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ON SOME ANALOGUES OF TITCHMARSH DIVISOR PROBLEM

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§ 1. Introduction

In [15] Titchmarsh posed and solved under the generalized Riemann Hypothesis, the problem of an asymptotic behavior of the number of the solutions of the equation $1 = p - n_1 n_2$ for a prime $p \leq x$ and natural numbers n_1 and n_2. When we put $\tau(n) = \sum_{d|n} 1$, then the above problem is to get an asymptotic law for the sum

$$\sum_{p \leq x} \tau(p-1) .$$

Later Linnik [11] solved this unconditionally using his dispersion method. The proof without the dispersion method is also known (Cf. [3] and [14]). Here we are concerned with an asymptotic behavior of the sum

$$\sum_{p_1 \leq x^\delta, p_2 \leq x^{1-\delta}} \tau(p_1 p_2 - 1) ,$$

where p_1 and p_2 run over primes and δ is in $0 < \delta \leq 1/2$. Linnik's dispersion method solves this for $0 < \delta < 1/6$. But it does not work for other values of δ. Barban [1] solved this for $\delta = 1/2$. Here we shall prove

THEOREM 1. *Suppose that δ is in $0 < \delta \leq 1/2$ and $\delta \log x$ tends to ∞ as x tends to ∞. Then we have*

$$\sum_{p_1 \leq x^\delta, p_2 \leq x^{1-\delta}} \tau(p_1 p_2 - 1) = \frac{315}{2\pi^4} \frac{\zeta(3)}{\delta(1-\delta)} \frac{x}{\log x} + O(x\delta^{-1}(\log x)^{-2}(\log\log x + \delta^{-1}))$$

uniformly for δ, where $\zeta(s)$ is the Riemann zeta function.

We shall also prove

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THEOREM 2.

$$\sum_{p_1p_2\le x}\tau(p_1p_2-1)=315\pi^{-4}\zeta(3)x\log\log x+O(x)\ .$$

To prove our theorems we need the following mean value theorems. We shall state them in the more general form than we need in this paper. For simplicity we put

$$E(y\,;a,d)=\sum_{\substack{p\le y\\ p\equiv a\ (\mathrm{mod}\ d)}}\cdot 1-\frac{\mathrm{Li}\,(y)}{\varphi(d)}\ ,$$

where

$$\mathrm{Li}\,(y)=\int_2^y\frac{dx}{\log x}+O(1)$$

and $\varphi(d)$ is the Euler function. Then we shall prove

THEOREM 3. *Suppose that* $\sum_{m\le x}|b(m)|^2\ll x(\log x)^C$ *with some positive absolute constant* C. *Then for any positive constants* A *and* $b\,(<1)$, *there exists a positive constant* B *such that*

$$\sum_{d\le Q}\mathop{\mathrm{Max}}_{(a,d)=1}\Bigg|\sum_{\substack{1\le m\le x^\delta\\ (m,d)=1}}b(m)E(x^{1-\delta}\,;am^*,d)\Bigg|\ll x(\log x)^{-A}$$

uniformly for δ *in* $0\le\delta<1-(\log x)^{-b}$, *where* $Q=x^{1/2}(\log x)^{-B}$ *and* $mm^*\equiv 1\ (\mathrm{mod}\ d)$.

The conclusion still holds even if we replace $E(x^{1-\delta}\,;am^*,d)$ by $E(x/m\,;am^*,d)$. We call this Theorem 3′. In § 2 we shall list up and prove some lemmas. We shall prove Theorem 3 in § 3, Theorem 1 in § 4 and Theorem 2 in § 5. We shall also give some remarks in § 6.

§ 2. Some lemmas

LEMMA 1. *For an arbitrarily given small positive* ε, *for all* $d\ll x^{1-\varepsilon}$ *and* $(a,d)=1$, *we have*

$$\sum_{\substack{pq\le x\\ pq\equiv a\ (\mathrm{mod}\ d)}}\cdot 1\ll\frac{x\log\log x}{\varphi(d)\log x}\ ,$$

where p *and* q *run over primes.*

Proof. Let η be a small positive number less than $\varepsilon/2$. Now the left hand side is

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$$\ll \sum_{\substack{x^\eta < p, q \\ pq \le x \\ pq \equiv a \pmod d}} 1 + \sum_{\substack{p \le x^\eta \\ (p,d)=1}} \sum_{\substack{q \le x/p \\ q \equiv a p^* \pmod d}} 1 = \sum\nolimits_1 + \sum\nolimits_2 ,$$

say.

$$\sum\nolimits_1 \ll x/(\varphi(d) \log x)$$

by Selberg's sieve method as usual.

$$\sum\nolimits_2 \ll \sum_{p \le x^\eta} x/(p \log x \cdot \varphi(d)) \ll x \log\log x/(\varphi(d) \log x)$$

by the Brun-Titchmarsh theorem. Hence we get our conclusion. Q.E.D.

