A SINGULAR BOUNDARY VALUE PROBLEM FOR A NON-SELF-ADJOINT DIFFERENTIAL OPERATOR

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If $(x^2 + 1)^{\frac{1}{2}}g(x) \in L^1(-\infty, \infty)$ the differential expression $l(y) = -y'' + g(x)y$ generates a closed operator L on $L^2(-\infty, \infty)$, with domain D consisting of those functions $y \in L^2$ with absolutely continuous derivatives and such that $l(y) \in L^2$. The case where $g(x)$ is real-valued has been extensively investigated and yields an expansion of any $f \in L^2$ in terms of the characteristic functions of L. We shall investigate the case where g is complex-valued.

We shall find that there is a function $W(s)$, analytic for $Im\, s > 0$ and continuous for $Im\, s \geqslant 0$, such that the squares of its zeros in $Im\, s > 0$ constitute a bounded set which is the point spectrum of L. The continuous spectrum of L is the set of $\lambda \geqslant 0$. In proving an expansion theorem real zeros of W cause difficulties and it is necessary to assume (Case II) that W has only a finite number of zeros in $Im\, s \geqslant 0$. The simplest form of the expansion is obtained if W has no real zeros except possibly at $s = 0$, and this must be a simple zero (Case I).

Naimark (**2**) has considered the same differential operator on $[0, \infty)$ with a boundary condition at 0 and obtains similar results. He uses a modification of a technique for singular self-adjoint problems (**1**, chap. 9), while we shall use a modification of the Cauchy Integral technique used for non-singular non-self-adjoint problems (**1**, chap. 12) and for general self-adjoint problems (**3**).

In § 1 we investigate the properties of certain solutions of $l(y) = \lambda y$ and introduce $W(s)$. We construct the Green's function and investigate the spectrum of L in § 2. An expansion of the Green's function for the general case is given in § 3, while in § 4 and § 5 we deal with Cases I and II respectively.

1. Solutions of $l(y) = \lambda y$. We shall set $\lambda = s^2$ and denote $Re\, s$ by σ and $Im\, s$ by τ. Also $\lambda^{\frac{1}{2}}$ will denote the root of λ with $0 \leqslant \arg\lambda^{\frac{1}{2}} < \pi$ and K will denote any constant whose value is unimportant.

It is easily seen by using variation of constants that a solution of $l(y) = s^2y$ will satisfy an integral equation of the form

Received December 17, 1956: in revised form March 31, 1958. This paper contains the results of the author's doctoral thesis at the Massachusetts Institute of Technology, with some corrections suggested by the referee. The author wishes to thank Professor Norman Levinson for his suggestion of the topic and his assistance thereafter. The final revision of this paper was carried out while the author was supported by a contract with the United States Air Force Office of Scientific Research.

$$(1.1)\qquad y(x) = c_1 e^{isx} + c_2 e^{-isx} + \frac{e^{isx}}{2is}\int_{x_1}^{x} e^{-is\xi} g(\xi)y(\xi)\,d\xi - \frac{e^{-isx}}{2is}\int_{x_2}^{x} e^{is\xi} g(\xi)y(\xi)\,d\xi,$$

and conversely. In estimating solutions of (1.1) we shall make frequent use of the following lemma, which we state without proof.

LEMMA 1.1. *If ϕ and ψ are piecewise continuous functions on $[a, b]$ and χ is integrable and non-negative on $[a, b]$ then*

$$\phi(x) \leqslant \psi(x) + \int_a^x \chi(\xi)\phi(\xi)\,d\xi, \qquad x \in [a, b],$$

implies

$$\phi(x) \leqslant \psi(x) + \int_a^x \chi(\xi)\psi(\xi) \exp\left[\int_\xi^x \chi(u)du\right] d\xi, \qquad x \in [a, b],$$

and

$$\phi(x) \leqslant \psi(x) + \int_x^b \chi(\xi)\phi(\xi)\,d\xi, \qquad x \in [a, b],$$

implies

$$\phi(x) \leqslant \psi(x) + \int_x^b \chi(\xi)\psi(\xi) \exp\left[\int_x^\xi \chi(u)du\right] d\xi, \qquad x \in [a, b].$$

LEMMA 1.2. *The solutions $y(x, s)$ of* (1.1) *with $c_1 = -c_2 = 1/2is$, $x_1 = x_2 = 0$, and $\tilde{y}(x, s)$ of* (1.1) *with $c_1 = c_2 = \frac{1}{2}$, $x_1 = x_2 = 0$ exist for all x and s, and for any fixed x are entire, and even functions of s.*

As this result follows from well-known theorems we omit the proof. We note that y and $\tilde{y}$ satisfy the initial conditions

$$(1.2)\qquad \begin{array}{ll} y(0, s) = 0 & y'(0, s) = 1 \\ \tilde{y}(0, s) = 1 & \tilde{y}'(0, s) = 0 \end{array}$$

and that a modification of Lemma 1.1 yields the following estimates:

$$(1.3)\qquad |y(x, s)| \leqslant \frac{K|x|e^{|\tau x|}}{1 + |sx|}, \quad |\tilde{y}(x, s)| \leqslant \left(1 + \frac{K|x|}{1 + |sx|}\right)e^{|\tau x|}.$$

LEMMA 1.3. *The solutions $y_1(x, s)$ of* (1.1) *with $c_1 = 1$, $c_2 = 0$, $x_1 = x_2 = \infty$, and $y_2(x, s)$ of* (1.1) *with $c_1 = 0$, $c_2 = 1$, $x_1 = x_2 = -\infty$ exist for all x and for $\tau \geqslant 0$. For any fixed x they are continuous in s for $\tau \geqslant 0$ and analytic in s for $\tau > 0$.*

Proof. We shall prove the result for $y_1(x, s)$ only as the proof for $y_2(x, s)$ is similar. Setting $\phi_0(x, s) = 0$ and

$$\phi_{n+1}(x, s) = e^{isx} - \int_x^\infty \frac{\sin s(x - \xi)}{s} g(\xi)\phi_n(\xi, s)\,d\xi$$

and using the inequality

$$\frac{|\sin sx|}{|s|} \leqslant K|x|e^{|\tau x|}(1 + |sx|)^{-1}$$

we see that the successive approximations exist for all x and for $\tau \geqslant 0$. For fixed x, $\phi_n(x, s)$ is continuous in s for $\tau \geqslant 0$ and analytic in s for $\tau > 0$. An induction yields

$$|\phi_{n+1}(x, s) - \phi_n(x, s)| \leqslant \left[K \int_x^\infty \xi |g(\xi)| d\xi\right]^n e^{-\tau x}/n!$$

for $x \geqslant 0$. This implies the uniform convergence of the successive approximations for $x \geqslant 0$, $\tau \geqslant 0$. An application of Lemma 1.1 proves the uniqueness and when we define

$$y_1(x, s) = y_1'(0, s) y(x, s) + y_1(0, s) \tilde{y}(x, s)$$

for $x < 0$ the regularity follows from the uniform convergence for $x \geqslant 0$ and from the known properties of $y(x, s)$ and $\tilde{y}(x, s)$. The fact that the integral equation is also satisfied for $x < 0$ follows from a few manipulations with the definition of $y_1(x, s)$ for $x < 0$.

