A PROBLEM ON CYCLIC SUBGROUPS OF FINITE GROUPS

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(Received 5th June 1972)

1. Introduction

Let G be a finite group and let S be a subgroup of G with

$$\text{core}(S) = \bigcap_{x \in G} x^{-1}Sx = 1.$$

We say (G, S) has *property* (*) if there exists $x \in G$ such that $S \cap x^{-1}Sx = 1$. A question which immediately arises is the following; what conditions on G, S ensure that (G, S) has property (*)?

It has been shown by J. S. Brodkey (**1**) that (G, S) has property (*) if S is an abelian Sylow p-subgroup of G for some prime p. Brodkey's argument can easily be extended to the case where S is an abelian Hall subgroup of the group G. (See also (**2**).)

It has been shown by N. Ito (**3**) that if G is soluble and S is a Sylow p-subgroup of G, then (G, S) has property (*) except possibly when $p = 2$ or p is a Mersenne prime.

In this note we consider the case where S is cyclic. We show that (G, S) has property (*) if G is simple and that if G is soluble and $S \cap F(G) = 1$, then (G, S) has property (*).

Our results suggest that (G, S) has property (*) if S is cyclic and $S \cap F(G) = 1$, but we have not been able to prove this in general.

The notation is standard. We recall that if G is a finite group $F(G)$ denotes the maximal nilpotent normal subgroup of G.

2. Preliminary results

Lemma 1. *Let $q, p_1, \ldots, p_n$ be distinct prime numbers. For each i let a_i be the least positive integer for which $q^{a_i} = 1 \bmod p_i$. Then* $\sum_{i=1}^{n} (1/q^{a_i}) < 1 - 1/q$.

Proof. Let $w(m)$ denote the number of distinct prime divisors of the positive integer m.

Then

$$\sum_{i=1}^{n} (1/q^{a_i}) < \sum_{i=1}^{\infty} \frac{w(q^i - 1)}{q^i} < \sum_{i=1}^{\infty} \frac{\log_2 q^i}{q^i}$$
$$= q \log_2 q/(q-1)^2.$$

This implies the result for $q \geqq 5$.

https://doi.org/10.1017/S0013091500010014 Published online by Cambridge University Press

Next

$$\sum_{i=1}^{n} (\tfrac{1}{3}^{a_i}) < \tfrac{1}{3} + \tfrac{1}{27} + \sum_{i=3}^{\infty} \frac{i}{3^i} < \tfrac{2}{3}.$$

Finally

$$\sum_{i=1}^{n} (\tfrac{1}{2}^{a_i}) < \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{128} + \tfrac{1}{256} + \sum_{i=9}^{\infty} (i/2^i) < \tfrac{1}{2}.$$

Lemma 2. *Let $q_1 < q_2 < q_3 \ldots$ be the sequence of prime numbers. Then*

$$\sum_{i=1}^{\infty} (1/q_i^2) < \tfrac{2}{3}.$$

Proof. The result follows immediately from the fact that

$$\sum_{n=1}^{\infty} (1/n^2) = \pi^2/6.$$

3. Cyclic group action on a nilpotent group

This section is devoted to a proof of the following result:

Theorem 1. *Let Q be a finite nilpotent group and let A be a cyclic group of automorphisms of Q. Then there exists $v \in Q$ such that $va \neq v$ for any $a \neq 1$ in A.*

Proof. Assume the result is false and let (Q, A) be a counterexample for which $|Q| + |A|$ is minimal. Let $|A| = p_1^{b_1} \ldots p_k^{b_k}$ be the canonical decomposition of $|A|$ as a product of distinct prime powers. The minimality of $|A|$ forces $b_1 = \ldots = b_k = 1$. Let q be a prime divisor of $|Q|$ and let Q_0 be the Sylow q-subgroup of Q.

(1) $Q_0 = Q$. For suppose not. Let Q_1 be the Hall q-complement of Q. Let $A_0 = \{a \in A \mid a$ acts trivially on $Q_0\}$, and let $A = A_0 \times A_1$. Now there exists $v_0 \in Q_0$ such that $v_0 a_1 \neq v_0$ for any $a_1 \in A_1 - \{1\}$ and there exists $v_1 \in Q_1$ such that $v_1 a_0 \neq v_1$ for any $a_0 \in A_0 - \{1\}$. Let $v = v_0 v_1$. Then $va \neq v$ for any $a \in A - \{1\}$. This establishes (1).

(2) Contradiction. Let a_i be an element of A of order p_i and let

$$Q_1 = \{w \in Q \mid wa_i = w\}.$$

Then $Q = \cup Q_i$ (set-theoretic union). Let $|Q| = q^n$, $|Q:Q_i| = q^{n_i}$. Since a_i permutes the elements of $Q - Q_i$ into orbits of length p_i, $q^{n_i} \equiv 1 \bmod p_i$ if $p_i \neq q$. In particular, $n_i \geqq d_i$ where d_i is the least positive integer for which $q^{d_i} \equiv 1 \bmod p_i$ if $p_i \neq q$. But now,

$$\sum_{i=1}^{k} 1/q^{n_i} \leqq 1/q + \sum_{p_i \neq q} 1/q^{n_i} \leqq 1/q + \sum_{p_i \neq q} 1/q^{d_i} < 1$$

by Lemma 1. On the other hand, the equation $Q = \cup Q_i$ implies that

$$|Q| \leqq \sum_{i=1}^{k} |Q_i|$$

and thus $\sum_{i=1}^{k} 1/q^{n_i} \geqq 1$. The contradiction is established.

4. Intersection theorems

Let G be a finite group and let A be a cyclic subgroup of G. Let $|A| = p_1^{a_1} \dots p_k^{a_k}$ where $p_1, \dots, p_k$ are distinct primes and $a_1, \dots, a_k$ positive integers. Let A_i be the subgroup of A of order p_i and let N_i be the normaliser of A_i in G. Then (G, A) has property (*) if and only if G is not the *set-theoretic union* of the groups $N_1, \dots, N_k$. Let $|G:N_i| = n_i$. We see that (G, A) has property (*) if $\Sigma 1/n_i \leqq 1$. If G is simple, then $n_i > \max\{p_i\}$, so we get

Proposition 1. *If G is simple and A is cyclic, then (G, A) has property* (*).

Suppose now that $A \cap F(G) = 1$. Let K_i be the core of N_i, that is K_i is the largest normal subgroup of G contained in N_i. Since G/K_i is a permutation group on n_i symbols and $A_i \nleqq K_i$, $n_i \geqq p_i^{a_i} + 1$. Lemma 2 implies

Proposition 2. *If A is a cyclic subgroup of G such that $A \cap F(G) = 1$ and no Sylow subgroup of A has prime order, then (G, A) has property* (*).

Finally we show

Proposition 3. *Let A be a cyclic subgroup of a finite group G such that $C_G(F(G))$ is abelian. Assume that $A \cap F(G) = 1$. Then (G, A) has property* (*).

Proof. Since $A \cap F(G) = 1$ and $C_G(F(G)) = Z(F(G))$, A acts faithfully on $F(G)$ by conjugation. The result follows from Theorem 1.

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