LEMMA 2. *Let m be an integer different from* 1. *Then we have*

$$\left| \sum_{\chi:\, d}{}^{*} \chi(m) \right| \le |(m-1, d)| ,$$

where in the summation, χ runs over all primitive characters mod d.

Proof. We denote the sum in the left hand side by $S^*(d, m)$. We put $S(d, m) = \sum_{\chi \ne \chi_0} \chi(m)$, where χ runs over all non-principal characters mod d. We know for $(m, d) = 1$,

$$S(d, m) = \begin{cases} \varphi(d) - 1 & \text{if } m \equiv 1 \pmod d \\ -1 & \text{otherwise .} \end{cases}$$

Suppose that $d = \prod p^\nu$ and $(m, d) = 1$. Then $S^*(d, m) = \prod_{p|d} S^*(p^\nu, m)$. Suppose that $p \ne 2$. We denote the primitive character attached to χ by χ^*. Then

$$\begin{aligned} S(p^\nu, m) &= \sum_{\chi \ne \chi_0} \chi(m) = \sum_{\chi \ne \chi_0} \chi^*(m) = \sum_{\nu \ge j \ge 1} S^*(p^j, m) \\ &= S^*(p^\nu, m) + S(p^{\nu-1}, m) . \end{aligned}$$

Hence for $\nu \ge 1$,

$$\begin{aligned} S^*(p^\nu, m) &= S(p^\nu, m) - S(p^{\nu-1}, m) \\ &= \begin{cases} \varphi(p^\nu) - \varphi(p^{\nu-1}) & \text{if } p^\nu | m-1 \\ -\varphi(p^{\nu-1}) & \text{if } p^{\nu-1} \| m-1 \\ 0 & \text{otherwise .} \end{cases} \end{aligned}$$

Next for $p = 2$ and for $\nu \ge 2$,

$$S(2^\nu, m) = \sum_{\nu \ge j \ge 2} S^*(2^j, m) = S^*(2^\nu, m) + S(2^{\nu-1}, m) .$$

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Hence for $\nu \geq 1$, we get

$$S^*(2^\nu, m) = \begin{cases} \varphi(2^\nu) - \varphi(2^{\nu-1}) & \text{if } 2^\nu \mid m-1 \\ -\varphi(2^{\nu-1}) & \text{if } 2^{\nu-1} \| m-1 \\ 0 & \text{otherwise .} \end{cases}$$

Hence we have

$$\begin{aligned} |S^*(d, m)| &\leq \prod_{p \mid d} |S^*(p^\nu, m)| \\ &\leq \prod_{p^\nu \mid m-1} (\varphi(p^\nu) - \varphi(p^{\nu-1})) \prod_{p^{\nu-1} \| m-1} \varphi(p^{\nu-1}) \\ &\leq \prod_{p^\nu \mid m-1} p^\nu \prod_{p^{\nu-1} \| m-1} p^{\nu-1} \\ &\leq |(m-1, d)| \, . \end{aligned}$$

Q.E.D.

LEMMA 3.

$$\sum_{D < d \leq Q} \frac{1}{\varphi(d)} \sum_{\chi : d}^{*} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 \ll \left(Q + \frac{N}{D} \right) \sum_{n=M+1}^{M+N} |a_n|^2$$

(Cf. (10) of [6] or (3) of [2]).

LEMMA 4. *For $d \leq x^{1/2+\varepsilon}$, and for $(a, d) = 1$,*

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod d}} \tau^2(n) \ll x \Big(\prod_{p \mid d} (1 - p^{-1}) \cdot \log (x/d) \Big)^3 \Big/ d \, .$$

(Cf. Lemma 1.1.3 of [11]).