Applying Lemma 1.1 yields estimates on $y_1(x, s)$ and $y_2(x, s)$ which allow us to draw certain conclusions about the asymptotic behaviour of these two solutions:

$$(1.4) \qquad |y_1(x, s)| \leqslant \exp\left[-\tau x + K \int_x^\infty \xi |g(\xi)| d\xi\right], \qquad x \geqslant 0,$$

$$(1.5) \qquad |y_1(x, s)| \leqslant \exp\left[-\tau x + \frac{1}{|s|} \int_x^\infty |g(\xi)| d\xi\right], \qquad s \neq 0,$$

$$(1.6) \qquad |y_2(x, s)| \leqslant \exp\left[\tau x + K \int_{-\infty}^x |\xi g(\xi)| d\xi\right], \qquad x \leqslant 0,$$

$$(1.7) \qquad |y_2(x, s)| \leqslant \exp\left[\tau x + \frac{1}{|s|} \int_{-\infty}^x |g(\xi)| d\xi\right], \qquad s \neq 0.$$

LEMMA 1.4. *The solutions $y_1(x, s)$ and $y_2(x, s)$ have the following asymptotic behaviour:*

$$(1.8) \quad y_1(x, s) = e^{isx}(1 + o(1)), \quad y_1'(x, s) = e^{isx}(is + o(1)) \qquad \text{as } x \to \infty.$$

$$(1.9) \quad y_1(x, s) = e^{isx}\left(1 + O\left(\frac{1}{s}\right)\right), \quad y_1'(x, s) = ise^{isx}\left(1 + O\left(\frac{1}{s}\right)\right) \quad \text{as } |s| \to \infty.$$

$$(1.10) \quad y_2(x, s) = e^{-isx}(1 + o(1)), \quad y_2'(x, s) = e^{-isx}(-is + o(1)) \quad \text{as } x \to -\infty.$$

$$(1.11) \quad y_2(x, s) = e^{-isx}\left(1 + O\left(\frac{1}{s}\right)\right), \quad y_2'(x, s) = -ise^{-isx}\left(1 + O\left(\frac{1}{s}\right)\right) \quad \text{as } |s| \to \infty.$$

Formulas (1.8) *and* (1.10) *hold uniformly in s for $\tau \geqslant 0$ and* (1.9) *and* (1.11) *hold uniformly in x, for all x.*

Proof. For (1.8) and (1.10) we use (1.4) and (1.6) respectively in the appropriate form of (1.1) and its derivative. For (1.9) and (1.11) we use (1.5) and (1.7) respectively in the same way.

https://doi.org/10.4153/CJM-1958-043-1 Published online by Cambridge University Press

LEMMA 1.5. *The Wronskian* $W(s) = y_1y_2' - y_1'y_2$ *is not identically zero, and can be calculated from the formulas*

$$W(s) = -2is + \int_{-\infty}^{\infty} e^{-isx}g(x)y_1(x, s)\,dx$$
$$= -2is + \int_{-\infty}^{\infty} e^{isx}g(x)y_2(x, s)\,dx.$$

Proof. It follows immediately from the regularity properties of y_1 and y_2 that $W(s)$, which is independent of x, is continuous in s for $\tau \geqslant 0$ and analytic in s for $\tau > 0$. Direct computation with the integral equations defining y_1 and y_2 yields

$$W(s) = -2is + \int_{-\infty}^{x} e^{is\xi}g(\xi)y_2(\xi, s)\,d\xi + \int_{x}^{\infty} e^{-is\xi}g(\xi)y_1(\xi, s)\,d\xi + R \tag{1.12}$$

where

$$|R| = \left| -\int_{x}^{\infty}\frac{\sin s(x-\xi)}{s}\,g(\xi)y_1(\xi, s)\,d\xi\cdot\int_{-\infty}^{x}\cos s(x-\xi)g(\xi)y_2(\xi, s)\,d\xi + \int_{x}^{\infty}\cos s(x-\xi)g(\xi)y_1(\xi, s)\,d\xi\cdot\int_{-\infty}^{x}\frac{\sin s(x-\xi)}{s}\,g(\xi)y_2(\xi, s)\,d\xi\right|$$
$$\leqslant \frac{2}{|s|}\left[\int_{x}^{\infty}|g(\xi)|\exp\left(\frac{1}{|s|}\int_{\xi}^{\infty}|g(u)|du\right)d\xi\right]\cdot\left[\int_{-\infty}^{x}|g(\xi)|\exp\left(\frac{1}{|s|}\int_{-\infty}^{\xi}|g(u)|du\right)d\xi\right]$$
$$= 2|s|\left[\exp\left(\frac{1}{|s|}\int_{x}^{\infty}|g(u)|du\right)-1\right]\cdot\left[\exp\left(\frac{1}{|s|}\int_{-\infty}^{x}|g(u)|du\right)-1\right].$$

Here (1.5) and (1.7) have been used and it now follows that for $s \neq 0$

$$\lim_{x\to\infty} R = \lim_{x\to-\infty} R = 0.$$

Using (1.4) and (1.6) we see that $|y_j(0, s)| \leqslant K$ and $|y_j'(0, s)| \leqslant K + |s|$ for $j = 1, 2$ and thus using (1.3) we obtain the further estimates

$$|e^{-isx}y_1(x, s)| \leqslant K(1 + |x|), \quad |e^{isx}y_2(x, s)| \leqslant K(1 + |x|)$$

for all x and $\tau \geqslant 0$. Thus

$$\int_{-\infty}^{\infty} e^{-isx}g(x)y_1(x, s)\,dx$$

and

$$\int_{-\infty}^{\infty} e^{+isx}g(x)y_2(x, s)\,dx$$

converge uniformly in s for $\tau \geqslant 0$ and thus are continuous functions of s. Using this fact and the results about R obtained above it follows that we may take the limit of (1.12) as $x \to \infty$ or as $x \to -\infty$ and obtain the desired formulas for $s \neq 0$. The result follows for $s = 0$ by continuity.

To see that $W(s)$ is not identically zero we note that

$$\begin{aligned}|W(s)| &= \left|-2is + \int_{-\infty}^{\infty} e^{-isx} g(x) y_1(x,s)\,dx\right| \\ &\geqslant 2|s| - \int_{-\infty}^{\infty} |g(x)| \exp\left(\frac{1}{|s|}\int_x^{\infty} |g(u)|\,du\right) dx \\ &= |s|\left[3 - \exp\left(\frac{1}{|s|}\int_{-\infty}^{\infty} |g(u)|\,du\right)\right].\end{aligned}$$

Thus for large $|s|$, $W(s)$ cannot be zero.

