AN ESTIMATE OF SECTIONAL CURVATURES OF HYPERSURFACES WITH POSITIVE RICCI CURVATURES

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Let M be a hypersurface in Euclidean space and let the Ricci curvature of M be bounded below by some nonnegative constant. In this paper, we estimate the sectional curvature of M in terms of the lower bound of Ricci curvature and the upper bound of mean curvature.

1. Introduction

It is well-known that if Ricci curvature of a complete Riemannian manifold M is bounded below by a positive constant then it is compact. If M is a hypersurface then the principal curvatures are defined at each point. In this paper, we estimate the bounds of sectional curvature in terms of the lower bound of Ricci curvature and the upper bound of mean curvature. Throughout this paper, M^n or M is an n-dimensional complete connected hypersurface embedded in R^{n+1}, where $n \geq 3$. And we denote the mean curvature (the trace of the second fundamental form) by H and the sectional curvature by K.

2. Estimates of sectional curvature

Lemma 2.1. *The Ricci curvature is nonnegative on M^n if and only if the sectional curvature is nonnegative on M^n.*

Proof. Fix $p \in M$. Let $\{e_i\}_{1 \leq i \leq n}$, $\{\lambda_i\}_{1 \leq i \leq n}$ be the principal vectors and the corresponding principal curvatures respectively, at p. We may assume that $\lambda_1 \leq \cdots \leq \lambda_n$. Let H be the mean curvature at P, that is, $H = \lambda_1 + \cdots + \lambda_n$. We may assume that $H \geq 0$. Suppose $Ric(x,x) = 0$ for each unit vector $x \in T_p(M)$. Then $Ric(e_i, e_i) = \lambda_i(H - \lambda_i) = 0$. Hence either all λ_i's are 0 or only one λ_i is nonzero. Hence the sectional curvature is 0. Suppose $Ric(x,x) > 0$ for each unit vector. We may assume that $H \geq 0$. If $\lambda_i < 0$ for some i, then $Ric(e_i, e_i) = \lambda_i(H - \lambda_i) < 0$, which is a contradiction. So it follows that all λ_i's are positive. Let x, y be unit vectors with $x \perp y$ and $x = \sum_{i=1}^{n} a_i e_i, y = \sum_{j=1}^{n} b_j e_j$. Then

$$K(x,y) = \langle A(x,x), A(y,y) \rangle - \langle A(x,y), A(x,y) \rangle$$

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$$= \left(\sum_{i=1}^{n} a_i^2 \lambda_i \right) \left(\sum_{j=1}^{n} b_j^2 \lambda_j \right) - \left(\sum_{i=1}^{n} a_i b_i \lambda_i \right)^2$$

$$= \sum_{i<j} (a_i b_j - a_j b_i)^2 \lambda_i \lambda_j,$$

where A is the second fundamental form. Hence the sectional curvature K at p is positive. The proof of the other direction is clear.

Corollary 2.2. *If $a \leq Ric(x, x)$ for some positive constant a and for each unit vector $x \in TM$, then M is diffeomorphic to S^n.*

Proof. Since $0 < a \leq Ric(x, x)$, M is compact. And the second fundamental form A is definite everywhere on M by Lemma 2.1. So the spherical map of Gauss $M \to S^n$ is a diffeomorphism ([3]).

The following lemma is a corollary of the Gauss equation, but we give another elementary proof.

Lemma 2.3. *If $a \leq Ric(x, x)$ for some positive constant a and for each unit vector $x \in TM$, then $(n^2 a/(n-1) \leq H^2$, where H is the mean curvature of M.*

Proof. Let $\{\lambda_i\}$, $\{e_i\}$ denote as in the proof of (2.1) above. We may assume that $H > 0$. Then all λ_i's are positive. Since $a \leq \lambda_1(H - \lambda_1)$ and $H > 0$, we have

$$\lambda_1 \geq \frac{H - \sqrt{H^2 - 4a}}{2}$$

and $H^2 \geq 4a$. Since $\lambda_1 \leq H/n$, we have

$$\frac{n(H - \sqrt{H^2 - 4a})}{2} \leq n\lambda_1 \leq H,$$

that is,

$$n(H - \sqrt{H^2 - 4a}) \leq 2H.$$

So we have

$$H^2 \geq \frac{n^2 a}{n-1}.$$

Theorem 2.4. *If $o < a \leq Ric(x, x)$ for all unit vectors $x \in TM$ and the mean curvature $|H| \leq H_0$ for some constant H_0, then we have*

$$0 < \frac{H_0(H_0 - \sqrt{H_0^2 - 4a})}{2} - a \leqq K < \frac{H_0^2}{4},$$

where K is the sectional curvature of M^n.

Proof.   Let $\{\lambda_i\}$, $\{e_i\}$ be as in the proof of (2.1) above. Let x, y be unit vectors with $x \perp y$ and $x = \sum_{i=1}^n a_i e_i$, $y = \sum_{j=1}^n b_j e_j$. Then by the proof of (2.1), we have

$$K(x, y) = \sum_{i < j} (a_i b_j - a_j b_i)^2 \lambda_i \lambda_j.$$

Since $\sum_{i=1}^n a_i e_i$, $\sum_{j=1}^n b_j e_j$ are orthonormal vectors,

$$1 = (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) - (a_1 b_1 + \cdots + a_n b_n)^2$$

$$= \sum_{i \neq j} a_i^2 b_j^2 - 2 \sum_{i < j} a_i b_i a_j b_j$$

$$= \sum_{i < j} (a_i b_j)^2 - 2 \sum_{i < j} a_i b_j a_j b_i + \sum_{i < j} (a_j b_i)^2$$

$$= \sum_{i < j} (a_i b_j - a_j b_i)^2.$$

We may assume that $H > 0$. Then $0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$ by (2.1). Then we have $\lambda_1 \lambda_2 \leqq K(x, y) \leqq \lambda_{n-1} \lambda_n$. By the proof of (2.3),

$$\frac{H - \sqrt{H^2 - 4a}}{2} \leqq \lambda_1,$$

so that

$$\frac{H(H - \sqrt{H^2 - 4a})}{2} - a \leqq \lambda_1^2,$$

and

$$H^2 \geqq \frac{n^2 a}{n - 1}.$$

If we define

$$g(X) = \frac{X - \sqrt{X^2 - 4aX}}{2},$$

then $g'(X) < 0$ when $X > 4a$. By letting $H^2 = X$, we see that

$$\frac{H(H - \sqrt{H^2 - 4a})}{2} = \frac{H^2 - \sqrt{H^4 - 4aH^2}}{2}$$

is monotone decreasing from $na/(n-1)$ to a on the interval $[\sqrt{n^2 a/(n-1)}, \infty)$ as a function of H. So we have

$$0 < \frac{H_0(H_0 - \sqrt{H_0^2 - 4a})}{2} - a \leqq \lambda_1^2 \leqq \lambda_1 \lambda_2 \leqq K.$$

On the other hand,

$$\lambda_{n-1}\lambda_n < (H - \lambda_n)\lambda_n \leqq \frac{(H - \lambda_n + \lambda_n)^2}{4} = \frac{H^2}{4} \leqq \frac{H_0^2}{4},$$

so we have $K(x, y) < H_0^2/4$.

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