_1 \cup \cdots \cup \alpha_p$. For each arc α_k, put $\partial\alpha_k = a_k^+ \cup a_k^-$. A subarc $N(a_k^\pm; \partial C)$ runs over the disk $F \cap S$ and $F - S$. Then we mark $a_k^\pm$ with an arrow so that it runs from $F \cap S$ to $F - S$ (see Figure 6).

Suppose now that F is not π_1-essential. We derive a contradiction by constructing a graph G that possesses an impossible property. The following claim is needed to establish some properties of $C \cap E$ that will be useful in our construction.

Claim. For an outermost arc α_k and the corresponding outermost disk δ_l, both arrows at $a_k^\pm$ turn out from δ_l (as in the right-hand side of Figure 6).

To prove the claim, we may assume without loss of generality that $\delta_l \subset B_1$. First, suppose that both arrows at $a_k^\pm$ turn into δ_l (see Figure 7). There exists an arc α'_k that connects a_k^+ and a_k^- on $F \cap S$. Also, the loop $\alpha_k \cup \alpha'_k$ bounds a disk δ'_l in B_2. We may now deduce that the extended disk $\delta_l \cup \delta'_l$ towards B_2 is a compressing disk for $(F_1)^{\sim}$, since we have assumed that the number of components of $C \cap E$ is minimal. This contradicts our assumption that F_1 is π_1-essential.

Next, suppose that one arrow at $a_k^\pm$ turns into δ_l and another turns out from δ_l (see Figure 8). Similarly, there exists an arc α'_k that connects a_k^+ and a_k^- on $F \cap S$, and the loop $\alpha_k \cup \alpha'_k$ bounds a disk δ'_l in B_2. Then the extended disk $\delta_l \cup \delta'_l$ towards B_2 is a ∂-compressing disk for $(F_1)^{\sim}$ since we have assumed that the number of components of $C \cap E$ is minimal. In either case, we have a contradiction and our claim is established.

We construct our graph G on C as follows. We assign a vertex v_l to each subdisk δ_l, and connect two vertices by an edge e_k if the two corresponding subdisks have a common arc α_k of $C \cap E$. Note that G is a tree, since any arc α_k separates δ. Since, by our first claim, both arrows at the boundary of an outermost arc turn out from the corresponding outermost disk, we can assign a natural orientation to the corresponding outermost edge. We call this orientation of the edge e_k the *orientation induced by* α_k (see Figure 6).

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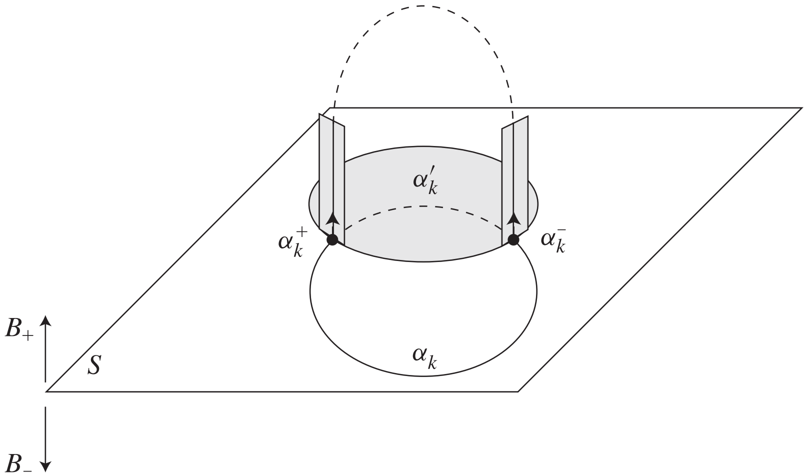

FIGURE 7. Both arrows at $a^\pm_k$ turn into δ_l.