Corollary 1.1. *The zeros of $W(s)$ form an at most countable, bounded subset of $\tau > 0$, with limit points only on the real axis $\tau = 0$.*

This follows immediately from the analyticity of $W(s)$ in $\tau > 0$ and the fact that $W(s) = 0$ implies

$$|s| < \int_{-\infty}^{\infty} |g(x)|\,dx.$$

We now complete our discussion of the asymptotic behaviour of $y_1(x, s)$ and $y_2(x, s)$.

Lemma 1.6.

$$y_1(x,s) = e^{isx}\left(-\frac{W(s)}{2is} + o(1)\right) \qquad \text{as } x \to -\infty \tag{1.13}$$

$$y_2(x,s) = e^{-isx}\left(-\frac{W(s)}{2is} + o(1)\right) \qquad \text{as } x \to +\infty \tag{1.14}$$

uniformly in s for $\tau \geqslant \delta > 0$.

Proof. We shall prove only (1.14) as (1.13) is similar.

$$\begin{aligned}y_2(x,s) &= e^{-isx} + \int_{-\infty}^{x} \frac{\sin s(x-\xi)}{s} g(\xi) y_2(\xi,s)\,d\xi \\ &= e^{-isx}\left[1 + \int_{-\infty}^{x} e^{+isx}\frac{\sin s(x-\xi)}{s} g(\xi) y_2(\xi,s)\,d\xi\right] \\ &= e^{-isx}\left[1 - \int_{-\infty}^{x} \frac{e^{is\xi}}{2is} g(\xi) y_2(\xi,s)\,d\xi \right. \\ &\qquad \left. + \int_{-\infty}^{x} \frac{e^{2is(x-\xi)}}{2is} g(\xi) e^{is\xi} y_2(\xi,s)\,d\xi\right] \\ &= e^{-isx}\left[-\frac{W(s)}{2is} + \int_{x}^{\infty} \frac{e^{is\xi}}{2is} g(\xi) y_2(\xi,s)\,d\xi \right. \\ &\qquad \left. + \int_{-\infty}^{x} \frac{e^{2is(x-\xi)}}{2is} g(\xi) e^{is\xi} y_2(\xi,s)\,d\xi\right].\end{aligned}$$

Now, the integral over the range from x to ∞ approaches zero as $x \to \infty$ uniformly in s for $\tau \geqslant \delta > 0$ from the proof of Lemma 1.5, and for $x \geqslant 0$

$$\left| \int_{-\infty}^{x} \frac{e^{2is(x-\xi)}}{2is} g(\xi) e^{is\xi} y_2(\xi, s)\, d\xi \right|$$
$$\leqslant \frac{1}{2\delta} \int_{-\infty}^{x} e^{-2\delta(x-\xi)} |g(\xi)| \exp\left[\frac{1}{\delta} \int_{-\infty}^{\xi} |g(u)|\, du \right] d\xi$$
$$\leqslant K \int_{-\infty}^{x/2} e^{-2\delta(x-\xi)} |g(\xi)|\, d\xi + K \int_{x/2}^{x} e^{-2\delta(x-\xi)} |g(\xi)|\, d\xi$$
$$\leqslant K e^{-\delta x} \int_{-\infty}^{x/2} |g(\xi)|\, d\xi + K \int_{x/2}^{x} |g(\xi)|\, d\xi,$$

which certainly approaches 0 as $x \to \infty$.

2. The Green's Function. If $W(s) \neq 0$ we may define

$$K(x, \xi, s) = \frac{1}{W(s)} \begin{cases} y_1(x, s) y_2(\xi, s) & \xi \leqslant x \\ y_1(\xi, s) y_2(x, s) & x \leqslant \xi. \end{cases}$$

THEOREM 2.1. *If λ is not real and non-negative, and $W(\lambda^{\frac{1}{2}}) \neq 0$ then the Green's function $G(x, \xi, \lambda)$ for $l(y) - \lambda y = f$ on the interval $-\infty < x < \infty$ is $K(x, \xi, \lambda^{\frac{1}{2}})$, that is, if $f \in L^2(-\infty, \infty)$ then $y = \int_{-\infty}^{\infty} K(x, \xi, \lambda^{\frac{1}{2}}) f(\xi) d\xi$ belongs to D and $l(y) - \lambda y = f$ almost everywhere.*

Proof. Let $\lambda^{\frac{1}{2}} = s = \sigma + i\tau$ as usual. By assumption $\tau > 0$ and $W(s) \neq 0$ so $y_1(x, s)$ and $y_2(x, s)$ are linearly independent solutions of $l(y) = \lambda y$. It follows from variation of constants that the general solution of $l(y) - \lambda y = f$ is

$$y(x) = c_1 y_1(x, s) + c_2 y_2(x, s)$$
$$+ \frac{y_1(x, s)}{W(s)} \int_{-\infty}^{x} f(\xi) y_2(\xi, s)\, d\xi + \frac{y_2(x, s)}{W(s)} \int_{x}^{\infty} f(\xi) y_1(\xi, s)\, d\xi$$
$$= c_1 y_1(x, s) + c_2 y_2(x, s) + \int_{-\infty}^{\infty} K(x, \xi, s) f(\xi)\, d\xi,$$

where the existence of the integrals is trivial. As $|K(x, \xi, s)| \leqslant K \exp[-\tau|x-\xi|]$, $y(x) = \int_{-\infty}^{\infty} K(x, \xi, s) f(\xi) d\xi$ is bounded by the convolution of a function in L^1 and a function in L^2. Thus $y \in L^2(-\infty, \infty)$. As $W(s) \neq 0$ it follows from (1.14) and (1.15) that

$$c_1 y_1(x, s) + c_2 y_2(x, s) \notin L^2(-\infty, \infty)$$

unless $c_1 = c_2 = 0$. So y is the unique L^2 solution of $l(y) - \lambda y = f$ and $y \in D$ follows easily from direct considerations.

COROLLARY 2.1. *L is a closed operator.*

Proof. If $y_n \in D$, $y_n \to y$, and $Ly_n \to f$ both in L^2, then we must show that $y \in D$ and $Ly = f$. As $W(s) \not\equiv 0$ there is an $s_0 = \sigma_0 + i\tau_0$ with $\tau_0 > 0$ and $W(s_0) \neq 0$, and we have

$$y_n(x) = \int_{-\infty}^{\infty} K(x, \xi, s_0)[Ly_n - s_0^2 y_n]\, d\xi.$$

Thus

$$y(x) = \int_{-\infty}^{\infty} K(x, \xi, s_0)[f - s_0^2 y]\, d\xi$$

almost everywhere.

As the convergence is in L^2 we may replace the limit y by an equivalent member of L^2 so that the above relation is an equality and we have $y \in D$ and $Ly = f$.