LEMMA 5. *For all $d \ll x^{2/3-\varepsilon}$, $(a, d) = 1$ and for $0 < \delta \leq 1/2$,*

$$\sum_{\substack{pq \equiv a \pmod d \\ p \leq x^\delta,\, q \leq x^{1-\delta}}} \cdot 1 \ll \delta^{-1} x / (\varphi(d)(\log x)^2) \, .$$

(Cf. Lemma 3.6 of [1])

§ 3. Proof of Theorem 3 and 3′

3-1. We shall prove only Theorem 3 since Theorem 3′ can be proved in a similar manner (Cf. [4] and [5]). By Bombieri's mean value theorem we may suppose that $\delta \geq A' \log\log x / \log x$ for a sufficiently large constant A'. For simplicity we put $x' = x^\delta$, $x'' = x^{1-\delta}$, $\ell = \log x$ and $\pi(x, \chi) = \sum_{p \leq x} \chi(p)$. We also put $Q = x^{1/2} \ell^{-B}$ with sufficiently large B which will be chosen appropriately in the following, $Q_j = 2^j \ell^D$ for $j = 0, 1, 2, \cdots, J$, where J satisfies $2^{J-1} \ell^D < Q \leq 2^J \ell^D$ and D is a sufficiently large constant. We always denote an arbitrarily small positive number by ε, a sufficiently large constant by E and some positive absolute constants by C. Now for $d \le Q$, $(m, d) = 1$ and $(a, d) = 1$,

$$\begin{aligned} E(x''; am^*, d) &= \frac{1}{\varphi(d)} \sum_{\chi \neq \chi_0} \bar{\chi}(a)\chi(m)\pi(x'', \chi) \\ &\quad + \frac{1}{\varphi(d)}\Big(\sum_{p \le x''} \cdot 1 - \mathrm{Li}\, x''\Big) - \frac{1}{\varphi(d)} \sum_{\substack{p \le x'' \\ p \mid d}} \cdot 1 \\ &= \frac{1}{\varphi(d)} \sum_{\chi \neq \chi_0} \bar{\chi}(a)\chi(m)\pi(x'', \chi) + O(x'' \ell^{-E} \varphi(d)^{-1}) \end{aligned}$$

by the prime number theorem. Since

$$\pi(x'', \chi^*) - \sum_{\substack{p \mid d \\ p \nmid d^* \\ p \le x''}} \chi^*(p) = \pi(x'', \chi) ,$$

we have

$$\begin{aligned} &\sum_{d \le Q} \mathop{\mathrm{Max}}_{(a,d)=1} \Big| \sum_{\substack{m \le x' \\ (m,d)=1}} b(m) E(x''; am^*, d) \Big| \\ &\ll \sum_{d \le Q} \frac{1}{\varphi(d)} \sum_{\chi \neq \chi_0} \Big| \sum_{\substack{m \le x' \\ (m,d)=1}} b(m)\chi^*(m)\pi(x'', \chi^*) \Big| \\ &\quad + \sum_{d \le Q} \frac{1}{\varphi(d)} \mathop{\mathrm{Max}}_{(a,d)=1} \Big| \sum_{\chi \neq \chi_0} \bar{\chi}^*(a) \sum_{\substack{m \le x' \\ (m,d)=1}} b(m)\chi^*(m) \sum_{\substack{p \mid d \\ p \nmid d^* \\ p \le x''}} \chi^*(p) \Big| \\ &\quad + x\ell^{-E} = S_1 + S_2 + x\ell^{-E} , \end{aligned}$$

say, where d^* is the conductor of χ.

3-2. We shall estimate S_2 first. Using Lemma 2, we get

$$\begin{aligned} S_2 &\ll \sum_{d \le Q} \frac{1}{\varphi(d)} \mathop{\mathrm{Max}}_{(a,d)=1} \sum_{\substack{d^* \mid d \\ d^* > 1}} \sum_{\substack{p \mid d/d^* \\ p \le x''}} \sum_{\substack{m \le x' \\ (m,d)=1}} |b(m)| \Big| \sum_{\chi : d^*}{}^{*} \chi(a^* mp) \Big| \\ &\ll \sum_{d \le Q} \frac{1}{\varphi(d)} \mathop{\mathrm{Max}}_{(a,d)=1} \sum_{\substack{d^* \mid d \\ d^* > 1}} \sum_{\substack{p \mid d/d^* \\ p \le x''}} \sum_{\substack{m \le x' \\ (m,d)=1}} |b(m)| \, |(mp - a, d^*)| \\ &\ll \ell^C \mathop{\mathrm{Max}}_{1 \le a \le Q} S_a , \end{aligned}$$