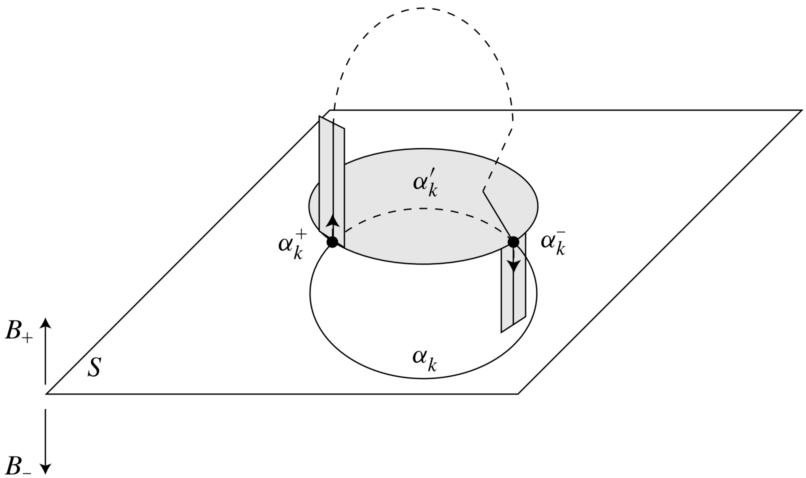

FIGURE 8. One arrow at a^+_k turns into δ_l and another arrow at a^-_k turns out from δ_l.

We say that a vertex of G has *depth* x if it becomes a vertex of degree one or zero after removing all vertices of depth less than x, where x is a natural number. We define vertices corresponding to the outermost subdisks of C as of depth 1. See Figure 9, where the depth of each vertex is indicated.

Our second claim will help us contradict our assumption that $(F)^{\tilde{}}$ is compressible.

Claim. Every edge of G has an induced orientation and every vertex has an edge oriented outwards.

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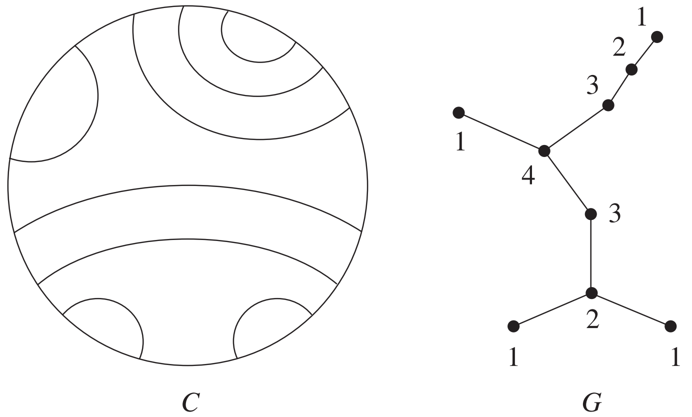

FIGURE 9. An example of $C \cap E$ on C and the corresponding graph G.

We prove this claim by induction on the depth of v_l. If v_l has depth 1, then this is nothing but our previous claim.

Next, suppose that this claim holds for all vertices of depth less than x, and that v_l has depth x. Let $N_{<x}(v_l)$ be the set of vertices that are adjacent to v_l and have depth less than x. Since G has no cycles, any vertex in $N_{<x}(v_l)$ has an edge oriented outwards to v_l. Without loss of generality, we may assume that $\delta_l \subset B_1$.

If v_l becomes a degree-zero vertex after all vertices of depth less than x are removed, then the extended disk δ'_l of δ_l towards B_2 is a compressing disk for $(F_1)^{\sim}$. If v_l becomes a degree-one vertex after all vertices of depth less than x are removed, then let e_k be the edge connecting v_l to a vertex that is not contained in $N_{<x}(v_l)$, and let α_k be the corresponding arc. First, suppose that both arrows at $a_k^{\pm}$ turn into δ_l. Then the extended disk δ'_l of δ_l towards B_2 is a compressing disk for $(F_1)^{\sim}$. Next, suppose that one of the arrows at $a_k^{\pm}$ turns into δ_l and another turns out from δ_l. Then the extended disk δ'_l of δ_l towards B_2 is a ∂-compressing disk for $(F_1)^{\sim}$. In either case, we have a contradiction.