LEMMA 2.1. *The adjoint L^* of L is the operator with domain D defined for $y \in D$ by*

$$L^*y = -y'' + \overline{g(x)}y = l^*(y).$$

Proof. We first note that if $y \in D$ then y and y' both tend to 0 as x approaches either $+\infty$ or $-\infty$. This follows from

$$y(x) = \int_{-\infty}^{\infty} K(x, \xi, s_0)[Ly - s_0^2 y]\, d\xi \qquad \text{for} \quad y \in D.$$

Thus if $z \in L^2$ and there exists $z^* \in L^2$ such that $(Ly, z) = (y, z^*)$ for all $y \in D$ then

$$(y, z^*) - (s_0^2 y, z) = (y, z^* - \overline{s_0^2} z)$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} K(x, \xi, s_0)(Ly(\xi) - s_0^2 y(\xi))\, d\xi\right](\overline{z^*(x)} - s_0^2\overline{z(x)})\, dx$$

$$= \int_{-\infty}^{\infty} (Ly(\xi) - s_0^2 y(\xi))\overline{\left[\int_{-\infty}^{\infty} \overline{K(x, \xi, s_0)}(z^*(x) - \overline{s_0^2} z(x))dx\right]}\, d\xi$$

$$= (Ly - s_0^2 y, z_1).$$

However $(Ly - s_0^2 y, z) = (y, z^*) - (s_0^2 y, z) = (Ly - s_0^2 y, z_1)$ so $z = z_1$ almost everywhere. Thus the domain D^* of L^* consists of functions in L^2 of the form

$$z_1(x) = \int_{-\infty}^{\infty} \overline{K(\xi, x, s_0)}(z^*(\xi) - \overline{s_0^2} z_1(\xi))\, d\xi$$

and $L^* z_1 = z^*$. This implies that $z_1 \in D$, and an easy calculation shows that $-z_1'' + \overline{g(x)} z_1 = z^*$, which completes the proof.

THEOREM 2.2. *The spectrum of L consists of an at most countable, bounded set of characteristic values and a continuous spectrum on the non-negative real axis $\lambda \geqslant 0$.*

Proof. If λ is not real and non-negative we have seen that λ is in the resolvent set of L unless $W(\lambda^{\frac{1}{2}}) = 0$. If $W(\lambda^{\frac{1}{2}}) = 0$, λ is obviously a characteristic value with characteristic function $y_1(x, \lambda^{\frac{1}{2}})$. Corollary 1.1 immediately yields the statement about the point spectrum except for $\lambda = 0$.

If $y(x)$ is a characteristic function for $\lambda = 0$ then by using a representation in terms of the Green's function it is easy to see that y is bounded and approaches 0 at $\pm\infty$. Thus

$$y(x) = -\int_x^\infty (x-\xi)g(\xi)y(\xi)\,d\xi$$

and using Lemma 1.1 we see that $y = 0$ and thus $\lambda = 0$ cannot be a characteristic value.

We see that $(L - \sigma^2)^{-1}$ is not bounded for $\sigma > 0$ by attempting to construct $y_0 \in L^2(-\infty, \infty)$ such that

$$l(y_0) - \sigma^2 y_0 = \begin{cases} \overline{y(x, \sigma)} & |x| \leqslant a \\ 0 & |x| > a. \end{cases}$$

Thus we see that the positive real axis is in the spectrum, and as the spectrum is closed zero belongs to the spectrum.

To see that the residual spectrum is empty we note that it must lie in the non-negative real axis and if σ^2 is in the residual spectrum it is in the point spectrum of L^*. This would mean that $l^*(y) = \sigma^2 y$ has a solution belonging to L^2. Taking the conjugate of this solution we see that σ^2 lies in the point spectrum of L, which contradicts the assumption.

3. An Expansion of the Green's Function. We shall first use the Cauchy Integral to obtain an expansion of the Green's function, and then use this to obtain our expansion theorem.

Let $C_{R,\delta}$ denote the contour in the s-plane consisting of the straight line $\tau = \delta > 0$ from $\sigma = -(R^2 - \delta^2)^{\frac{1}{2}}$ to $\sigma = (R^2 - \delta^2)^{\frac{1}{2}}$, and the circular arc $s = Re^{i\theta}$ from $\theta = \eta = \sin^{-1}\delta/R$ to $\theta = \pi - \eta$. We choose δ and R_0 so that if λ_0 is a characteristic value $Im\, \lambda_0^{\frac{1}{2}} \neq \delta$ and $R_0^2 > |\lambda_0|$; and consider

$$I_{R,\delta} = \oint_{C_{R,\delta}} \frac{K(x,\xi,s)}{s^2 - \lambda}\, s\, ds \tag{3.1}$$

for $R \geqslant R_0$, and $\lambda^{\frac{1}{2}}$ within the contour.

In evaluating (3.1) by residues we see that the singularities of the integrand occur at $s = \lambda^{\frac{1}{2}}$, and at the square roots of the characteristic values. If λ_1, $\lambda_2, \ldots$ are the characteristic values arranged in order so that $Im\, \lambda_1^{\frac{1}{2}} \geqslant Im\, \lambda_2^{\frac{1}{2}} \geqslant \ldots$; we see that for any δ and $R \geqslant R_0$ there is an integer $n(\delta)$ such that $Im\, (\lambda_{n(\delta)})^{\frac{1}{2}} > \delta > Im\, (\lambda_{n(\delta)+1})^{\frac{1}{2}}$, and the value of $I_{R,\delta}$ is thus independent of R for $R \geqslant R_0$. As $K(x, \xi, s)$ is the ratio of two functions, each of which is analytic for $\tau > 0$ we see that the singularities at

$$s^2 = \lambda_1, \lambda_2, \ldots$$

must be poles. If we have

$$K(x,\xi,s) = \sum_{p=1}^{m_i} G_p^{(i)}(x,\xi)(s^2 - \lambda_i)^{-p} + F(x,\xi,s)$$

for s^2 sufficiently close to λ_i where F is analytic in s at $s = \lambda_i^{\frac{1}{2}}$, it is easily seen that the residue of the integrand in (3.1) at $s = \lambda_i^{\frac{1}{2}}$ is

$$-\tfrac{1}{2} \sum_{p=1}^{m_i} G_p^{(i)}(x,\xi)(\lambda - \lambda_i)^{-p}.$$

Since the residue at $s = \lambda^{\frac{1}{2}}$ is $\frac{1}{2}G(x, \xi, \lambda)$ we have

$$(3.2) \qquad I_{R,\delta} = \pi i G(x, \xi, \lambda) - \pi i \sum_{i=1}^{n(\delta)} \sum_{p=1}^{m_i} G_p^{(i)}(x, \xi)(\lambda - \lambda_i)^{-p}.$$

Now if we evaluate $I_{R,\delta}$ directly we have

$$I_{R,\delta} = \int_\eta^{\pi-\eta} \frac{K(x, \xi, Re^{i\theta}) iR^2 e^{2i\theta}}{R^2 e^{2i\theta} - \lambda} d\theta + \int_{-(R^2-\delta^2)^{\frac{1}{2}}}^{(R^2-\delta^2)^{\frac{1}{2}}} \frac{(\sigma + i\delta) K(x, \xi, (\sigma + i\delta))}{(\sigma + i\delta)^2 - \lambda} d\sigma.$$

