GROUPS OF MATRICES WITH INTEGER EIGENVALUES

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(Received 26 June 1969)

Communicated by G. E. Wall

1

Let F be an algebraic number field, and S a subgroup of the general linear group $GL(n, F)$. We shall call S a *U-group* if S satisfies the condition (U): Every $x \in S$ is a matrix all of whose eigenvalues are algebraic integers. This is equivalent to either of the following conditions:

a) the eigenvalues of each matrix x are all units as algebraic numbers;

b) the characteristic polynomial for x has all its coefficients integers in F.

In particular, then, every group of matrices with entries in the integers of F is a U-group.

Our aim is to examine the structure of completely reducible soluble U-groups. We use the results given by Suprunenko [1] for soluble and nilpotent linear groups, and obtain some special conditions that must be satisfied by completely reducible soluble U-groups. We show that such groups are polycyclic, and we obtain some arithmetical conditions that must be satisfied by primitive irreducible soluble U-groups, depending on the degree of the group and the field F. The results obtained depend on results for irreducible abelian and nilpotent U-groups, which we examine separately.

2. Abelian U-groups

The structure of abelian linear groups over the integers of an algebraic number field has been described by Dade [2]. In this section we give a generalisation of his result to completely reducible U-groups.

2.1 THEOREM. *Let A be an irreducible abelian U-group in $GL(n, F)$, and let the degree $[F : Q]$ of F over Q be d. Then A is finitely generated, of rank at most $nd-1$, and A_T, the torsion subgroup of A, is cyclic of order t, where $\phi(t)$ (the Euler function) divides nd.*

Note: the estimate for $|A_T|$ depends only on the fact that A is an irreducible abelian subgroup of $GL(n, F)$, not on the condition (U).

PROOF. Let $[A]$ be the linear hull of A over F (i.e. $[A]$ is the subalgebra of the full matrix algebra $M(n, F)$ generated by elements of A). $[A]$ is irreducible, and therefore is a simple ring ([3], p. 56). Since $[A]$ is also a commutative ring with unity, the fact that it is simple makes it a field. Let $a_1, \cdots, a_s$ be a basis for $[A]$ over F. Let V be an n-dimensional F-module, and identify $\mathrm{End}_F V$ and $M(n, F)$. Let $v \in V$, $v \neq 0$. Because $[A]$ is a field, $va_1, \cdots, va_s$, are linearly independent over F. They span a subspace W of V which is invariant under $[A]$. Since $[A]$ is irreducible, $W = V$, and the dimension $[[A] : F]$ of $[A]$ over F is equal to n.

The group A is therefore isomorphic to a subgroup of the multiplicative group of a finite extension E of F such that $[E : Q] = nd$. The condition (U) satisfied by the elements of A implies that each $a \in A$ corresponds to a unit in the integers of E. By Dirichlet's theorem on units in an algebraic number field ([4], Ch. XI), the rank of the group of units of E cannot exceed $nd-1$.

Suppose A has an element of order $r > 1$. Then E contains a primitive r-th root of unity, ζ, say, and $\phi(r) = [Q(\zeta) : Q]$ divides $[E : Q] = nd$. Since each finite multiplicative group in a field is cyclic, we conclude that A_T is a cyclic group of order t such that $\phi(t)$ divides nd. By an obvious argument we obtain the following corollary.

2.2 COROLLARY. *A completely reducible abelian U-group in $GL(n, F)$ requires at most nd generators, where $d = [F : Q]$.*

3. Irreducible nilpotent U-groups

Let N be an irreducible nilpotent U-group in $GL(m, F)$.

3.1 THEOREM. *If the class of N is c, then*

$$c \leqq \begin{cases} 2m(1+\log_2 d) & \text{if } d > 1 \\ 2.12m & \text{if } d = 1 \end{cases}$$

where $d = [F : Q]$.

Note: This estimate depends only on the fact that $N \subseteq GL(m, F)$, not on the condition (U).

PROOF. Let $N = \gamma_1(N) \supset \gamma_2(N) \supset \cdots \supset \gamma_{c+1}(N) = 1$ be the lower central series for N, and let s be the smallest index such that $\gamma_s(N)$ is abelian. Since $[\gamma_i(N), \gamma_j(N)] \subset \gamma_{i+j}(N)$ we must have $s \leqq [c/2]+1$.

By Clifford's theorem, $\gamma_s(N)$ is completely reducible over F. Suppose $\gamma_s(N)$ has r homogeneous components. We shall show that $c \leqq 2r(1+\log_2 md/r)$.