where we put

$$S_a = \sum_{k_1 k_2 \le Q} 1/(k_1 k_2) \sum_{\substack{p \mid k_2 \\ p \le x'' \\ (a,p)=1}} \sum_{m \le x'} |b(m)| \, |(mp - a, k_1)| .$$

$$\begin{aligned} S_a &\ll \sum_{k_1 \le Q} 1/k_1 \sum_{\substack{m \le x', p \le x'' \\ (a,p)=1}} |b(m)| \, |(mp - a, k_1)| \sum_{\substack{k_2 \le Q \\ p | k_2}} 1/k_2 \\ &\ll \ell^C \sum_{\substack{m \le x', p \le x'' \\ (a,p)=1}} |b(m)|/p \sum_{d \,|\, |mp-a|} d \sum_{d | k_1, k_1 \le Q} 1/k_1 \\ &\ll \ell^C \sum_{\substack{m \le x', p \le x'' \\ (a,p)=1}} |b(m)| \, \tau(|mp - a|)/p \\ &\ll \ell^C x'^{1/2} \Big(\sum_{m \le x'} \sum_{\substack{p \le x'' \\ (a,p)=1}} \tau^2(|mp - a|)/p \Big)^{1/2} \\ &\ll \ell^C x'^{1/2} S_a'^{1/2} \,, \end{aligned}$$

say. If $0 < \delta \le 1/2$, then $S'_a \ll x^{\varepsilon} \ell^C x' \ll x$.

If $1/2 \le \delta < 1$, then

$$S'_a \ll \sum_{\substack{p \le x'' \\ (a,p)=1}} 1/p \sum_{m \le x'} \tau^2(|mp - a|) \ll x\ell^C$$

using Lemma 4. Hence always we get $S_a \ll x^{(1+\delta)/2} \ell^C$, and

$$S_2 \ll x^{(1+\delta)/2} \ell^C \ll x\ell^{-E} \qquad \text{uniformly for } 0 < \delta \le 1 - (\log x)^{-b} \,.$$

3-3. Next we shall estimate S_1.

$$S_1 \ll \ell \operatorname*{Max}_{1 < b \le Q} S_{1,b} \,,$$

where we put

$$S_{1,b} = \sum_{0 \le j \le J} S_{1,b}(j)$$

with

$$S_{1,b}(j) = \sum_{Q_{j-1} < d \le Q_j} \frac{1}{\varphi(d)} \sum_{\chi : d}{}^{*} \Big| \sum_{\substack{m \le x' \\ (m,b)=1}} \chi(m) b(m) \pi(x'', \chi) \Big|$$

for $0 \le j \le J$ and $Q_{-1} = 1$.

By Siegel-Walfisz theorem (Cf. p. 134 and 144 of [13]), we get

$$S_{1,b}(0) \ll x\ell^{-E} \,.$$

Now

$$\begin{aligned} S_{1,b}(j) &\ll \Big(\sum_{Q_{j-1} < d \le Q_j} \frac{1}{\varphi(d)} \sum_{\chi : d}{}^{*} \Big| \sum_{m \le x'} \chi(m) b(m) \Big|^2 \Big)^{1/2} \\ &\quad \cdot \Big(\sum_{Q_{j-1} < d \le Q_j} \frac{1}{\varphi(d)} \sum_{\chi : d}{}^{*} |\pi(x'', \chi)|^2 \Big)^{1/2} \\ &= \textstyle\sum_1^{1/2} \sum_2^{1/2} \,, \end{aligned}$$

say. By Lemma 3, we get

$$\sum_1 \ll (Q_j + x'Q_j^{-1}) \sum_{m \le x'} |b(m)|^2 \ll x'\ell^C(Q_j + x'Q_j^{-1}) .$$

Similarly we get

$$\sum_2 \ll (Q_j + x''Q_j^{-1})x'' .$$

Hence $S_{1,b}(j) \ll x\ell^{-E}$ by taking B and D sufficiently large. Hence $S_1 \ll x\ell^{-E}$. Combining this with the estimate of S_2, we get our conclusion.