Since $|K(x, \xi, s)| \leqslant K|W(s)|^{-1} \exp[-\tau|\xi - x|]$, and, for $|s|$ sufficiently large, $|W(s)| > |s|$; we see that

$$\left| \int_\eta^{\pi-\eta} \frac{K(x, \xi, Re^{i\theta}) iR^2 e^{2i\theta}}{R^2 e^{2i\theta} - \lambda} d\theta \right| \leqslant \frac{K}{R}$$

for R sufficiently large, and

$$\left| \frac{(\sigma + i\delta) K(x, \xi, (\sigma + i\delta))}{(\sigma + i\delta)^2 - \lambda} \right| \leqslant \frac{K}{\sigma^2 + 1}.$$

Thus we have

$$\lim_{R \to \infty} I_{R,\delta} = \int_{-\infty}^{\infty} \frac{(\sigma + i\delta) K(x, \xi, (\sigma + i\delta))}{(\sigma + i\delta)^2 - \lambda} d\sigma,$$

where the integral converges absolutely and uniformly for $Im\, \lambda^{\frac{1}{2}} \geqslant \delta_1 > \delta$ and any x, ξ. Combining this result with (3.2) and the remark that $I_{R,\delta}$ is independent of R for $R \geqslant R_0$ we have the following theorem.

THEOREM 3.1. *With the notation introduced above, and under the restrictions on δ introduced above, we have*

$$(3.3) \qquad G(x, \xi, \lambda) = \sum_{i=1}^{n(\delta)} \sum_{p=1}^{m_i} G_p^{(i)}(x, \xi)(\lambda - \lambda_i)^{-p} + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{(\sigma + i\delta) K(x, \xi, (\sigma + i\delta))}{(\sigma + i\delta)^2 - \lambda} d\sigma.$$

It will be convenient to write this in a different form, which will exhibit the symmetry between L and L^*.

LEMMA 3.1. *There are sets of functions $\chi_j^{(i)}(x)$, $\psi_j^{(i)}(x)$ for $j = 0, 1, \ldots,$ $m_i - 1$ such that*

$$(3.4) \qquad l(\chi_j^{(i)}) - \lambda_i \chi_j^{(i)} = \chi_{j-1}^{(i)}, \quad l^*(\psi_j^{(i)}) - \bar{\lambda}_i \psi_j^{(i)} = \psi_{j-1}^{(i)}$$

for $j = 0, 1, \ldots, m_i - 1$, where $\chi_{-1}^{(i)} \equiv \psi_{-1}^{(i)} \equiv 0$, and

$$(3.5) \qquad G_p^{(i)}(x, \xi) = -\sum_{j=0}^{m_i-p} \chi_j^{(i)}(x) \overline{\psi_{m_i-p-j}^{(i)}(\xi)}.$$

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Also, each $\chi_j^{(i)}$ *and* $\psi_j^{(i)}$ *is bounded by* $K \exp[-\tau_i|x|/2]$ where $\tau_i = Im\,\lambda_i^{\frac{1}{2}}$, *and*

$$(3.6) \qquad (\chi_j^{(i)}, \psi_r^{(k)}) = \int_{-\infty}^{\infty} \chi_j^{(i)}(x)\, \overline{\psi_r^{(k)}(x)}\, dx = \delta_{ik}\delta_{m_i-j-1\, r}.$$

Proof. Let $\Gamma(x, \xi, \lambda)$ denote the function which is given by $\tilde{y}(x, \lambda^{\frac{1}{2}})y(\xi, \lambda^{\frac{1}{2}})$ for $\xi \leqslant x$ and by $\tilde{y}(\xi, \lambda^{\frac{1}{2}})y(x, \lambda^{\frac{1}{2}})$ for $x \leqslant \xi$. This is trivially an entire function of λ and $G(x, \xi, \lambda) - \Gamma(x, \xi, \lambda)$ is of class C^2 as a function of x or ξ. Thus if C_i is a circle with centre at λ_i, enclosing no other points of the spectrum of L we see that

$$G_p^{(i)}(x, \xi) = \frac{1}{2\pi i}\oint_{C_i} (\lambda - \lambda_i)^{p-1}[G(x, \xi, \lambda) - \Gamma(x, \xi, \lambda)]\, d\lambda.$$

From this we see that $G_p^{(i)}(x, \xi)$ is of class C^2 as a function of x or ξ, and from this it is easily seen that as a function of x, $l(G_p^{(i)}) - \lambda_i G_p^{(i)} = G_{p+1}^{(i)}$, and as a function of ξ

$$l^*(\overline{G_p^{(i)}}) - \overline{\lambda_i}\, \overline{G_p^{(i)}} = \overline{G_{p+1}^{(i)}},$$

where

$$G_{m_i+1}^{(i)} \equiv 0.$$

From these equations it follows that $G_p^{(i)}$ can be given in the form (3.5) by functions $\chi_j^{(i)}$ and $\psi_j^{(i)}$ satisfying (3.4).

Now we also have

$$G_p^{(i)}(x, \xi) = \frac{1}{2\pi i}\oint_{C_i} (\lambda - \lambda_i)^{p-1} G(x, \xi, \lambda)\, d\lambda,$$

from which we see, by taking the radius of C_i sufficiently small, that $|G_p^{(i)}(x, \xi)| \leqslant K \exp[-\tau_i|x - \xi|/2]$. Using this and an induction we find that $\chi_j^{(i)}$ and $\psi_j^{(i)}$ are both bounded by $K \exp[-\tau_i|x|/2]$. In order to prove (3.6) we note that

$$(L\chi_j^{(i)}, \psi_r^{(k)}) = \lambda_i(\chi_j^{(i)}, \psi_r^{(k)}) + (\chi_{j-1}^{(i)}, \psi_r^{(k)})$$

and

$$(\chi_j^{(i)}, L^*\psi_r^{(k)}) = \lambda_r(\chi_j^{(i)}, \psi_r^{(k)}) + (\chi_j^{(i)}, \psi_{r-1}^{(k)}),$$

so that

$$(\lambda_i - \lambda_r)(\chi_j^{(i)}, \psi_r^{(k)}) = (\chi_j^{(i)}, \psi_{r-1}^{(k)}) - (\chi_{j-1}^{(i)}, \psi_r^{(k)}).$$