In any irreducible nilpotent linear group the index of the centre is finite ([1], p. 64). $|N : Z(N)|$ finite implies $|\gamma_2(N)|$ finite ([6], problem 5.24). If $s \neq 1$, $\gamma_s(N)$ is therefore finite, and is a subgroup of a direct product of r cyclic groups of order t_1, where $\phi(t_1)$ divides md/r. Let Q_s be the Sylow q-subgroup of $\gamma_s(N)$, and suppose $|Q_s| = q^l$. Let Q_{s+j} be the Sylow q-subgroup of $\gamma_{s+j}(N)$. Then

$Q_{s+l} = 1$. Let p^α be the highest prime power dividing t_1. Then for each Q_s, $l \leqq r\alpha$, and so $\gamma_{s+r\alpha}(N) = 1$, and $c+1 \leqq s+r\alpha \leqq [c/2]+1+r\alpha$. Hence $c \leqq 2r\alpha$. Since $\phi(t_1)$ divides md/r, $p^{\alpha-1}$ divides md/r, and so $\alpha-1 \leqq \log_2 md/r$. We have now $c \leqq 2r(1+\log_2 md/r)$, and the result follows from this if we consider the maximum value of $2x(1+\log_2 md/x)$ over $1 \leqq x \leqq m$.

3.2 COROLLARY. There exist maximal irreducible nilpotent U-groups in $GL(m, F)$.

This follows from 3.1 by an application of Zorn's Lemma.

3.3 *Divisors of* $|N : Z(N)|$: All prime factors of $|N : Z(N)|$ divide the exponent m_2 of $Z_2(N)/Z(N)$, and m_2 divides m ([1], Chapter III, Lemmas 19 and 22). Also, if $xZ(N)$ is of order k in $Z_2(N)/Z(N)$, there exists $y \in N$ such that $[x, y]$ has order k ([1], Chapter III, Lemma 20). In our case we have an additional condition on m_2. $[x, y]$ lies in the torsion subgroup of $Z(N)$, which is cyclic, of order t_2. m_2 divides t_2, and $\phi(t_2)$ divides md.

3.4 In particular, *if md is odd, there are no non-abelian irreducible nilpotent U-groups in $GL(m, F)$.*

PROOF. If md is odd, $t_2 = 1$ or 2, since $\phi(t_2)$ divides md, and m_2 must be odd. Hence $m_2 = 1$ and N is necessarily abelian.

3.5 *Structure of N/A, where A is a maximal normal abelian subgroup of N.*

LEMMA. (i) *N/A is isomorphic to a nilpotent permutation group $\hat{N}$ of degree k, where k divides m.*

(ii) *If N is primitive (see [5], p. 346), $\hat{N}$ is semiregular (i.e. a permutation group in which only the identity leaves any symbol fixed).*

PROOF. (i) A is finitely generated. Choose a finite set of generators for A, and adjoin their eigenvalues to F. The field E obtained is a normal extension of F. If we consider N as a subgroup of $GL(m, E)$, N is completely reducible, and all its irreducible components are of equal degree. ([5], Theorems 69.4 and 70.15). A is also completely reducible over E. Since A is abelian, and each of a set of its generators can be diagonalised in $GL(m, E)$, A is reducible to a diagonal group.

Let W be a minimal invariant space for N in V^E (where V^E is an m-dimensional E-module, and we have identified $\mathrm{End}_E V$ with $M(m, E)$). Then $\dim W$ divides m. Let $\tau : x \to x|W$ (the restriction of $x \in N$ to W). Let $y \in \ker \tau \cap Z(N)$. $Z(N)$ is isomorphic to one of its own irreducible components. Hence $y|W = 1$ implies $y = 1$, and we have $\ker \tau \cap Z(N) = 1$. τ is therefore faithful. Define $N^* = N|W$, $A^* = A|W$. We shall prove the result for N^*/A^*.

A^* is reducible to a diagonal group. Let $W_1, \cdots, W_k$ be the distinct eigenspaces for A in W. $A^*|W_1, \cdots, A^*|W_k$ are the homogeneous components of A^*, and the spaces $W_1, \cdots, W_k$ are permuted by the elements of N^* (see [5], p. 345).

Since A^* is its own centralizer in N^* ([6], problem 6.36) we have $N^*/A^* \simeq \hat{N}$, a nilpotent permutation group on k symbols. k divides dim W, which divides m.

(ii) If N is a primitive group in $GL(m, F)$, A is isomorphic to one of its own irreducible components over F. Hence if $a \in A$, $a-1$ is either zero or invertible. If $x^* \in N^*$ fixes an eigenspace W_1 of A^* in W, then $[x, a]|W_1 = 1$ for all $a \in A^*$. This implies $[x, a] = 1$ for all $a \in A$, and so $x \in A$. $\hat{N}$ is therefore semiregular.