Q.E.D.

§4. Proof of Theorem 1

We put for simplicity $x_1 = x^\delta$, $x_2 = x^{1-\delta}$, $\ell = \log x$ and $Q = x^{1/2}\ell^{-B}$ with a sufficiently large constant B. Now

$$\begin{aligned}\sum_{p_i \le x_i} \tau(p_1p_2 - 1) &= \sum_{p_i \le x_i} \sum_{d \mid p_1p_2-1} \cdot 1 \\ &= 2\sum_{d \le Q} \sum_{\substack{p_i \le x_i \\ d \mid p_1p_2-1}} \cdot 1 + O\Big(\sum_{Q<d\le\sqrt{x}} \sum_{\substack{d \mid p_1p_2-1 \\ p_i \le x_i}} \cdot 1\Big) \\ &= \sum_1 + O(\sum_2) ,\end{aligned}$$

say.

$$\begin{aligned}\sum_1 &= 2\sum_{d\le Q} \sum_{\substack{p_1 \le x_1 \\ (p_1,d)=1}} \Big(\sum_{\substack{p_2 \le x_2 \\ p_2 \equiv p_1^*(\mathrm{mod}\ d)}} \cdot 1 - \mathrm{Li}\, x_2/\varphi(d)\Big) \\ &\quad + 2\sum_{d \le Q} \sum_{\substack{p_1 \le x_1 \\ (p_1,d)=1}} \mathrm{Li}\, x_2/\varphi(d) \\ &= O(x\ell^{-A}) + 2\,\mathrm{Li}\, x_1\, \mathrm{Li}\, x_2 \Big(\sum_{d \le Q} \frac{1}{\varphi(d)}\Big) \\ &= \mathrm{Li}\, x_1\, \mathrm{Li}\, x_2\, 315\zeta(3)(2\pi^4)^{-1} \log x \\ &\quad + O(\mathrm{Li}\, x_1\, \mathrm{Li}\, x_2 \log\log x) .\end{aligned}$$

On the other hand by Lemma 5,

$$\sum_2 \ll \delta^{-1} x \log\log x/(\log x)^2 \qquad \text{uniformly for } 0 < \delta \le 1/2 .$$

Hence we get our conclusion. Q.E.D.

Remark. To prove our theorem 1 just for any δ in $0 < \delta < 1/2$ we do not need Barban's Lemma 3.6 (namely, Lemma 5 in §2). Because by the Brun-Titchmarsh theorem we get

$$\sum_2 \ll x \log\log x/(\log x)^2 .$$

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§ 5. Proof of Theorem 2

Let δ be any number in $0 < \delta < 1$. We put $x' = x^{\delta}$, $x'' = x^{1-\delta}$, $Q = x^{1/2}(\log x)^{-B}$ and $F(\delta, x) = \sum_{p \le x^{\delta}} (p \log (x/p))^{-1}$. Now

$$\begin{aligned}
&\sum_{pq \le x} \tau(pq-1) \\
&\qquad = \sum_{\substack{p \le x' \\ pq \le x}} \tau(pq-1) + \sum_{\substack{q \le x'' \\ pq \le x}} \tau(pq-1) - \sum_{\substack{p \le x' \\ q \le x''}} \tau(pq-1) \\
&\qquad = \sum\nolimits_1 + \sum\nolimits_2 - \sum\nolimits_3 ,
\end{aligned}$$

say. By Theorem 1 $\sum_3 \ll x(\log x)^{-1}$.

$$\begin{aligned}
\sum\nolimits_1 &= 2 \sum_{d<Q} \sum_{\substack{p \le x' \\ (p,d)=1}} \mathrm{Li}\,(x/p)/\varphi(d) \\
&\quad + O\Bigg(\sum_{d \le Q} \sum_{\substack{p \le x' \\ (p,d)=1}} \Bigg(\sum_{\substack{q \le x/p \\ q \equiv p^*(\mathrm{mod}\ d)}} \cdot 1 - \mathrm{Li}\,(x/p)/\varphi(d)\Bigg)\Bigg) \\
&\quad + O\Bigg(\sum_{Q < d \le \sqrt{x}} \sum_{\substack{pq \le x \\ pq \equiv 1 (\mathrm{mod}\ d)}} \cdot 1\Bigg) \\
&= 315\zeta(3)(2\pi^4)^{-1} x \log x F(\delta, x) + O(x \log\log x F(\delta, x)) \\
&\quad + O(x(\log\log x)^2(\log x)^{-1}) \\
&\quad + O\Bigg(x(\log x)^{-1} \sum_{d<Q} \frac{1}{\varphi(d)} \cdot \sum_{\substack{p \le x' \\ p \mid d}} \frac{1}{p}\Bigg) .
\end{aligned}$$

