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A NOTE ON A THEOREM OF KY FAN

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Fan ([2, Theorem 2]) has proved the following theorem:

Let K be a nonempty compact convex set in a normed linear space X. For any continuous map f from K into X, there exists a point $u \in K$ such that

$$\|u - f(u)\| = \operatorname*{Min}_{x \in K} \|x - f(u)\|$$

In this note, we prove that the above theorem is true for a continuous condensing map defined on a closed ball in a Banach space. We also prove that it is true for a continuous condensing map defined on a closed convex bounded subset of a Hilbert space.

Now, we introduce our notations and definitions:

Let B be a nonempty bounded subset of a metric space X. We shall denote (after Kuratowski [5]) by $a(B)$ the infimum of the numbers r such that B can be covered by a finite number of subsets of X of diameter less than or equal to r.

Let S be a nonempty subset of X and let f be a map from S into X. If for every nonempty bounded subset B of S with $a(B) > 0$, we have $a(f(B)) < a(B)$, then f will be called condensing ([7]). If there exists k, $0 \le k \le 1$, such that for each nonempty bounded subset B of S we have $a(f(B)) \le k\, a(B)$, then f is called k-set-contractive ([5]).

Let X, Y be two normed linear spaces, S a nonempty subset of X, f a map from S into Y, f is called nonexpansive if for each x, $y \in S$, we have $\|f(x) - f(y)\| \le \|x - y\|$.

LEMMA. ([4] or [7])

Let S be a nonempty closed convex bounded subset of a Banach space X. If f is a continuous condensing map from S into S, then f has a fixed point in S.

THEOREM 1. *Let S be a closed ball with center at origin and radius r in a Banach space X. If f is a continuous condensing map from S into X, then there exists a point $u \in S$ such that*

$$\|u - f(u)\| = \operatorname*{Min}_{x \in S} \|x - f(u)\|$$

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Proof. Define

$$R(x)=\begin{cases} x\ , & \text{if } \|x\|\le r \\ \dfrac{rx}{\|x\|}, & \text{if } \|x\|\ge r \end{cases}$$

Then R is a continuous 1-set-contractive map ([6, Proposition 9]) from X onto S. Define $F(x)=R(f(x))$, F is a continuous map from S into S. Moreover, for each nonempty bounded subset B of S, with $a(B)>0$, we have

$$a(F(B))=a(R(f(B)))\le a(f(B))<a(B).$$

Thus F is a condensing map. By Lemma, there exists $u\in S$ such that $F(u)=u$. Now,

$$\|u-f(u)\|=\|F(u)-f(u)\|=\|R(f(u))-f(u)\|$$

$$=\begin{cases} \|f(u)-f(u)\|=0, & \text{if } \|f(u)\|\le r \\ \left\|\dfrac{rf(u)}{\|f(u)\|}-f(u)\right\|=\|f(u)\|-r, & \text{if } \|f(u)\|\ge r \end{cases}$$

For each $x\in S$, we have $\|f(u)\|-r\le\|f(u)\|-\|x\|\le\|x-f(u)\|$ Hence

$$\|u-f(u)\|=\operatorname*{Min}_{x\in S}\|x-f(u)\|.$$

THEOREM 2. *Let S be a nonempty closed convex subset of a Hilbert space X. Let f be a continuous condensing map from S into X. If $f(S)$ is bounded, then there exists a point $u\in S$ such that*

$$\|u-f(u)\|=\operatorname*{Min}_{x\in S}\|x-f(u)\|$$

Proof. By ([3]), there exists a continuous map p from X into S, such that for each $y\in X$, we have

$$\|p(y)-y\|=\operatorname*{Min}_{x\in S}\|x-y\|$$

By ([1]) p is nonexpansive in Hilbert space. Then $p\circ f$ is a continuous condensing map from *clco* $p\circ f(S)$ (where *clco* A denote the closed convex hull of A) into *clco* $p\circ f(S)$. By Lemma, there exists $u\in S$ such that $p\circ f(u)=u$. Hence

$$\|u-f(u)\|=\|p(f(u))-f(u)\|=\operatorname*{Min}_{x\in S}\|x-f(u)\|.$$

REMARK. Only continuous map can not assure Theorem 1 is true. We use a well-known example to illustrate our case. Let S be the closed unit ball in the Hilbert space 1_2. Let

$$f(x)=(\sqrt{(1-\|x\|^2)}, x_1, x_2, \ldots, x_n, \ldots)$$

where $x=(x_1, x_2, \ldots, x_n, \ldots)\in S$. Since $\|f(x)\|=1$ therefore $f(S)\subset S$. If there were a point $u\in S$ such that

$$\|u-f(u)\|=\operatorname*{Min}_{x\in S}\|x-f(u)\|.$$

it must be a fixed point of f. But it is easily seen that f has no fixed point in S.

Added in proof. We can adopt the same technique used by F. E. Browder (On a sharpened form of the Schauder fixed-point theorem, Proc. Natl. Acad. Sci. U.S.A., Vol. 74, No. 11, pp. 4749–4751, November 1977) to obtain the following theorems by using our Theorem 1 and Theorem 2: The hypotheses are the same as Theorem 1 (or Theorem 2) if in addition, for each $x\in S$ with $x\neq f(x)$, there exists y in $I_S(x)=\{x+c(z-x)\mid \text{for some } z\in S, \text{ some } c>0\}$ such that $\|y-f(x)\|<\|x-f(x)\|$, then f has a fixed point in S. We also note that if f is weakly inward (i.e. $f(x)$ lie in the closure of $I_S(x)$ for each $x\in S$), then for each $x\in S$ with $x\neq f(x)$ we can choose y in $I_S(x)$ such that $\|y-f(x)\|<\|x-f(x)\|$.

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