3.6 In particular, *suppose $\hat{N}$ is transitive, and $E = Q$ or $Q(\theta)$, where θ is complex of degree* 2 *over Q. Then N is finite.*

PROOF. $Z(N^*)$ is a group of scalar matrices $f \cdot 1, f \in E$, if $\hat{N}$ is transitive. By Dirichlet's Theorem, the group of units of E is finite, and so $Z(N^*)$ is finite. Hence $Z(N)$ and $|N : Z(N)|$ are both finite, and the result follows.

3.7 THEOREM. *If the class of N is* 2, *then N has a faithful absolutely irreducible representation in $GL(s, E)$ where E is the field defined in* 3.5, *and s divides m. It follows that $|N : Z(N)| = s^2$.*

PROOF. We shall show that the group N^* defined in 3.6 is absolutely irreducible.

(i) If the class of N is 2, $\hat{N}$ is semiregular. For: Let W be the space defined in 3.5, and W_1 an eigenspace for A^* in W. We have already: if $x^* = x|W \in N^*$ fixes W_1, then $[x, a]|W_1 = 1$ for all $a \in A$. Since the class of N is 2, $[x, a] \in Z(N)$. $Z(N)$ is isomorphic to one of its own irreducible components, and so $[x, a] = 1$ for all $a \in A$. Hence $x \in A$, and $\hat{N}$ must then be semiregular.

(ii) Let $w \neq 0 \in W_1$ and let $1 = x_1, x_2, \cdots, x_5$ be a complete set of coset representatives for A in N. Let L_1 be the space spanned by w, $x_i^* = x_i|W$, and define $L_j = L_1 x_j^*$ $j = 1, \cdots, s$. By (i) the L_j belong to distinct eigenspaces of A^* in W. They are permuted transitively by the elements of N^*. The space $L = L_1 \oplus \cdots \oplus L_s$ is a nonzero invariant space for N^* in W, and so $L = W$.

The construction of L shows that the centralizer of N^* in $M(s, E)$ can contain scalar matrices only. N^* is therefore the required representation of N. (see [5], p. 202).

(iii) $|N : Z(N)| = s^2$. This can be deduced from [1] Chapter I, Lemma 10. The following more elegant argument is due to Professor J. D. Dixon.

The linear hull of N^* over E has dimension s^2 ([5], Theorem 27.8). We can therefore find elements $x_1^*, \cdots, x_{s^2}^* \in N^*$ that form a basis for $M(s, E)$. Since $Z(N^*)$ is a group of scalar matrices, $x_1^*, \cdots, x_{s^2}^*$ are in distinct cosets of $Z(N^*)$ in N^*. We show that they form a complete set of coset representatives for $Z(N^*)$ in N^*.

Let $x^* \in N^*$, $x^* \notin Z(N^*)$. Then there exists $y^* \in N^*$ such that $[x^*, y^*] = z^* \in Z(N^*)$, $z^* = \zeta \cdot 1$, $\zeta \neq 1$, i.e. $(y^*)^{-1}x^*y^* = \zeta x^*$, $\zeta \neq 1$. Trace x^* = trace $(y^*)^{-1}x^*y^*$ = trace $\zeta x^* = \zeta$ trace x^*. Since $\zeta \neq 1$, trace $x^* = 0$. Now let x^* be

https://doi.org/10.1017/S1446788700009812 Published online by Cambridge University Press

any element of N^*. $x^* = \sum_{i=1}^{s^2} \alpha_i x_i^*$, $\alpha_i \in E$. At least one $\alpha_j \neq 0$. Trace $x^*(x_j^*)^{-1} = \sum_{i=1}^{s^2} \alpha_i$ trace $(x_i^*(x_j^*)^{-1}) = s\alpha_j \neq 0$. As we have just seen, this implies $x^*(x_j^*)^{-1} \in Z(N^*)$, and gives the result.

4. Completely reducible soluble U-groups

4.1 Let S be an irreducible soluble U-group in $GL(n, F)$. Suppose S is maximal with respect to the property of being soluble.

Suppose S is imprimitive. Let V be an n-dimensional F-module, and identify $\mathrm{End}_F V$ with $M(n, F)$. Let $V = V_1 \oplus \cdots \oplus V_k$ be a complete decomposition of V into systems of imprimitivity for S (cf. [1], p. 7). By an argument similar to that used in the proof of Lemmas 3 and 4 of [1], Chapter I, S has a normal subgroup G for which each V_i, $i = 1, \cdots, k$, is an invariant space, such that S/G is isomorphic to a maximal soluble permutation group of degree k. G is the direct product of the groups $G|V_i$, $i = 1, \cdots, k$. Each $G|V_i$ is isomorphic to $G|V_1$, which is a maximal irreducible primitive soluble U-group in $GL(n/k, F)$.