Hence e_k has an orientation induced by α_k, and v_l has an edge oriented outwards. Our claim follows by induction.

This second claim leads us to a contradiction since G is a tree. Hence $(F)^{\sim}$ is incompressible. It follows, by an elementary cutting-and-pasting argument, that K is nonsplit. If $(F)^{\sim}$ is ∂-compressible, then it is ∂-parallel annulus by Lemma 3.1. Thus F is not a ∂-π_1-injective Möbius band. Hence one of F_1 and F_2 is also not a ∂-π_1-injective Möbius band. This contradicts our assumption that both of F_1 and F_2 are π_1-essential. Therefore F must be essential. □

4. Proofs

PROOF OF THEOREM 2.8. Suppose that a diagram D is σ-adequate and σ-homogeneous for some state σ. Then the σ-state graph G_σ may be decomposed into maximal blocks $G_1, \ldots, G_n$, each of which has no loop and has all of its edges of the same sign.

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Let $F_1, \dots, F_n$ be the σ-state surfaces corresponding to $G_1, \dots, G_n$. Then for each i, the boundary ∂F_i represents an alternating diagram that is reduced and prime since G_i has no loop and the block decomposition is maximal. By Lemma 3.3, F_i is π_1-essential for each i. It follows by Lemma 3.4 that F is also π_1-essential. □

PROOF OF THEOREM 2.15. Let K be a knot or link that admits a σ-adequate and σ-homogeneous diagram D without nugatory crossings. By Theorem 2.8, a σ-state surface F_σ is π_1-essential.

Suppose first that K is nontrivial. Then any diagram of K has at least one crossing. Hence D is nontrivial. Conversely, suppose that D is nontrivial. Since D has at least one crossing and does not have nugatory crossings, there exists a component of F_σ that is not a disk. This shows that K is nontrivial.

Suppose now that K is nonsplit. Then any diagram of K is connected. Hence D is nonsplit. Conversely, suppose that D is nonsplit. Since D is connected, F_σ is also connected. By a cutting-and-pasting argument on a splitting sphere, K is nonsplit.

5. Problems

Here, we list four problems that we would like to solve in the future.

(1) Show that there exists a knot that has no σ-adequate and σ-homogeneous diagram. Furthermore, characterize the nature of knots and links that have σ-adequate and σ-homogeneous diagrams.
(2) Determine primeness, satelliteness, fibredness, smallness and tangle decomposability from a given σ-adequate and σ-homogeneous diagram.
(3) Show that, for a given knot, the number of all σ-adequate and σ-homogeneous diagrams without nugatory crossings is finite.
(4) Classify knots and links that have σ-adequate and σ-homogeneous diagrams.

We believe that essential state surfaces will be useful for solving these problems.

6. Addendum

Since our initial version of this paper, written in 2006, some progress has been made on related matters. We summarize some of those results here.

In [10, Theorem 3], Futer *et al.* used our main Theorem 2.8 to verify the Garoufalidis conjecture on a relation between the boundary slopes of a knot and its colored Jones polynomials.

In [6], Curtis and Taylor used an unpublished result of Adams and Kindred, which is based on the work here, to show that, for an alternating knot, the minimal integral boundary slope is given by the signature plus twice the minimum degree of the Jones polynomial. They also showed that the maximal integral boundary slope is given by the signature plus twice the maximum degree of the Jones polynomial.

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MAKOTO OZAWA, Department of Natural Sciences,
Faculty of Arts and Sciences, Komazawa University,
1-23-1 Komazawa, Setagaya-ku, Tokyo, 154-8525, Japan
e-mail: w3c@komazawa-u.ac.jp

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