From the fact that $\chi_{-1}^{(i)} \equiv \psi_{-1}^{(k)} \equiv 0$ it follows that $(\chi_0^{(i)}, \psi_0^{(k)}) = 0$ for $k \neq i$ so an easy induction yields $(\chi_j^{(i)}, \psi_r^{(k)}) = 0$ if $k \neq i$. To deal with the case $k = i$ note that

$$(l - \lambda)\left(-\sum_{k=0}^{j} \chi_k^{(i)}(x)(\lambda - \lambda_i)^{k-j-1}\right) = \chi_j^{(i)}(x);$$

and thus

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$$-\sum_{k=0}^{j} \chi_k^{(i)}(x)(\lambda-\lambda_i)^{k-j-1} = \int_{-\infty}^{\infty} G(x,\xi,\lambda)\chi_j^{(i)}(\xi)\,d\xi,$$

as the right hand side is the unique L^2 solution of $l(y) - \lambda y = \chi_j^{(i)}$ and the left hand side is such a solution. Thus

$$-\sum_{k=0}^{j} \chi_k^{(i)}(x)(\lambda-\lambda_i)^{k-j-1} = \sum_{p=1}^{m_i} (\lambda-\lambda_i)^{-p} \int_{-\infty}^{\infty} G_p^{(i)}(x,\xi)\chi_j^{(i)}(\xi)\,d\xi + F(x,\lambda)$$

where $F(x, \lambda)$ is analytic at $\lambda = \lambda_i$. From this we have

$$\sum_{k=0}^{m_i-p} \chi_k^{(i)}(x)(\chi_j^{(i)}, \psi_{m_i-p-k}^{(i)}) = -\int_{-\infty}^{\infty} G_p^{(i)}(x,\xi)\chi_j^{(i)}(\xi)\,d\xi = \begin{cases} 0 & p > j+1 \\ \chi_{j+1-p}^{(i)}(x) & p \leqslant j+1. \end{cases}$$

As

$$\chi_0^{(i)}, \ldots, \chi_{m_i-1}^{(i)}$$

are easily seen to be linearly independent we have

$$(\chi_j^{(i)}, \psi_{m_i-p-k}^{(i)}) = \begin{cases} 0 & p > j+1 & \\ 0 & p \leqslant j+1 & k \neq j+1-p \\ 1 & p \leqslant j+1 & k = j+1-p. \end{cases}$$

Combining this with the result for functions corresponding to different characteristic values we have (3.6).

We might remark that it can be shown that

$$\chi_j^{(i)}(x) = \sum_{k=0}^{j} a_{j-k}^{(i)} \frac{1}{k!} \frac{d^k}{d\lambda^k} y_1(x, \lambda^{\frac{1}{2}})_{\lambda=\lambda_i}$$

for suitable constants $a_k^{(i)}$, and a similar result for $\psi_j^{(i)}$.

4. The Expansion in Case I. In order to obtain an expansion analogous in form to that which holds when L is self-adjoint, we must modify the integral in (3.3) so that it involves only solutions of $l(y) = \lambda y$ for $\lambda \geqslant 0$. The obvious way to do this is to evaluate the limit as $\delta \to 0$, but this may lead to two difficulties:

(i) $n(\delta)$ may become infinite and the discrete portion of the expansion may diverge.

(ii) The integral in (3.3) may not exist for $\delta = 0$ if W has real zeros.

Although the convergence difficulties do not arise, $n(\delta)$ may become infinite even if L is self-adjoint. We shall construct two examples, both with $g(x) = 0$ for $|x| \geqslant b$, to show that W can have real zeros of sufficiently high order that the integral in (3.3) will not exist even as a principal value for $\delta = 0$.

As $g(x) = 0$ for $|x| \geqslant b$, $W(s) = -e^{isb}[y_1'(-b, s) + isy_1(-b, s)]$ and if W is to have a zero of order m at $s = s_0$ we find that we must have

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$$y_1^{(k)\prime}(-b, s_0) + is_0 y_1^{(k)}(-b, s_0) + iy_1^{(k-1)}(-b, s_0) = 0, \qquad k = 0, 1, \ldots, m-1$$

where

$$y_1(x, s) = \sum_{n=0}^{\infty} y_1^{(n)}(x, s_0)(s - s_0)^n \quad \text{and} \quad y_1^{(-1)}(x, s_0) \equiv 0.$$

Example I. Third order zero at $s = 0$. We set $y_1(x, 0) = e^{i\theta(x)}$ so that $g(x) = i\theta''(x) - [\theta'(x)]^2$ for $|x| \leqslant b$ and require that $\theta \in C^\infty$, $\theta(-b) = -\frac{1}{2}\pi$, $\theta(b) = 2\pi$, $\theta^{(n)}(-b) = \theta^{(n)}(b) = 0$ for $n > 0$, and $\int_{-b}^{b} \sin 2\theta(x)dx = 0$.

Example II. Second order zero at $s = 1$. We set $y_1(x, 1) = e^{ib}f(x)$ so that $g(x) = 1 + f''(x)/f(x)$ for $|x| \leqslant b$ and require that $f(x) \neq 0$ for $|x| \leqslant b$, $f^{(n)}(b) = i^n$, $f^{(n)}(-b) = (-i)^n f(-b)$, $f \in C^\infty$, and

$$[f(-b)]^2 = -1 + 2i\int_{-b}^{b}[f(x)]^2 dx.$$

Here we may obtain an explicit $f(x)$ as a polynomial if we do not require that $g \in C^\infty$ at $x = \pm b$, that is, set

$$4b^3 f(x) = b(2 - ib)\,[\alpha(x - b)^2 + (x + b)^2] + x(1 - ib)\,[\alpha(x - b)^2 - (x+b)^2]$$

and choose α so that

$$\alpha^2 = -1 + 2i\int_{-b}^{b}[f(x)]^2 dx.$$

Thus, even if g is a C^∞ function of compact support, the integral in (3.3) may still not exist for $\delta = 0$. We shall now add the assumptions of Case I that, for sufficiently small $|s|$, $|W(s)| \geqslant K|s|$ and that W has no real zeros except possibly $s = 0$.

With these assumptions $n(\delta)$ must remain finite as $\delta \to 0$ and we shall suppose that $n(\delta) = n$ (its maximum) for $\delta < \delta_0$. Thus for $\delta < \delta_0$ the integral in (3.3) is independent of δ, and for $\delta < \frac{1}{2}\delta_0$ the integrand is bounded by $K(\sigma^2 + 1)^{-1}$ where K is independent of δ. Thus we may set $\delta = 0$ in (3.3) to obtain

$$G(x, \xi, \lambda) = \sum_{i=1}^{n} \sum_{p=1}^{m_i} G_p^{(i)}(x, \xi)(\lambda - \lambda_i)^{-p} + \frac{1}{\pi i}\int_0^\infty \frac{[\sigma K(x, \xi, \sigma) - \sigma K(x, \xi, -\sigma)]}{\sigma^2 - \lambda}\, d\sigma. \tag{4.1}$$

LEMMA 4.1. *We have*

$$\sigma K(x, \xi, \sigma) - \sigma K(x, \xi, -\sigma) = \frac{2i\sigma^2}{W(\sigma)W(-\sigma)}\,[y_1(x, \sigma)y_1(\xi, -\sigma) + y_2(x, \sigma)y_2(\xi, -\sigma)].$$

Proof. From the definition of $K(x, \xi, \sigma)$ we have

$$\sigma K(x, \xi, \sigma) - \sigma K(x, \xi, -\sigma) = \frac{\sigma}{W(\sigma)W(-\sigma)} \begin{cases} W(-\sigma)y_1(x, \sigma)y_2(\xi, \sigma) - W(\sigma)y_1(x, -\sigma)y_2(\xi, -\sigma), & \xi \leqslant x \\ W(-\sigma)y_1(\xi, \sigma)y_2(x, \sigma) - W(\sigma)y_1(\xi, -\sigma)y_2(x, -\sigma), & x \leqslant \xi. \end{cases}$$