4.2 Let S be a primitive irreducible soluble U-group in $GL(n, F)$. We describe S by describing the factors in the series

$$1 \lhd A \lhd B \lhd C \lhd S$$

where A is a maximal normal abelian subgroup of S, C the centraliser of A in S, and B the Fitting subgroup of C. Suprunenko ([1], Chapter I) uses a similar decomposition to describe primitive soluble linear groups, except for a different choice of B. Our choice of B allows us to use information about irreducible nilpotent U-groups.

4.3 *The group A*: By Clifford's Theorem A is completely reducible over F. Since S is primitive, all the irreducible components of A are equivalent, and so A is isomorphic to an irreducible abelian U-group in $GL(t, F)$,where t divides n. The results of 2.1 then apply to A.

4.4 LEMMA. *B is nilpotent, of class at most* 2.

PROOF. The Fitting subgroup of any linear group is nilpotent ([9], Theorem 1 (ii)). Let the class of B be c, $B = \gamma_1(B) \supset \gamma_2(B) \supset \cdots \supset \gamma_{c+1}(B) = 1$ the lower central series for B, and r the smallest index such that $\gamma_r(B)$ is abelian. $\gamma_r(B) \subset C$, and so $\gamma_r(B) \cdot A$ is abelian, and normal in S. By the maximality of A, $\gamma_r(B) \subset A = Z(B)$. We have therefore $c \leqq r$, and, by the argument used in 3.1, $r \leqq [c/2]+1$. Hence $c \leqq 2$.

4.5 Since S is primitive, B is isomorphic to one of its own irreducible components, and so, if $c = 2$, we can apply 3.7, with s a divisor of n/t (where t is the degree of an irreducible component of A). The primes dividing s must satisfy the conditions of 3.3.

4.6 In particular, *if nd is odd,* $B = A = C = H$, *where H is the Fitting subgroup of S.*

Proof. By 3.4, *nd* odd implies B and H are both abelian. If $C \neq A$, C/A contains a non-trivial characteristic abelian subgroup K/A, and K is necessarily nilpotent, giving a contradiction.

4.7 *The group* B/A: Suppose B/A is non-trivial. Since B is the Fitting subgroup of C, B/A is the maximal normal abelian subgroup of C/A. B/A is equal to its own centralizer in C/A (cf. [1], Chapter I, proof of Theorem 4). By [1], Chapter I, Lemma 15, the Sylow q-subgroups of B/A are elementary abelian q-groups.

4.8 *The groups* C/B *and* S/B: By [1], Chapter I, Theorem 11, if $s = q_1^{\alpha_1} \cdots q_k^{\alpha_k}$, C/B is isomorphic to a soluble subgroup of the direct product of the symplectic groups $\mathrm{Sp}(2\alpha_1, q_1), \cdots, \mathrm{Sp}(2\alpha_k, q_k)$.

By an argument similar to that of 3.5 we obtain: S/C is isomorphic to a soluble semiregular permutation group of degree t. (cf. [1], p. 12). For these two factors we obtain no special restrictions.

4.9 Theorem. *A completely reducible soluble U-group S in* $GL(n, F)$ *satisfies the maximum condition for subgroups.*

We shall prove the equivalent condition that all subgroups of S are finitely generated.

Proof. (i) If S is a primitive irreducible soluble U-group, it is a finite extension of a finitely generated abelian group, and the result follows. This extends to the maximal imprimitive irreducible case by 4.1, and therefore to any irreducible soluble U-group in $GL(n, F)$.

(ii) If S is completely reducible, with $V = V_1 \oplus \cdots \oplus V_k$ a direct sum of minimal S-invariant subspaces, then S is isomorphic to a subgroup of $S|V_1 \times S|V_2 \times \cdots \times S|V_k$. Each $S|V_i$, $i = 1, \cdots, k$, is an irreducible soluble U-group in $GL(n_i, F)$, where $n_i = \dim V_i$. The result then holds for each $S|V_i$, and hence for $S|V_1 \times \cdots \times S|V_k$ and for S.

4.10 Corollary. *If S is any completely reducible soluble U-group in* $GL(n, F)$, *we can apply two theorems of Hirsch to conclude*:

(i) *S is polycyclic* [7], p. 193.

(ii) *If S is infinite, S has a normal subgroup H such that* $|S : H|$ *is finite, and H has a normal series* $H = H_0 \supset H_1 \supset \cdots \supset H_k = 1$, *in which each factor* H_{i-1}/H_i, $i = 1, \cdots, k$, *is an infinite cyclic group* [8], *p*. 188. We can actually take H to be a finitely generated torsion-free abelian group, since we have a bound on the orders of torsion elements in a maximal normal abelian subgroup of finite index in S.

Note. It follows from Mal'cev's Theorem ([1], p. 31) that any completely reducible soluble linear group is an extension of an abelian group by a finite group. 4.9 and 4.10 can therefore be made to follow directly from 2.1 (but without intermediate results 4.4–4.6). I am indebted to the referee for this comment.

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