The last term is $\ll x$. In a similar way we get

$$\begin{aligned}
\sum\nolimits_2 &= 315(2\pi^4)^{-1}\zeta(3) x \log x F(1-\delta, x) \\
&\quad + O(x \log\log x F(1-\delta, x)) + O(x) .
\end{aligned}$$

Now

$$\begin{aligned}
F(\delta, x) &= \int_{3/2}^{x'} \frac{1}{t \log (x/t)} d\Big(\sum_{p \le t} 1\Big) \\
&= (\log\log x + \log \delta - \log (1-\delta))/\log x + O((\log x)^{-1}) .
\end{aligned}$$

Hence

$$F(\delta, x) + F(1-\delta, x) = 2 \log\log x/\log x + O(1/\log x) .$$

Hence we get our conclusion. Q.E.D.

§ 6. Concluding remarks

6-1. Theorem 1 and 2 for the sum of $\tau(N - p_1p_2)$ or $\tau(p_1p_2 - a)$ can be similarly proved.

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6-2. More generally, if $k \geq 1$, $\delta_1 + \delta_2 + \cdots + \delta_k = 1$, $\delta_i > 0$ for each i and $\delta_j + \delta_\ell > 3/4$ for some j, ℓ in $1 \leq j,\ \ell \leq k$, then we have

$$\sum_{p_i \leq x^{\delta_i}} \tau(p_1 p_2 \cdots p_k - 1)$$
$$= \frac{315}{2\pi^4} \frac{\zeta(3)}{\delta_1 \delta_2 \cdots \delta_k} \frac{x}{(\log x)^{k-1}} + O(x \log\log x/(\log x)^k) .$$

For $k = 2$, this is nothing but our Theorem 1. (Cf. [1] and [9] for previous weaker results.).

Further, under the same condition of $\delta_1, \delta_2, \cdots, \delta_k$, we have an asymptotic formula for the sum

$$\sum_{p_i \leq x^{\delta_i}} \tau_m(p_1 p_2 \cdots p_k - a) \qquad \text{for almost all } a$$

and for each $m \geq 3$, where

$$\tau_m(n) = \sum_{n = d_1 d_2 \cdots d_m} \cdot 1 .$$

(Cf. [16] for $k = 1$ and for $m \geq 3$.)

6-3. In a similar manner to the proof of Theorems 3 and 3′, we get the following inequality; for any positive constants A and $b\,(< 1)$, if $\sum_{m \leq x} |b(m)|^2 \ll x(\log x)^C$, $b(m) \ll x^{1-\delta-\beta}$ for $m \leq x^\delta$, $\beta = (\log x)^{-f}$ with some f in $b < f < 1$, then there exists a positive constant B such that

$$\sum_{d \leq Q} \operatorname*{Max}_{(a,d)=1} \operatorname*{Max}_{1 \leq y \leq x} \left| \sum_{\substack{mp \leq y,\, m \leq x^\delta \\ mp \equiv a \pmod d}} b(m) - \frac{1}{\varphi(d)} \sum_{\substack{mp \leq y \\ m \leq x^\delta}} b(m) \right| \ll x(\log x)^{-A}$$

uniformly for δ in $0 \leq \delta < 1 - (\log x)^{-b}$, where $Q = x^{1/2}(\log x)^{-B}$. Using this, Theorems 3 and 3′ and Hooley's argument in [8], we can show an asymptotic formula for the number of the solutions of the equation

$$N = p_1 p_2 + x^2 + y^2 \qquad \text{for } p_1 p_2 \leq N .$$

We do not need Linnik's dispersion method. (Cf. [11] and [12] for a proof of this using the dispersion method.) As is seen in [11] or [12] we may improve the remainder term in Theorem 2 if we use the dispersion method.

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