If we denote $y_i(x, \pm\sigma)y_j'(x, \pm\sigma) - y_i'(x, \pm\sigma)y_j(x, \pm\sigma)$ by $W(y_i(\pm\sigma), y_j(\pm\sigma))$ and note that

$$W(y_1(\sigma), y_1(-\sigma)) = -W(y_2(\sigma), y_2(-\sigma)) = -2i\sigma,$$

then

$$y_2(\xi, \sigma) = \frac{2i\sigma}{W(-\sigma)} y_1(\xi, -\sigma) - \frac{W(y_2(+\sigma), y_1(-\sigma))}{W(-\sigma)} y_2(\xi, -\sigma)$$

and

$$y_1(x, -\sigma) = \frac{W(y_1(-\sigma), y_2(+\sigma))}{W(\sigma)} y_1(x, \sigma) - \frac{2i\sigma}{W(\sigma)} y_2(x, +\sigma).$$

Thus

$$\begin{aligned} W(-\sigma)y_1(x, \sigma)y_2(\xi, \sigma) &- W(\sigma)y_1(x, -\sigma)y_2(\xi, -\sigma) \\ &= 2i\sigma y_1(x, \sigma)y_1(\xi, -\sigma) - W(y_2(+\sigma), y_1(-\sigma))y_1(x, \sigma)y_2(\xi, -\sigma) \\ &- W(y_1(-\sigma), y_2(+\sigma))y_1(x, \sigma)y_2(\xi, -\sigma) + 2i\sigma y_2(x, \sigma)y_2(\xi, -\sigma) \\ &= 2i\sigma[y_1(x, \sigma)y_1(\xi, -\sigma) + y_2(x, \sigma)y_2(\xi, -\sigma)]. \end{aligned}$$

A similar computation yields the same result for $x \leqslant \xi$.

COROLLARY 4.1. *$G(x, \xi, \lambda)$ can be written in the form*

$$(4.2) \quad G(x, \xi, \lambda) = -\sum_{i=1}^{n} \sum_{p=1}^{m_i} \sum_{j=0}^{m_i - p} \chi_j^{(i)}(x)\, \overline{\psi_{m_i-p-j}^{(i)}(\xi)}(\lambda - \lambda_i)^{-p} + \frac{2}{\pi}\int_0^\infty \sum_{i=1}^{2} \frac{\phi_i(x, \sigma)\overline{\theta_i(\xi, \sigma)}}{\sigma^2 - \lambda}\, d\sigma$$

where

$$\phi_i(x, \sigma) = \frac{\sigma}{W(\sigma)} y_i(x, \sigma) \quad \text{and} \quad \theta_i(x, \sigma) = \frac{\sigma}{W^*(\sigma)} y_i^*(x, \sigma).$$

Here * denotes the corresponding quantity associated with the adjoint equation.

Proof. As

$$y_1^*(x, \sigma) = e^{i\sigma x} - \int_x^\infty \frac{\sin \sigma(x - \xi)}{\sigma} \overline{g(\xi)} y_1^*(\xi, \sigma)\, d\xi$$

it follows immediately that

$$y_1^*(x, \sigma) = \overline{y_1(x, -\sigma)}.$$

Similarly

$$y_2^*(x, \sigma) = \overline{y_2(x, -\sigma)}$$

so

$$W^*(\sigma) = \overline{W(-\sigma)}$$

and using these relations with Lemma 4.1 in (4.1) we obtain (4.2).

We are now in a position to prove an expansion theorem.

THEOREM 4.1. *If $f \in L^p$ for $p \geqslant 1$ then*

$$(4.3) \qquad f(x) = \sum_{i=1}^{n} \sum_{j=0}^{m_i-1} \chi_j^{(i)}(x)(f, \psi_{m_i-j-1}^{(i)}) + (l-\lambda)\frac{2}{\pi}\int_{-\infty}^{\infty} f(\xi)\int_0^{\infty}\sum_{i=1}^{2}\frac{\phi_i(x,\sigma)\overline{\theta_i(\xi,\sigma)}}{\sigma^2-\lambda}\,d\sigma\,d\xi,$$

almost everywhere, for any λ not in the spectrum.

Proof. If $f \in L^p$ for $p \geqslant 1$ it is easily seen that if λ is not in the spectrum of L then

$$\int_{-\infty}^{\infty} G(x, \xi, \lambda) f(\xi)\, d\xi$$

exists and is the unique L^p solution of $l(y) - \lambda y = f$. Using (4.2) to calculate the integral and applying $l - \lambda$ we immediately obtain (4.3).

The last term of (4.3) is not in a very convenient form, but in order to simplify it we must impose some restrictions on f. If $f \in L^1$ then the order of integration in the last term may be inverted, and setting

$$f_i(\sigma) = \int_{-\infty}^{\infty} f(x)\, \overline{\theta_i(x, \sigma)}\, dx$$

we obtain

$$(4.4) \qquad f(x) = \sum_{i=1}^{n} \sum_{j=0}^{m_i-1} \chi_j^{(i)}(x)(f, \psi_{m_i-j-1}^{(i)}) + (l-\lambda)\frac{2}{\pi}\int_0^{\infty}\sum_{i=1}^{2}\frac{\phi_i(x,\sigma) f_i(\sigma) d\sigma}{\sigma^2-\lambda}.$$

We define D_1 to be the class of functions $f \in L^1$ with derivatives which are absolutely continuous on every finite interval and such that $l(f) \in L^1$. Then for $f \in D_1$ choose $-a^2 < 0$ not in the spectrum of L and set $h = l(f) + a^2 f$. Then it is easily seen that f and f' approach 0 as x approaches $\pm\infty$ so

$$\int_{-\infty}^{\infty} h(x)\, \overline{\theta_i(x, \sigma)}\, dx = (\sigma^2 + a^2)\int_{-\infty}^{\infty} f(x)\, \overline{\theta_i(x, \sigma)}\, dx$$

and thus

$$|f_i(\sigma)| \leqslant K(\sigma^2 + a^2)^{-1}\int_{-\infty}^{\infty} |h(x)| dx.$$

So for $f \in D_1$ the operation of $l - \lambda$ in (4.4) may be perfomed under the integral sign to obtain

$$(4.5) \quad f(x) = \sum_{i=1}^{n} \sum_{j=0}^{m_i-1} \chi_j^{(i)}(x)(f, \psi_{m_i-j-1}^{(i)}) + \frac{2}{\pi}\int_0^{\infty}\sum_{i=1}^{2} \phi_i(x,\sigma) f_i(\sigma)\, d\sigma.$$

We also have an analogue of the Parseval equality, and a corresponding expansion theorem associated with L^*.

THEOREM 4.2. *As well as* (4.5) *we have, for* $f \in D_1$

$$(4.6)\quad f(x) = \sum_{i=1}^{n} \sum_{j=0}^{m_i-1} \psi_j^{(i)}(x)(f, \chi_{m_i-j-1}^{(i)}) + \frac{2}{\pi}\int_0^\infty \sum_{i=1}^{2} \theta_i(x, \sigma) f_i^*(\sigma)\, d\sigma$$

where

$$f_i^*(\sigma) = \int_{-\infty}^{\infty} f(x)\overline{\phi_i(x, \sigma)}\, dx;$$

and if $f, g \in D_1$

$$(4.7)\quad (f, g) = \sum_{i=1}^{n} \sum_{j=0}^{m_i-1} (f, \psi_{m_i-j-1}^{(i)})(\chi_j^{(i)}, g) + \frac{2}{\pi}\int_0^\infty \sum_{i=1}^{2} f_i(\sigma)\overline{g_i^*(\sigma)}\, d\sigma$$
$$= \sum_{i=1}^{n} \sum_{j=0}^{m_i-1} (f, \chi_{m_i-j-1}^{(i)})(\psi_j^{(i)}, g) + \frac{2}{\pi}\int_0^\infty \sum_{i=1}^{2} f_i^*(\sigma)\overline{g_i(\sigma)}\, d\sigma.$$

Proof. The proof of (4.6) is analogous to that of (4.5) and to obtain the two forms of (4.7) we note that if $f \in D_1$ it is in L^2 as well as L^1 and take the inner products of (4.5) and (4.6) with g. In doing this the order of integration in the last term can trivially be inverted to obtain the results.

5. The Expansion in Case II. Here we may assume that for some $a > 0$, $e^{a|x|}g(x) \in L^1$, but it is a consequence of this about the zeros of $W(s)$ which we use. If $e^{a|x|}g(x) \in L^1$; $y_1(x, s)$, $y_2(x, s)$, and thus $W(s)$ are analytic for $\tau > -\frac{1}{2}a$. In conjunction with Corollary 1.1 this implies that W has only a finite number of zeros in $\tau \geqslant 0$, and this is the assumption we make.

Suppose that the real zeros of W are $\sigma_1, \sigma_2, \ldots, \sigma_q$, and perhaps $\sigma_0 = 0$, arranged so that $0 = \sigma_0^2 < \sigma_1^2 < \ldots < \sigma_q^2$. Choose r so that $r < 2(\sigma_{i+1}^2 - \sigma_i^2)$ for $i = 0, 1, \ldots, q-1$ and so that $r < \min\, [Im(\lambda_i^{\frac{1}{2}})]^2$ for all λ_i in the point spectrum $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. We define the contour C by $\tau = f(\sigma)$ where $f(\sigma) = 0$ for $|\sigma^2 - \sigma_j^2| \geqslant r$ (j running from 0 to q or 1 to q according as $W(0)$ is or is not 0), and $f(\sigma) = (r^2 - (\sigma^2 - \sigma_j^2)^2)^{\frac{1}{4}}$ for $|\sigma^2 - \sigma_j^2| \leqslant r$. Now if $\delta < \min[Im(\lambda_i^{\frac{1}{2}})]$ the integral in (3.3) along $\tau = \delta$ is equal to the integral along C.

As the portion of C lying along $\tau = 0$ is symmetric about 0 we may transform it to an integral over $L = \{\sigma | \sigma \geqslant 0,\ |\sigma^2 - \sigma_j^2| \geqslant r\}$ with the same integrand as (4.1). The sum of the integrals over the indentations about σ_i and $-\sigma_i$ can be transformed by a change of variable into $\frac{1}{2}\oint_{C_i} G(x, \xi, \mu)$ $(u - \lambda)^{-1} d\mu$ where C_i is the circle of radius r about σ_i^2. Note that $G(x, \xi, \mu)$ is discontinuous where C_i crosses the real axis. This proves

THEOREM 5.1. *Under the hypothese of Case II we have:*

$$(5.1)\quad G(x, \xi, \lambda) = -\sum_{i=1}^{n} \sum_{p=1}^{m_i} \sum_{j=0}^{m_i-p} \chi_j^{(i)}(x)\overline{\psi_{m_i-p-j}^{(i)}(\xi)}\, (\lambda - \lambda_i)_{-p}$$
$$+ \frac{2}{\pi}\int_L \sum_{i=1}^{2} \frac{\phi_i(x, \sigma)\overline{\theta_i(\xi, \sigma)}}{\sigma^2 - \lambda}\, d\sigma + \sum_j \frac{1}{2\pi i}\oint_{C_j} \frac{G(x, \xi, \mu)}{\mu - \lambda}\, d\mu.$$

The only change from (4.2) is that the integral in the second term is taken over L rather than over $[0, \infty)$, and an extra sum is introduced. The other results carry over in the same way, replacing $[0, \infty)$ by L and adding a new summation. We shall merely indicate the forms these sums must take by considering a sample term

$$\frac{1}{2\pi i}\oint_{C_j}\frac{G(x, \xi, \mu)}{\mu - \lambda}\,d\mu.$$

In expanding $f \in D_1$ we have

$$\begin{aligned}(L - \lambda)\int_{-\infty}^{\infty}\frac{f(\xi)}{2\pi i}\oint_{C_j}\frac{G(x, \xi, \mu)}{\mu - \lambda}\,d\mu\,d\xi \\ = (L - \lambda)\frac{1}{2\pi i}\oint_{C_j}\frac{1}{\mu - \lambda}\int_{-\infty}^{\infty}G(x, \xi, \mu)f(\xi)\,d\xi\,d\mu \\ = \frac{1}{2\pi i}\oint_{C_j}\int_{-\infty}^{\infty}G(x, \xi, \mu)f(\xi)\,d\xi\,d\mu,\end{aligned}$$

and in the analogue of the Parseval equality we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\overline{g(x)}\oint_{C_j}\int_{-\infty}^{\infty}G(x, \xi, \mu)f(\xi)\,d\xi\,d\mu\,dx \\ = \frac{1}{2\pi i}\oint_{C_j}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x, \xi, \mu)f(\xi)\overline{g(x)}\,d\xi\,dx\,d\mu.\end{aligned}$$

The formulas arising from L^* are the same with G replaced by G^*.

A transformation of

$$H_j(x) = \frac{1}{2\pi i}\oint_{C_j}\int_{-\infty}^{\infty}G(x, \xi, \mu)f(\xi)\,d\xi\,d\mu$$

yields

$$H_j(x) = [L - \sigma_j^2]^{-t}\frac{2}{\pi}\int_{(\sigma_j^2 - r)^{\frac{1}{2}}}^{(\sigma_j^2 + r)^{\frac{1}{2}}}(\sigma^2 - \sigma_j^2)^t\sum_{k=1}^{2}\phi_k(x, \sigma)f_k(\sigma)\,d\sigma$$

where t is sufficiently large that $(s^2 - \sigma_j^2)^t K(x, \xi, s)$ is continuous at $\pm\sigma_j$, but one cannot carry the (unbounded) operator $[L - \sigma_j^2]^{-t}$ under the integral sign. In particular cases one can also evaluate the limit as $r \to 0$ in terms of the principal value of $\int_0^\infty \ldots d\sigma$ and a sum of terms which appear to involve characteristic functions, but do not.

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